On existence of solutions for nonconvex optimal control problems

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Abstract: We present new sufficient conditions for existence of solutions to some nonconvex and noncoercive Lagrange optimal control problems.

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1 Introduction

Consider the nonconvex Lagrange optimal control problem $(\mathcal{OCP})$ which consists in minimizing the integral functional

$$J(x,u) := \int_a^b f_0(x(t), u(t))\, dt$$

over all pairs $(x(\cdot), u(\cdot))$, with $x(\cdot) \in W^{1,1}([a,b], \mathbb{R}^n)$ and $u : [a,b] \to \mathbb{R}^m$ measurable, satisfying

$$x(t) \in \Omega \subset \mathbb{R}^n \ \ \forall t \in [a,b]$$
$$x(a) = A, \ \ x(b) = B$$
$$u(t) \in U(x(t)) \ \ for \ \ a.e. \ \ t \in [a,b]$$
$$x'(t) = f(x(t), u(t)) \ \ for \ \ a.e. \ \ t \in [a,b],$$

where

$$U : \Omega \to 2^{\mathbb{R}^m} \setminus \emptyset$$
$$f_0 : graph\,(U) \to [0, \infty)$$
$$f : graph\,(co\,U) \to \mathbb{R}^n.$$

Under adequate hypotheses – which we will state below – our strategy to prove existence of solutions to $(\mathcal{OCP})$ consists in the following steps:

Step 1. Take a solution $(\overline{x}_c(\cdot), \overline{u}_c(\cdot))$ for the associated convexified problem $(\mathcal{OCP}_c)$:

$$minimize$$
$$J_c(x,u) := \int_a^b f_0^{**}(x(t), u(t))\, dt$$

where $f_0^{**}(s,\cdot)$ is the bipolar of $f_0(s,\cdot)$, defined by $epi\, f_0^{**}(s,\cdot) = \overline{co}\, epi\, f_0(s,\cdot)$ (namely: the epigraph of $f_0^{**}(s,\cdot)$ is the closed convex hull of the epigraph of $f_0(s,\cdot)$), over all pairs $(x(\cdot), u(\cdot))$, with $x(\cdot) \in W^{1,1}([a,b], \mathbb{R}^n)$ and $u : [a,b] \to \mathbb{R}^m$ measurable, satisfying

$$x(t) \in \Omega \ \ \forall t \in [a,b]$$
$$x(a) = A, \ \ x(b) = B$$
$$u(t) \in co\,U(x(t)) \ \ for \ \ a.e. \ \ t \in [a,b]$$
$$x'(t) = f(x(t), u(t)) \ \ for \ \ a.e. \ \ t \in [a,b].$$

Step 2. Reformulate $(\mathcal{OCP}_c)$ as an adequate Lagrange problem of the calculus of variations $(\mathcal{CVP}_c)$:

$$minimize$$
$$I_c(x) := \int_a^b L^{**}(x(t), x'(t))\, dt$$

on

$$\mathcal{X}^n_{A,B} := \left\{ \begin{array}{l} x(\cdot) \in W^{1,1}([a,b], \mathbb{R}^n) : \\ x(a) = A, \ \ x(b) = B \end{array} \right\},$$

for which $\overline{x}_c(\cdot)$ is a solution.

Step 3. Prove existence of a solution $\overline{x}(\cdot)$ for the nonconvex problem $(\mathcal{CVP})$:

$$minimize$$
$$I(x) := \int_a^b L(x(t), x'(t))\, dt$$
$$on \ \ \mathcal{X}^n_{A,B}.$$

Step 4. Use such $\overline{x}(\cdot)$ to obtain a solution $(\overline{x}(\cdot), \overline{u}(\cdot))$ for the nonconvex problem $(\mathcal{OCP})$.

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In this paper we present new hypotheses under which – through the above-stated strategy – we can prove existence of solutions to our nonconvex optimal control problem $(\mathcal{OCP})$.

To begin with we describe our recent result of existence of minimizers to the integral $I(\cdot)$. This will be a crucial tool to prove the main result of this paper.

