Portfolio Optimization
in the Financial Market with Regime Switching under Constraints,
Transaction Costs and Different Rates for Borrowing and Lending

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Abstract: - In this work, we consider the optimal portfolio selection problem under hard constraints on trading amounts, transaction costs and different rates for borrowing and lending when the dynamics of the risky asset returns are governed by a discrete-time approximation of the Markov-modulated geometric Brownian motion. The states of Markov chain are interpreted as the states of an economy. The problem is stated as a dynamic tracking problem of a reference portfolio with desired return. Our approach is tested on a set of a real data from Russian Stock Exchange MICEX.

Key-Words: - Investment portfolio, Regime switching models, Transaction costs

1 Introduction
In recent years considerable interest has been focused on regime-switching, or Markov-modulated, models to describe the behavior of the economic systems [4,13]. Models with Markov jumps are successfully used to describe the price dynamics of the risky asset for investment. This is due to the fact that these models can explain some important features of real financial markets.

Motivated by the great importance of these models from both theoretical and practical perspectives, optimization techniques to various portfolio selection problems under Markov regime-switching models have been intensively studied in the literature [2,3,5,8,9,11,14-16,22-25].

In [11,14] dynamic optimal consumption and portfolio choice problem, where expected returns of a risky asset follow a hidden Markov chain is studied. In [24] a Markowitz’s mean-variance portfolio selection with regime-switching problem is presented. The proposed target function looks only at the final variance of the portfolio. Costa and Araujo [5] consider a multi-period generalized mean-variance model with Markov switching in the key market parameters. In this paper, all intermediate values of the mean and of the variance of the portfolio are involved in the objective. In [25] the authors investigate the optimal portfolio selection problem subject to a maximum value-at-risk constraint where the expected return and the volatility of the risky asset switch over time according to a continuous-time Markov chain. Sotomayor and Cadenillas [22] studied an optimal investment problem with the bankruptcy constraint under a Markovian regime-switching model for the asset price dynamics. Wu [23] investigates a non-self-financing mean-variance portfolio selection model in a Markov regime-switching jump-diffusion market with a stochastic cash-flow. In [15,16] the portfolio selection problem in the financial market with regime switching is stated as a dynamic tracking problem of a reference portfolio with desired return under quadratic performance criterion.

The vast majority of the existing literature on dynamic portfolio selection is based on using dynamic programming approach for determining the solution. However, that approach leads to the well-known “curse of dimensionality,” which hinders design of the decision strategies under constraints. Therefore, the most of the results presented in the literature are limited to the cases without trading constraints and transaction costs. Also, the rates of borrowing and lending are assumed to be the same. However, it’s well-known that realistic investment models must include these features [12].

In this paper, we consider the dynamic investment portfolio selection problem subject to hard constraints on trading amounts (a borrowing limit on the total wealth invested in the risky assets, and long- and shortsale restrictions on all risky
assets), taking into account the presence of quadratic transaction costs. Other realistic feature we incorporate is that in our model the rates of borrowing and lending are different (the rate of borrowing is greater than that of lending). We assume that the mean return, variance and covariance of the risky assets switch over time according to a Markov chain, whose states are interpreted as the states of the economy.

The problem is stated as a dynamic tracking problem of a reference portfolio with desired return. The investor’s objective is to choose the dynamic trading strategy to minimize the conditional mean-square error between the investment portfolio value and a reference (benchmark) portfolio, penalized for the transaction costs associated with trading. We consider quadratic transaction costs. The natural interpretation of a quadratic cost is that price impact is linear in the trade size, resulting in a quadratic cost [17].

In this work, we use the model predictive control (also known as receding horizon control) method in order to solve the problem. The major attraction of such technique lies in the fact that it can handle hard constraints on the inputs (manipulated variables) and states/outputs of a process and allows to avoid the “curse of dimensionality” [6-8,10,18-20].

The purpose of the present paper is to provide numerically tractable algorithm for practical applications. We want to demonstrate the performance of our model under real market conditions. We pay a particular attention to testing of our approach on a set of a real data from the Russian Stock Exchange MICEX.

This work is organized as follows. Section 2 presents portfolio model and the optimization problem formulation. The main results of this article are presented in Section 3 where we design the optimal investment strategy for the problem under consideration. In Section 4 the numerical modeling results are presented. This paper is concluded in Section 5 with some final remarks.

2 Portfolio Model and Optimization Problem

2.1 The Proposed Portfolio Model

Let us consider the investment portfolio of $n$ risky assets and one risk-free asset (e.g. a bank account or a government bond). Let $u_i(k)$, $(i=0,1,2,\ldots,n)$ denote the amount of money invested in the $i$th asset at time $k$; $u_0(k) \geq 0$ is the amount invested in a risk-free asset. Investor also can borrow the capital in case of need. The volume of the borrowing of the risk-free asset is equal to $u_{n+1}(k) \geq 0$. If $u_i(k) < 0$, $(i=1,2,\ldots,n)$, then we use short position with the amount of shorting $|u_i(k)|$.

The wealth process $V(k)$ satisfies

$$V(k) = \sum_{i=1}^{n} [1 + \eta_i(k+1)]u_i(k) + [1 + r_1]u_0(k) - [1 + r_2]u_{n+1}(k),$$

with initial value $V(0)$, where $r_1$ is the riskless lending rate, $r_2$ is the riskless borrowing rate ($r_1 < r_2$).

Using (1), the dynamics in (2) can be rewritten as follows:

$$V(k+1) = [1 + r_1]V(k) + \sum_{i=1}^{n} [\eta_i(k+1) - r_1]u_i(k) - [r_2 - r_1]u_{n+1}(k),$$

here $u_0(k) = V(k) - \sum_{i=1}^{n} u_i(k) + u_{n+1}(k)$ is the amount invested in a risk-free asset.

We impose the following constraints on the decision variables (a borrowing limit on the total wealth invested in the risky assets, and long- and short-sale restrictions on all risky assets)

$$u_{i}^{\text{min}}(k) \leq u_i(k) \leq u_{i}^{\text{max}}(k), \quad (i = 1, n),$$

$$0 \leq V(k) - \sum_{i=1}^{n} u_i(k) + u_{n+1}(k) \leq u_0^{\text{max}}(k),$$

$$0 \leq u_{n+1}(k) \leq u_{n+1}^{\text{max}}(k).$$

If $u_{i}^{\text{min}}(k) < 0$, $(i=1,2,\ldots,n)$, so we suppose that the amounts of the short-sale are restricted by $|u_{i}^{\text{min}}(k)|$; if the short-selling is prohibited then $u_{i}^{\text{min}}(k) \geq 0$, $(i=1,2,\ldots,n)$. The amounts of long-sale are restricted by $u_{i}^{\text{max}}(k)$, $(i=1,2,\ldots,n)$; $u_0^{\text{max}}(k) \geq 0$ defined the maximum amount of money we can invest in the risk-free asset; the borrowing amount is restricted by $u_{n+1}^{\text{max}}(k) \geq 0$. Note, that values $u_{i}^{\text{max}}(k)$, $(i=0,1,\ldots,n)$, $u_0^{\text{max}}(k)$, $(i=0,1,\ldots,n+1)$ are often depend on common wealth of portfolio in practice.
So that we can write $u^{\text{inv}}(k)=\beta_i V(k)$, $u^{\text{max}}(k)=\gamma_i V(k)$, where $\beta_i$, $\gamma_i$ are constant parameters.

The returns of assets in which we are able to invest are described by a discrete-time approximation of the geometric Brownian motion with parameters depended on the state of the Markov chain $\alpha(k)$

$$\eta_i[\alpha(k),k] = \mu_i[\alpha(k),k] + \sum_{j=1}^{\nu} \sigma_{ij}[\alpha(k),k] w_j(k), \quad (7)$$

where $\mu_i[\alpha(k),k]$ is the expected return of the $i$th risky asset; $\sigma_{ij}[\alpha(k),k]$ is the volatility of the $i$th risky asset; $\sigma_i[\alpha(k),k]$ is the volatility matrix, $[\alpha(k),k]$ is the sequence of states, $\{\alpha(k); k=0,1,2,\ldots\}$ is a finite-state discrete-time Markov chain taking values in $\{1,2,\ldots,\nu\}$ with transition probability matrix

$$P = \begin{bmatrix} P_{ij} \end{bmatrix}, \quad (i, j \in \{1,2,\ldots,\nu\})$$

and initial distribution

$$p_i = P[\alpha(0)=i], \quad (i=1,2,\ldots,\nu), \quad \sum_{i=1}^{\nu} p_i = 1.$$  

We assume that $\alpha(k)$ and $\omega(k)$ are mutually independent. From (7), the expected return, variance and covariance of the assets at time $k$ are affected by local or global factors, which are represented by the market operation mode $\alpha(k)$. When the market operation mode is $\alpha(k)=\alpha_g$, then $\mu_i[\alpha(k),k]=\mu_i^{(g)}$, $\sigma_i[\alpha(k),k]=\sigma_i^{(g)}$ represents the expected return of the $i$th asset, $\sigma_i[\alpha(k),k]$ is the volatility matrix of the returns.