2 The Lagrange problem of the calculus of variations

Let $L : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, \infty]$. We say that $L(s, \cdot)$ is almost convex provided

$$\begin{aligned}
&\forall \xi \text{ where } L^{**}(s,\xi) < L(s,\xi), \\
&\exists \lambda = \lambda(s,\xi) \in [0,1] \;\; \exists \Lambda = \Lambda(s,\xi) \in [1,\infty) \\
&\exists \alpha = \alpha(s,\xi) \in [0,1] \; : \\
&L^{**}(s,\xi) = (1-\alpha)\, L(s,\lambda\xi) + \alpha\, L(s,\Lambda\xi) \\
&\xi = (1-\alpha)\,(\lambda\xi) + \alpha\,(\Lambda\xi)
\end{aligned}$$

(we set $\lambda = 1 = \Lambda = \alpha$ where $L^{**}(s,\xi) = L(s,\xi)$ and use $0 \cdot \infty := 0$).

Remark 1 *If $L(s, \cdot)$ is almost convex, then*

(i) $L^{**}(s,0) = L(s,0)$

(ii) $L^{**}(s,\lambda\xi) = L(s,\lambda\xi)$ & $L^{**}(s,\Lambda\xi) = L(s,\Lambda\xi)$ $\forall \xi \in \mathbb{R}^n$

(iii) $L^{**}(s, \cdot)$ restricted to the segment $[\lambda\xi, \Lambda\xi]$ is affine.

Remark 2 *The concept of almost convexity was introduced by A. Cellina & A. Ornelas in the paper [3], for sets, to prove existence of solutions to nonconvex upper semicontinuous differential inclusions and time optimal control problems. For functions, it was introduced in [2] to prove results of existence of solutions to nonconvex problems of the calculus of variations.*

Proposition 3 *(See [1].) Let $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ be a Borel function bounded from below, having $L^{**}(\cdot,\cdot)$ Borel and $L(s,\cdot)$ almost convex lower semicontinuous (lsc) for each s.*

Assume also the following extra hypothesis (EH):

there exists a minimizer $\overline{x}_c(\cdot)$ of $I_c(\cdot)$ for which either the set
$E_0 := \{t \in [a,b] : \lambda(\overline{x}_c(t), \overline{x}_c'(t)) = 0\}$
has zero measure
or else:
$\overline{E_0} \setminus E_0$ *is a null set (with* $\overline{E_0}$ *the closure of* E_0*)*
and
$\exists\, c \in [a,b]$ *such that*
$L(\overline{x}(c), 0) \le L(\overline{x}(t), 0) \;\; \forall t \in [a,b]$.

Then there exists a minimizer $\overline{x}(\cdot)$ to the nonconvex integral $I(\cdot)$.

3 Main result

Assume that:

(H1) $\Omega \subset \mathbb{R}^n$ is closed

(H2) $U : \Omega \to 2^{\mathbb{R}^m} \setminus \emptyset$ is such that

- (H2.1) $graph(U) := \left\{ \begin{array}{l} (s,u) \in \mathbb{R}^n \times \mathbb{R}^m : \\ s \in \Omega,\ u \in U(s) \end{array} \right\}$ is closed
- (H2.2) $graph(co\,U)$ is closed

(H3) $f_0 : graph(U) \to [0,\infty)$ is lsc

(H4) $f_0^{**} : graph(co\,U) \to [0,\infty)$ is lsc

(H5) $f : graph(co\,U) \to \mathbb{R}^n$, $f(s,u) = A_0(s) + B_0(s)\,u$,

where, for every $s \in \Omega$,

$A_0(s)$ is a $n \times 1$ matrix
$B_0(s)$ is a $n \times m$ matrix
$A_0(\cdot)_{i1}$, $B_0(\cdot)_{ij} : \Omega \to \mathbb{R}$ are continuous $(i = 1,...,n;\ j = 1,...,m)$

(H6) for every $s \in \Omega$, the sets

$$f(s, co\,U(s))$$

$$\{(f(s,u), f_0(s,u) + v) : u \in U(s), v \ge 0\}$$

$$\{(f(s,u), f_0(s,u) + v) : u \in co\,U(s), v \ge 0\}$$

are closed

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(H7) $\exists\left(\overline{x}_c(\cdot),\overline{u}_c(\cdot)\right)$ solution for $(\mathcal{OCP}_c)$ and $I_c\left(\overline{x}_c,\overline{u}_c\right)<\infty$.