The Markovian chain defines the state of a market, e.g., a market in a state of high or low volatility and/or a market in a state of ascending or descending trend. We assume that at the instant of decision making, the current state of the market is known, i.e., the Markov state $\{\alpha(k)\}$ is observable. When practical problems are solved, indicators of a market state can be market indices.

### 2.2 Optimization Problem (Risk Function)

Our objective is to control the investment portfolio, via dynamics asset allocation among the $n$ stocks and the risk-free asset, as closely as possible tracking the deterministic benchmark

$$V^0(k+1) = [1 + \mu_0] V^0(k), \quad (8)$$

where $\mu_0$ is a given parameter representing the growth factor, the initial state is $V^0(0)=V(0)$.

We use the MPC methodology in order to define the optimal control portfolio strategy. For the given prediction horizon $m$, a sequence of predictive controls (trading volume amounts)

$$u(k/k), u(k+1/k), \ldots, u(k+m-1/k)$$

depending on the portfolio wealth and the market state at the current time $k$ is calculated at each step $k$. This sequence optimizes the criterion chosen by the investor for the prediction horizon. At the time $k$, $u(k)=u(k/k)$ is assumed to be control $u(k)$. To obtain the control at the next step $k+1$, the procedure is repeated, and the control horizon is one step shifted.

We consider the following objective with receding horizon (risk function)

$$J(k+m/k) = E\left[ \sum_{i=1}^{n} \left[V(k+i) - V^0(k+i)\right]^2 \right] - \rho(k,i) \left[V(k+i) - V^0(k+i)\right] V(k), \alpha(k) + \sum_{i=1}^{n} E\left[ \left[u(k+i/k) - u(k+i-1/k)\right]^2 R(k,i) \times \left[u(k+i/k) - u(k+i-1/k)\right] V(k), \alpha(k) \right],$$

where $m$ is the prediction horizon, $u(k+i/k)$ is the optimal control vector obtained on the previous step, $u(-1/0)=0$; $R(k,i)>0$ is a positive-definite symmetric matrix measuring the level of transaction costs, $\rho(k,i)>0$ is a positive weight coefficient; $E\left[a/b\right]$ is the conditional expectation of $a$ with respect to $b$. Notice that variable $V^0(k)$ is known for all time instant $k$ and may be considered as a pre-chosen parameter.

Let us explain the terms in the objective function (9). The first term represents the conditional mean-square error between the investment portfolio value and a reference (benchmark) portfolio, the second term penalizes wealth values that less than the desired value. The third term penalizes for transaction costs associated with trading amount $[u(k+i/k) - u(k+i-1/k)]$.

An important advantage of tracking a reference portfolio approach under quadratic criterion (9) is its capability to predict the trajectory of growth portfolio wealth, which would follow close to the deterministic (given by the investor) benchmark or beat it. It makes possible to obtain a smooth curve of the growth of the portfolio wealth on the entire investment horizon. It is one of the basic requirements for the trading strategies of investors in financial markets. The growth factor $\mu_0$ is selected by investor, based on the analysis of the financial market.
### 3 The Proposed Investment Strategy Design

The discrete-time Markov chain, taking values in \{1,2,...,v\}, with transition probability matrix \(P\) admits the following representation in the state space

\[\theta(k + 1) = P\theta(k) + \nu(k + 1),\] (10)

where \(\theta(\cdot) = [\theta(a(k), 1), \ldots, \theta(a(k), v)]^T\), \(\delta(a(k), j)\) is a Kronecker function; \(\{\nu(k)\}\) is a sequence of martingale increments with conditional moments

\[E[\nu(k + 1) / \theta(k)] = 0,\] (11)

\[C(k + 1) = E[\nu(k + 1) \nu^T(k + 1) / \theta(k)] = \text{diag}(P\theta(k)) - P\text{diag}(\theta(k))P^T.\] (12)

Taking (10) into consideration, equation (7) can be represented as follows

\[\eta'[\theta(k), k] = \mu[\theta(k), k] + \sum_{j=1}^{\nu} \sigma_j[\theta(k), k]w_j(k).\] (13)

The problem of minimizing the criterion (9) is equivalent to the quadratic control problem with criterion

\[J(k + m / k) = E[\sum_{i=1}^{n} V^2(k + i) - R_i(k)u(k + i)\nu^T(k + i) + \sum_{i=0}^{n-1} E[\nu(k + i / k) - \nu(k + i - 1 / k)]^T \cdot R(k, i) \times \nu[k + i / k] - \nu[k + i - 1 / k]]V(k, \theta(k))],\]

where we eliminated the term that is independent of control variables, \(R_i(k+i)=2\nu^2(k+i)+\rho(p, i)\).

We have the following theorem.

**Theorem 1.** Let the wealth dynamics is given by (3) with risky asset returns followed by dynamics of the form (13) under constraints (4)-(6). Then the MPC policy with receding horizon \(m\), such that it minimizes the objective (14), for each instant \(k\) is defined by the equation

\[u(k) = I_{m+1}0_{n+1}...0_{n+1}U(k),\] (15)

where \(I_{m+1}\) is \((n+1)\)-dimensional identity matrix; \(0_{n+1}\) is \((n+1)\)-dimensional zero matrix; \(U(k) = I_U(k), u(k) = u^T(k+m-1 / k)\) is the set of predictive controls defined from the solving of quadratic programming problem with criterion

\[Y(k + m / k) = [2V(k)G(k) - F(k)]U(k) + \] (16)

\[U^T(k)\left[\bar{R}(k) + \bar{H}(k)\right]U(k)\]

under constraints (element-wise inequality)

\[U_{\text{min}}(k) \leq \bar{S}(k)U(k) \leq U_{\text{max}}(k),\] (17)

where

\[\bar{R}(k) = \begin{bmatrix} R(k, 0) + R(k, 1) & -R(k, 1) & \cdots & 0_{n+1 \times n+1} \\ -R(k, 1) & R(k, 1) + R(k, 2) & \cdots & 0_{n+1 \times n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n+1 \times n+1} & 0_{n+1 \times n+1} & \cdots & 0_{n+1 \times n+1} \end{bmatrix},\]

\[\bar{S}(k) = \text{diag}\{S(k), 0_{n+1 \times n+1}, \ldots, 0_{n+1 \times n+1}\},\]

\[U_{\text{min}}(k) = [u_{\text{min}}^T(k), 0_{n+1 \times n+1}, \ldots, 0_{n+1 \times n+1}^T],\]

\[U_{\text{max}}(k) = [u_{\text{max}}^T(k), 0_{n+1 \times n+1}, \ldots, 0_{n+1 \times n+1}^T],\]

\[u_{\text{min}}(k) = \ldots u_{n_{\text{min}}(k)} - V(k),\]

\[u_{\text{max}}(k) = \ldots u_{n_{\text{max}}(k)} - V(k).\]

\(H(k), G(k), F(k)\) are the block matrices

\[H(k) = [H_g(k)], G(k) = [G_i(k)], F(k) = [F_i(k)],\] (18)

\((t, f = 1, m),\)

and the blocks satisfy the following recursive equations

\[H_g(k) = Q_m(m-t) \times \sum_{q=1}^{v} \left[ e_q \text{diag}(P^T \theta(k))e_q \sum_{j=0}^{n} (B_j^{(q)}(k+t))^T B_j^{(q)}(k+t) \right],\]

\[H_g(k) = A^{f-t} Q_t(m-f) \times \sum_{q=1}^{v} \sum_{r=1}^{n} \left[ e_q \text{diag}(P^T \theta(k)) \left[ P^{T \times} \right] e_r \right] \times \left[ B_r^{(q)}(k+t) \right]^T B_r^{(q)}(k+f), t < f,\]

\[G_i(k) = A^{t} Q_{t-m} \sum_{q=1}^{v} \sum_{r=1}^{n} \left[ e_q P^T \theta(k) \right] B_r^{(q)}(k+t),\] (22)

\[F_i(k) = Q_{m-t} \sum_{q=1}^{v} \sum_{r=1}^{n} \left[ e_q P^T \theta(k) \right] B_r^{(q)}(k+t) - 2R(k)u(k-1),\] (23)
$Q(t) = A^2 Q(t-1) + 1, \quad Q(0) = 1,$