Moreover, defining for every $s\in\Omega$,

$$R(s,\gamma v):=\left\{\begin{array}{l} u\in U(s): \\ B_0(s)(\gamma v-u)=(1-\gamma)A_0(s)\end{array}\right\} \quad \forall v\in U(s),\ \forall\gamma\geq 0,$$

$$R_0(s,v_0):=\left\{\begin{array}{l} u\in co\,U(s): \\ B_0(s)(v_0-u)=0\end{array}\right\} \quad \forall v_0\in co\,U(s),$$

suppose

(H8)

$$\begin{array}{l}\forall u_0\in co\,U(s)\ \ with \\ \inf\ f_0^{**}\left(s,R_0\left(s,u_0\right)\right)<\inf\ f_0\left(s,R\left(s,u_0\right)\right) \\ \exists\,\lambda=\lambda\left(s,u_0\right)\in[0,1] \\ \exists\,\Lambda=\Lambda\left(s,u_0\right)\in[1,+\infty)\ \ for\ which \\ \inf\ f_0^{**}\left(s,R_0\left(s,u_0\right)\right)= \\ =(1-\alpha)\inf\ f_0\left(s,R\left(s,\lambda\,u_0\right)\right)+ \\ +\alpha\ \inf\ f_0\left(s,R\left(s,\Lambda\,u_0\right)\right)\end{array}$$

with $\alpha:=\frac{1-\lambda}{\Lambda-\lambda}$ and $0\cdot\infty=:0$; if

$$\inf\ f_0^{**}\left(s,R_0\left(s,u_0\right)\right)=\inf\ f_0\left(s,R\left(s,u_0\right)\right),$$

set $\alpha=1=\lambda=\Lambda$.

Finally,

(H9) in case the set

$$D_0:=\left\{t\in[a,b]:\lambda\left(\overline{x}_c(t),\overline{u}_c(t)\right)=0\right\}$$

has positive measure, assume

$$\overline{D_0}\setminus D_0 \text{ is a null set}$$

and $\exists\,c\in[a,b]$ for which we can find some $u_c\in U\left(\overline{x}(c)\right)$ satisfying:

$$A_0\left(\overline{x}(c)\right)+B_0\left(\overline{x}(c)\right)u_c=0$$

$$f_0\left(\overline{x}(c),u_c\right)\leq$$

$$\leq\inf\left\{\begin{array}{l} f_0\left(\overline{x}(t),u\right): \\ u\in U\left(\overline{x}(t)\right)\ \ \& \\ A_0\left(\overline{x}(t)\right)+B_0\left(\overline{x}(t)\right)u=0\end{array}\right\} \quad \forall t\in[a,b].$$

Then our main result is

Theorem 4 *Under* $(H1)-(H9)$ *problem* $(\mathcal{OCP})$ *has a solution.*

Proof: Let $L:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty]$ be the function defined by

$$L(s,\xi):=\inf\left\{f_0(s,u):u\in H(s,\xi)\right\},$$

where

$$\begin{aligned} H(s,\xi) &:= \left\{u\in U(s):\xi=f(s,u)\right\} \\ &= \left(f(s,\cdot)\right)^{-1}(\xi)\cap U(s).\end{aligned}$$

Hypotheses $(H1)$ to $(H6)$ guarantee that $L(\cdot)$ is a well-defined Borel function having $L^{**}(\cdot)$ Borel and $L(s,\cdot)$ lsc $\forall s$; $L(s,\cdot)$ is also almost convex $\forall s$, by $(H8)$.

On the other hand, $\overline{x}_c(\cdot)$ is a minimizer of $I_c(\cdot)$ for which, due to assumption $(H9)$, the extra hypothesis (EH) is satisfied.

Therefore, by proposition 3, there exists a minimizer $\overline{x}(\cdot)$ for $(\mathcal{CVP})$. It follows that there also exists a measurable $\overline{u}:[a,b]\to\mathbb{R}^m$ such that $\overline{x}'(t)=f\left(\overline{x}(t),\overline{u}(t)\right)$ for a.e. $t\in[a,b]$ and $\left(\overline{x}(\cdot),\overline{u}(\cdot)\right)$ is a solution for the nonconvex problem $(\mathcal{OCP})$. $\square$

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