$Q(t) = A Q(t-1) + R_i(k, m-t), \quad Q(0) = R_i(k, m),$  

$R_i(t) = 2 V^A(k+t) + \rho(t), \quad (t = \overline{1,m}),$

$A = 1 + r_i,$

$e_q = \left[0, \ldots, 0, 1, 0, \ldots, 0\right]_n, \quad (q = \overline{1,v}),$

$B_k^{(q)}(k) = \left[\mu_i^{(q)} - r_i \quad \mu_2^{(q)} - r_i \quad \ldots \quad \mu_n^{(q)} - r_i \quad r_i - r_j\right],$

$B_k^{(q)}(k) = \left[\sigma_{ij}^{(q)} \quad \ldots \quad \sigma_{nn}^{(q)} \quad 0\right], \quad (j = \overline{1,n}), (q = \overline{1,v}).$

A brief proof of this theorem is reported in the Appendix.

4 A Real-Data Numerical Example and Discussion

In this section, we present several numerical examples demonstrating the application of our approach to the portfolio of a real stock. We want to assess the performance of our model under real market conditions by computing the portfolio wealth over a long period of time. The data used for these examples are taken from the Russian Stock Exchange MICEX (www.finam.ru). They include the daily stock prices of the largest Russian companies and the values of the MICEX index.

We consider the situation of an investor who has to allocate one unit of wealth over the investment horizon of about 600 trading days among risky assets and one risk-free asset. The updating of the portfolio is executed once every trading day.

The risk-free asset is considered here as a bank account with $r_f=0.0001, \quad r_r=0.0002$ We also don’t take into account cross-sectional correlation between different assets, i.e., $\sigma_{ij}=0, \quad i \neq j$. We have experimented with more sophisticated scheme under assumption that cross-correlation between assets is presented. However, we found its impact on the tracking performance quite negative. This is expected, since we need to estimate a large number of parameters that introduces “estimation uncertainty” into the portfolio optimization strategy.

In our experiments, we used risky assets traded on the Russian Stock Exchange MICEX: Sberbank, Gazprom, VTB, LUKOIL, NorNickel, Rosneft, Gazprommef. All investment portfolios were composed of 5 risky assets. The relatively small size of the Russian Stock Exchange precluded experiments with a large number of assets. On the over hand, it presented additional challenges due to the high volatility of the index.

We assume that the market parameters depend on the market mode that switches according to a Markov chain among two states. We use MICEX Index to observe the current Markov state and to estimate the transition probability matrix. We assume that only volatilities of the returns are effected by local or global factors, which are represented by the market regime. Thus, State 1 represents low market volatility and State 2 represents high market volatility. Whenever the daily volatility of the index was below 0.015, we defined that day as low-volatility and set $\sigma_i^{(1)}=0.01, \quad (i=1,2,\ldots,n)$. Whereas the daily volatility of the index was above 0.015, we defined the day as a high-volatility and set $\sigma_i^{(2)}=0.02, \quad (i=1,2,\ldots,n)$. These values of volatilities were obtained by analyzing a real market behavior.

The transition probability matrix was estimated by the maximum likelihood method using the past 200 daily closing values of the MICEX index prior to the tracking period. The estimation of transition probability matrix was

$$P = \begin{bmatrix} 0.78 & 0.03 \\ 0.22 & 0.97 \end{bmatrix}.$$  

The Markov process is assumed to be a stationary multi-period process over the investment horizon.

We computed the expected returns using 13-day simple averaging of past historical return data and assume that the expected returns remain constant over the predictive horizon $m$. We use the adjusted procedure, updating the estimates at each decision time $k$, to adapt the portfolio to price changes on the market incorporating of newly arrived information.

We set the tracking target to return 0.15% per day ($\mu_0=0.0015$). We assumed an initial portfolio wealth of $V(0)=V^A(0)=1$. The matrix measuring the level of transaction costs is set as $R(k,i)=\text{diag}(10^{-4}, \ldots, 10^{-4})$ for all $k,i$, the weight coefficient $\rho(k,i)=0.1$ for all $k,i$. We impose constraints on the tracking portfolio problem with parameters $\beta_i=-0.6, \quad (i=1,\ldots,n), \quad \gamma_i=3, \quad (i=1,\ldots,n+1)$. Therefore we allow borrowing and short selling. A prediction horizon was $m=10$.

The optimization problem (16), (17) is solved by the quadprog.m function in MATLAB Optimization Toolbox.

We present the typical results of the experiments on Fig. 1-3. In the pictures below the portfolio was composed of five risky assets: LUKOIL, Gazprom, Sberbank, Sibneft, NorNickel. The investment period was from March 6, 2009 to October 30, 2013. Fig. 1 plots real portfolio and benchmark values. In Fig. 2 we have investments in the risky asset Gazprom. Fig. 3 illustrates the MICEX Index daily returns and the estimated states of the Markov chain.
Several insights can be garnered from the example illustrated above. It is important to acknowledge that in our experiments where we use rather simple methods for parameters estimation, the tracking performance appears to be rather efficient. One of the major attractions of proposed MPC algorithm lies in the fact that it appears to be rather insensitive to the estimated parameters that are fed into the model. Our approach does not require a heavy reliance on parametric estimation based on past data; instead it focuses on trying to capture the dynamic changes of the market over the tracking period and react accordingly. So our approach allows us to design strategies which are desensitized, i.e., robustified, to parameters’ estimation. It should, however, be realized that the results of an ensuing optimization model will always be affected by a level of “estimation uncertainty.” It is clear that one can use more sophisticated estimation schemes to improve the parameters estimation precision [21].

5 Conclusion
In this paper, we studied a discrete-time portfolio selection problem subject to Markovian jumps in the parameters. We proposed to use the MPC methodology in order to solve the problem. The optimal portfolio control strategy was derived under hard constraints on trading amounts, transaction costs and different rates for borrowing and lending. The advantage of using a receding horizon implementation is that at each decision stage we can profit from observations of actual market behavior during the preceding period and use information to feed fresh estimates to the model. We presented the numerical modeling results, based on a set of real data from the Russian Stock Exchange MICEX. We find that on actual data the proposed approach is reasonable. The value of the portfolio follows the value of the reverence portfolio, beating it most of the time and the constraints are satisfied.

Appendix
Proof of the Theorem 1: The portfolio dynamics (3) can be rewritten in the form
\[ V(k + 1) = AV(k) + B_j[\theta(k + 1), k + 1]u(k) + \sum_{j=1}^{n} B_j[\theta(k + 1), k + 1]w_j(k + 1)u(k), \] (24)
where
\[ u(k) = [u_1(k), \ldots, u_n(k)]^T, \]
\[ A = 1 + r_1, \]
\[ B_0[\theta(k), k] = [\theta(k), k] - r_1 \ldots \theta(k), k] - r_1 \ldots - r_2], \]
\[ B_j[\theta(k), k] = [\sigma_j[\theta(k), k] \ldots \sigma_n[\theta(k), k] 0]_1, (j = 1, n), \]
\[ B_j[\theta(q), k] = \sum_{q=1}^{v} \theta_q(k)B_{j(q)}(k), (j = 0, n). \]
Here \( \theta(k), (i=1,2,\ldots,v) \) are elements of the vector \( \theta(k); \{B_{j(q)}^{(q)}(k)\}, (j=0,\ldots,n), (q=1,\ldots,v) \) is a set of values of the matrix \( B_j[\theta(k), k] \).

The constraints (4)-(6) can be rewritten in matrix form (element-wise inequality)
\[ u_{\min}(k) \leq S(k)u(k) \leq u_{\max}(k), \quad (25) \]

where

\[
S(k) = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
-1 & -1 & \cdots & -1 & 1 \\
0 & 0 & \cdots & 0 & 1 
\end{bmatrix}.
\]

Thus, we have a control problem to minimize a functional (14), with system dynamics like (24) under constraints (25).

The objective (14) can be written in the form

\[ J(k+m/k) = E \left[ X^T(k+1)X(k+1) - \Delta_t(k+1)X(k+1)U(k) - 2u(k/k)R(k,0)u(k-1) + u^T(k-1)R(k,0)u(k-1)/V(k), \theta(k) \right], \]

subject to

\[ X(k+1) = \Psi V(k) + \Phi_0[\Xi(k+1),k+1]U(k) + \sum_{j=1}^{n} \Phi_j[\Xi(k+1),k+1] \text{diag} \{ W_j(k+1) \} U(k), \]

where

\[
X(k+1) = \begin{bmatrix}
V(k+1) \\
V(k+2) \\
\vdots \\
V(k+m) 
\end{bmatrix}, \quad \Psi = \begin{bmatrix}
A \\
A^2 \\
\vdots \\
A^m 
\end{bmatrix},
\]

\[
U(k) = \begin{bmatrix}
u^T(k/k) & u^T(k+1/k) & \cdots & u^T(k+m-1/k) 
\end{bmatrix}^T,
\]

\[ \Xi(k+1) = \begin{bmatrix}
\theta(k+1) \\
\theta(k+2) \\
\vdots \\
\theta(k+m) 
\end{bmatrix}, \quad W_j(k+1) = \begin{bmatrix}
w_j(k+1) \\
w_j(k+2) \\
\vdots \\
w_j(k+m) 
\end{bmatrix},
\]

\[ \Phi_j[\Xi(k+1),k+1] = \begin{bmatrix}
B_j[\theta(k+1),k+1] & 0_{0\times1} & \cdots \\
AB_j[\theta(k+1),k+1] & B_j[\theta(k+2),k+2] & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
A^{m-1}B_j[\theta(k+1),k+1] & A^{m-2}B_j[\theta(k+2),k+2] & \cdots & 0_{0\times1} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & B_j[\theta(k+m),k+m]
\end{bmatrix}, \]

\[ \Delta_t(k+1) = [R_t(k,1),R_t(k,2),\ldots,R_t(k,m)], \]

\[
\bar{R}(k) = \begin{bmatrix}
R(k,0) + R(k,1) & -R(k,1) & \cdots \\
-R(k,1) & R(k,1) + R(k,2) & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0_{n+1\times n+1} & 0_{n+1\times n+1} & \cdots & 0_{n+1\times n+1} \\
0_{n+1\times n+1} & 0_{n+1\times n+1} & \cdots & 0_{n+1\times n+1} \\
\vdots & \vdots & \ddots & \vdots \\
0_{n+1\times n+1} & 0_{n+1\times n+1} & \cdots & 0_{n+1\times n+1} \\
\end{bmatrix}.
\]

Using (27), we can rewrite (26) as follows

\[ J(k+m/k) = V^2(k)\Psi^T\Psi - \Delta_t^2V(k) + \left( \left[ \begin{array}{c} 2V(k)\Psi^T - \Delta_t \end{array} \right] \times \frac{E}{\Phi_0[\Xi(k+1),k+1]/\theta(k)} + L(k) \right) U(k) + \]

\[ +U^T(k) \left[ E \Phi_0^T[\Xi(k+1),k+1] \Phi_0[\Xi(k+1),k+1] + \right. +\sum_{j=1}^{n} \text{diag} \{ W_j(k+1) \} \Phi_j^T[\Xi(k+1),k+1] \Phi_j[\Xi(k+1),k+1] \times \]

\[ \times \frac{E}{\Phi_0[\Xi(k+1),k+1]/\theta(k)} + \bar{R}(k)U(k) + \]

\[ +u^T(k-1)R(k,0)u(k-1), \]

where

\[ L(k) = \left[ \begin{array}{c} 2R(k,0)u(k-1) \end{array} \right] 0_{1\times n} \cdots 0_{1\times 1}. \]

Denote the matrices

\[ H(k) = E \Phi_0^T[\Xi(k+1),k+1] \Phi_0[\Xi(k+1),k+1] + \]

\[ \sum_{j=1}^{n} \text{diag} \{ W_j(k+1) \} \Phi_j^T[\Xi(k+1),k+1] \Phi_j[\Xi(k+1),k+1] \times \frac{E}{\Phi_0[\Xi(k+1),k+1]/\theta(k)}, \]

\[ G(k) = \Psi^T E \Phi_0[\Xi(k+1),k+1]/\theta(k), \]

\[ F(k) = \Delta_t E \Phi_0[\Xi(k+1),k+1]/\theta(k) + L(k). \]

We have that the minimization of the criterion (26) under constraints (25) is equivalent to the quadratic programming problem with criterion

\[ Y(k+m/k) = \left[ 2V(k)G(k) - F(k) \right] U(k) + \]

\[ +U^T(k)H(k) + \bar{R}(k)U(k) \]

under constraints (19).

Straightforward calculations lead to the expressions (19)-(23) for the matrices \( H(k) \), \( G(k) \), \( F(k) \). This completes the proof.

References:


