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Valeri Mladenov
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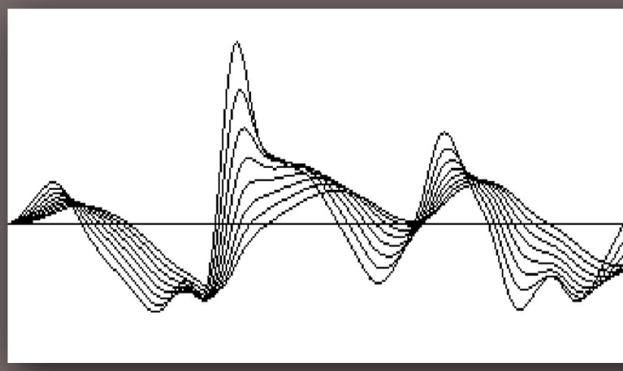
Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling

by Prof. Mihailo Lazarević



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Preface

In this monograph several aspects of fractional calculus will be presented ranging from its brief history over control applications and stability problems for time delay systems to applications in bio-engineering fields with illustrative examples.

The advantages of fractional calculus have been described and pointed out in the last few decades by many authors. Fractional calculus is based on derivatives and integrals of non integer arbitrary order, fractional differential equations and methods of their solution, approximations and implementation techniques. It has been shown that the fractional order models of real systems are regularly more adequate than usually used integer order models.

The monograph consists of seven chapters and an appendix where related a list of references include in the end of chapters.

The monograph begins in Chapter 1 with a brief historical review of the theory of fractional calculus and its applications. The theory of non-integer order differentiation and integration is almost as old as classical calculus itself, but nevertheless there seems to be an astonishing lack of knowledge of this field in most mathematicians. A look at the historical development can in parts explain the absence of this field in today's standard mathematics textbooks on calculus and in addition give the reader not familiar with this field a good access to the topics addressed in this monograph. In this chapter some well known definitions and properties of fractional order differ integrals are also stated.

Chapter 2 is devoted to the problem of discrete-time (digital) implementation of fractional order systems, i.e. fractional differ integrators, where two novel methods have been closely investigated: direct optimal and indirect. Both methods produce approximations of fractional differ integrators, which are then used to create approximations to more complex fractional order systems. It has been demonstrated by means of a number of numerical examples that both presented methods.

Some of stability problems for time delay systems have been discussed in the two following chapters (Chapters 3, 4). While Lyapunov methods have been developed for stability analysis and control law synthesis of integer linear systems and have been extended to stability of fractional systems, only few studies deal with non-Lyapunov stability of fractional systems. Here, finite-time stability of fractional order time-delay systems is considered in Chapter 3. Sufficient conditions for finite-time stability for (non) linear (non)homogeneous as well as perturbed fractional order time-delay systems are obtained and presented.

The problem of stability (simple stability and robust stability) of linear discrete-time fractional order systems is addressed in Chapter 4 where it is shown that some stability criteria for discrete time-delay systems could be applied with small changes to discrete fractional order state-space systems. The approach is based on the idea of constructing novel Lyapunov-Krasovskii functionals combined with free-weighting matrices or algebraic methods.

The next three Chapters (5, 6, 7) are related to applications of fractional calculus in bio-engineering fields. Chapter 5 is dedicated to the mathematical modeling of skin structure applying fractional calculus where it is proposed the skin structure as a more complex system consisting of several layers which describes series of structures via continuous generalizing (distributed order type) the Cole equation. According to this model and experimental data of the skin bioimpedance measurements, one may predict more complex equivalent electrical circuit and define new time parameters which correspond to each reduced Cole element.

In Chapter 6, a thermodynamically consistent rheological modified Zener model of viscoelastic body, i.e. standard fractional linear viscoelastic body is studied and presented. Proposed model comprises both fractional derivatives of stress and strain and the restrictions on the coefficients that follow from Clausius Duhem inequality. In that way, it should be included in both analytical and experimental projects ab initio, particularly in experiments in which newly developed materials are tested.

Finally, Chapter 7 concludes this monograph showing an useful modeling double DNA helix main chains of the free and forced fractional order vibrations applying fractional calculus. Different models are focusing on different aspects of the DNA molecule (biological, physical and chemical processes in which DNA is involved). The aim of this study was to model the DNA dynamics (vibrations of DNA chains) as a biological system in a specific boundary condition that are possible to occur in a life system during regular function of a DNA molecule.

I hope that this monograph will be value to Ph.D. students and fractional systems researchers as well as the other readers will find something in this monograph exciting.

Also, I want to thank very much Mrs. Ranki Gajic for the support in the preparation of the manuscript for English edition.

Belgrade, August, 2012

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Part I

“Introduction to Fractional Calculus”

Introduction to Fractional Calculus with Brief Historical Background

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Abstract: - The Fractional Calculus (FC) is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The first known reference can be found in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 where the question of meaning of the semi-derivative has been raised. This question consequently attracted the interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov and many others. Since the 19th century, the theory of fractional calculus developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations (FDE) and fractional dynamics. The applications of FC are very wide nowadays. It is safe to say that almost no discipline of modern engineering and science in general, remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc..In fact, one could argue that real world processes are fractional order systems in general. The main reason for the success of FC applications is that these new fractional-order models are often more accurate than integer-order ones, i.e. there are more degrees of freedom in the fractional order model than in the corresponding classical one. One of the intriguing beauties of the subject is that fractional derivatives (and integrals) are not a local (or point) quantities. All fractional operators consider the entire history of the process being considered, thus being able to model the non-local and distributed effects often encountered in natural and technical phenomena. Fractional calculus is therefore an excellent set of tools for describing the memory and hereditary properties of various materials and processes.

Key-Words: fractional calculus, historical background, Riemann-Liouville definition, Grunwald-Letnikov definition, Caputo definition

1. 1 Brief History of Fractional Calculus

Fractional calculus (FC) is an extension of ordinary calculus with more than 300 years of history. FC was initiated by Leibniz and L`Hospital as a result of a correspondence which lasted several months in 1695. In that year, Leibniz wrote a letter to L'Hospital raising the following question [1]:

> *"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hopital was somewhat curious about the above question and replied by another simple one to Leibniz: "What if the order will be 1/2?". Leibniz in a letter dated September 30, 1695, replied: "It will lead to a paradox, from which one day useful consequences will be drawn."*

That date is regarded as the exact birthday of the fractional calculus. The issue raised by Leibniz for a fractional derivative (semi-derivative, to be more precise) was an ongoing topic in decades to come [1,2]. Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best mathematical minds in Europe. Euler [2],wrote in 1730:

> "*When n is a positive integer and p is a function of x,* $p = p(x)$*, the ratio of* $d^n p$ *to* dx^n *can always be expressed algebraically. But what kind of ratio can then be made if* n *be a fraction*?"

Subsequent references to fractional derivatives were made by Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919, and others [3-5]. During the 19th century, the theory of fractional calculus was developed primarily in this way, trough insight and genius of great mathematicians. Namely, in 1819 Lacroix [6], gave the correct answer to the problem raised by Leibnitz and L'Hospital for the first time, claiming that $d^{1/2}x/dx^{1/2} = 2\sqrt{x/\pi}$. In his 700 pages long book on Calculus published in 1819, Lacroix developed the formula for *n*-th derivative of $y = x^m$, with *m* being a positive integer

$$D_x^n y = \frac{d^n}{dx^n}\left(x^m\right) = \frac{m!}{(m-n)!}x^{m-n}, m \geq n \tag{1}$$

Replacing the factorial symbol by Gamma function (3), he developed the formula for the fractional derivative of a power function

$$D_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha} \tag{2}$$

where α and β are fractional numbers and where the gamma function $\Gamma(z)$ [1] is defined for $z > 0$ as:

$$\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx \tag{3}$$

In particular, Lacroix calculated

$$D_x^{1/2}x = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = 2\sqrt{\frac{x}{\pi}} \tag{4}$$

Surprisingly, the previous definition gives *a nonzero value for the fractional derivative of a constant function* $(\beta = 0)$, since

$$D_x^\alpha 1 = D_x^\alpha x^0 = \frac{1}{\Gamma(1-\alpha)}x^{-\alpha} \neq 0 \tag{5}$$

Using linearity of fractional derivatives, the method of Lacroix is applicable to any analytic function by term-vise differentiation of its power series expansion. Unfortunately, this class of functions is too narrow in order for the method to be considered general.

It is interesting to note that simultaneously with these initial theoretical developments, first practical applications of fractional calculus can also be found. In a sense, the first of these was the discovery by Abel in

[*] See the Appendix A.1.

1823,[7-9]. Abel considered the solution of the integral equation related to the *tautochrone problem*[**]. He found that the solution could be accomplished via an integral transform, which could be written as a semi-derivative. More precisely, the integral transform considered by Abel was

$$K = \int_0^x (x-t)^{-1/2} f(t)\,dt\,, \quad K = const\,. \tag{6}$$

Abel wrote the right hand side of (6) by means of a fractional derivative of order $1/2$,

$$\sqrt{\pi}\left(\frac{d^{-1/2}}{dx^{-1/2}}(f(x))\right) \tag{7}$$

Abel's solution had attracted the attention of Joseph Liouville, who made the first major study of fractional calculus,[10-13]. The most critical advances in the subject came around 1832 when he began to study fractional calculus in earnest and then managed to apply his results to problems in potential theory. Liouville began his theoretical development using the well-known result for derivatives of integer order *n*

$$D_x^n e^{ax} = a^n e^{ax}\,. \tag{8}$$

Expression (8) can rather easily be formally generalized to the case of non-integer values of *n*, thus obtaining

$$D_x^\alpha e^{ax} = a^\alpha e^{ax} \tag{9}$$

By means of Fourier expansion, a wide family of functions can be composed as a superposition of complex exponentials.

$$f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x),\ \operatorname{Re} a_n > 0 \tag{10}$$

Again, by invoking linearity of the fractional derivative, Liouville proposed the following expression for evaluating the derivative of order α

$$D_x^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x}\,. \tag{11}$$

Formula (11) is known as *the Liouville's first formula* for a fractional derivative,[10,11]. However, this formula cannot be seen as a general definition of fractional derivative for the same reason Lacroix formula could not: because of its relatively narrow scope. In order to overcome this, Liouville labored to produce a second definition. He started with a definite integral (closely related to the gamma function):

$$I = \int_0^\infty u^{\beta-1} e^{-xu}\,du,\ \beta > 0,\ x > 0. \tag{12}$$

and derived what is now referred to as *the second Liouville's formula*

$$D_x^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta},\ \beta > 0 \tag{13}$$

None of previous definitions were found to be suitable for a general definition of a fractional derivative. In the consequent years, a number of similar formulas emerged. Greer [14], for example, derived formulas for the fractional derivatives of trigonometric functions using (9) in the form:

$$D_x^\alpha e^{iax} = a^\alpha \left(\cos\frac{\pi\alpha}{2} + i\sin\frac{\pi\alpha}{2}\right)(\cos ax + i\sin ax) \tag{14}$$

Joseph Fourier [15] obtained the following integral representations for $f(x)$ and its derivatives

[**] *The tautochrone problem consists of the determination of a curve in the (x, y) plane such that the time required for a particle to slide down the curve to its lowest point under the influence of gravity is independent of its initial position (x_0, y_0) on the curve.*

$$D_x^n f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) d\xi \int_{-\infty}^{+\infty} t^n \cos\left[t(x-\xi) + n\pi/2\right] dt, \tag{15}$$

By formally replacing integer n by an arbitrary real quantity α he obtained

$$D_x^\alpha f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) d\xi \int_{-\infty}^{+\infty} t^\alpha \cos\left[t(x-\xi) + \alpha\pi/2\right] dt. \tag{16}$$

Probably the most useful advance in the development of fractional calculus was due to a paper written by G. F. Bernhard Riemann [16] during his student days. Unfortunately, the paper was published only posthumously in 1892. Seeking to generalize a Taylor series in 1853, Riemann derived different definition that involved a definite integral and was applicable to power series with non-integer exponents

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt + \Psi(x) \tag{17}$$

In fact, the obtained expression is the most-widely utilized modern definition of fractional integral. Due to the ambiguity in the lower limit of integration *c,* Riemann added to his definition a "complementary" function $\Psi(x)$ where the present-day definition of fractional integration is without the troublesome complementary function. Since neither Riemann nor Liouville solved the problem of the complementary function, it is of historical interest how today's Riemann-Liouville definition was finally deduced.

The earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Ya. Sonin in 1869, [17] where he used Cauchy`s integral formula as a starting point to reach differentiation with arbitrary index. A. V. Letnikov [18] extended the idea of Sonin a short time later in 1872, [19]. Both tried to define fractional derivatives by utilizing a closed contour. Starting with Cauchy's integral formula for integer order derivatives, given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt, \tag{18}$$

the generalization to the fractional case can be obtained by replacing the factorial with Euler's Gamma function $\alpha! = \Gamma(1+\alpha)$. However, the direct extension to non-integer values α results in the problem that the integrand in (18) contains a branching point, where an appropriate contour would then require a branch cut which was not included in the work of Sonin and Letnikov. Finally, Laurent [20], used a contour given as an open circuit (known as *Laurent loop*) instead of a closed circuit used by Sonin and Letnikov and thus produced today's definition of the *Riemann-Liouville fractional integral*

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt, \quad \mathrm{Re}(\alpha) > 0. \tag{19}$$

In expression (19) one immediately recognizes Riemann's formula (17), but without the problematic complementary function. In nowadays terminology, expression (19) with lower terminal $c = -\infty$ is referred as Liouville fractional integral; by taking $c = 0$ the expression reduces to the so called Riemann fractional integral, whereas the expression (19) with arbitrary lower terminal c is called Riemann-Liouville fractional integral. Expression (19) is the most widely utilized definition of the fractional integration operator in use today. By choosing $c = 0$ in (19) one obtains the Riemann's formula (17) without the problematic complementary function $\Psi(x)$ and by choosing $c = -\infty$, formula (19) is equivalent to Liouville's first definition (10). These two facts explain why equation (19) is called *Riemann-Liouville fractional integral*. While the notation of fractional integration and differentiation only differ in the sign of the parameter α in (19), the change from fractional integration to differentiation cannot be achieved directly by inserting negative α at the right-hand side of (19). The problem originates from the integral at the right side of (19) which is divergent for negative integration orders. However, by analytic continuation it can be shown that

$$D_{c,x}^{\alpha} f(x) = D_{c,x}^{n-\beta} f(x) = D_{c,x}^{n} f(x) D_{c,x}^{-\beta} f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\beta)} \int_c^x (x-t)^{\beta-1} f(t)\, dt \right), \quad (20)$$

holds, which is known today as the definition of the *Riemann-Liouville fractional derivative*. In (20) $n = \lceil \alpha \rceil$ is the smallest integer greater than α with $0 < \beta = n - \alpha < 1$. For either $c = 0$ or $c = \infty$ the integral in (20) is the Beta-integral (see Appendix A.2) for a wide class of functions and thus easily evaluated.

Nearly simultaneously, Grunwald and Letnikov provided the basis for another definition of fractional derivative [21] which is also frequently used today. Disturbed by the restrictions of the Liouville`s approach Grunwald (1867) adopted the definition of a derivative as the limit of a difference quotient as its starting point. He arrived at definite-integral formulas for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit, and also that the Liouville's definition, in which no distinguishable lower limit appeared, correspond to a lower limit $-\infty$. Formally,

$${}^{GL}D_x^{\alpha} f(x) = \lim_{h \to 0} \frac{\left(\Delta_h^{\alpha} f\right)}{h^{\alpha}}(x) = \lim_{h \to 0} \frac{\sum_{k=0} (-1)^k \binom{\alpha}{k} f(x - kh)}{h^{\alpha}}, \quad \alpha > 0 \quad (21)$$

which is today called the *Grunwald-Letnikov fractional derivative*. In definition (21), $\binom{\alpha}{k}$ is the generalized binomial coefficient, wherein the factorials are replaced by Euler's Gamma function. Letnikov [18] also showed that definition (21) coincides, under certain relatively mild conditions, with the definitions given by Riemann and Liouville. Today, the Grunwald-Letnikov definition is mainly used for derivation of various numerical methods, which use formula (21) with finite sum to approximate fractional derivatives. Together with the advances in fractional calculus at the end of the nineteenth century the work of O. Heaviside [22] has to be mentioned. The operational calculus of Heaviside, developed to solve certain problems of electromagnetic theory, was an important next step in the application of generalized derivatives. The connection to fractional calculus has been established by the fact that Heaviside used arbitrary powers of p, mostly $\sqrt{p}$, to obtain solutions of various engineering problems.

Weyl [23] and Hardy,[24,25], also examined some rather special, but natural, properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes in 1917. Moreover, Weyl showed that the following fractional integrals could be written for $0 < \alpha < 1$ assuming that the integrals in (22) are convergent over an infinite interval

$$I_{+}^{\alpha} \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} \varphi(t)\, dt, \quad I_{-}^{\alpha} \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} \varphi(t)\, dt, \quad (22)$$

Specially, the Riemann-Liouville definition of a fractional integral given in (19) with lower limit $c = -\infty$, the form equivalent to the definition of fractional integral proposed by Liouville, is also often referred to as *Weyl fractional integral*. In the modern terminology one recognizes two distinct variants of all fractional operators, left sided and right sided ones. Weyl operators defined in (22) are sometimes also referred to as the left and right Liouville fractional integrals, respectively.

Later, in 1927 Marchaud [27] developed an integral version of the Grunwald-Letnikov definition (21) of fractional derivatives, using

$${}^{M}D_x^{\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\left(\Delta_t^l f\right)(x)}{t^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt, \quad \alpha > 0 \quad (23)$$

as fractional derivative of a given function f, today known as *Marchaud fractional derivative*. The term $\left(\Delta_t^l f\right)(x)$ is a finite difference of order $l > \alpha$ and c is a normalizing constant. Since this definition is related to the Grunwald-Letnikov definition,it also coincides with the Riemann-Liouville definition under certain

conditions. M. Riesz published a number of papers starting from 1938 [28, 29] which are centered around the integral

$$ {}^{R}I^{\alpha}\varphi = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{\varphi(t)}{|t-x|^{1-\alpha}} dt, \quad \operatorname{Re}\alpha > 0, \quad \alpha \neq 1,3,5,... \tag{24} $$

today known as *Riesz potential*. This integral (and its generalization in the *n*-dimensional Euclidean space) is tightly connected to Weyl fractional integrals (22) and therefore to the Riemann-Liouville fractional integrals by

$$ {}^{R}I^{\alpha} = \left(I_{+}^{\alpha} + I_{-}^{\alpha}\right)\left(2\cos(\alpha\pi/2)\right)^{-1} \tag{25} $$

In 1949 Riesz [29] also developed a theory of fractional integration for functions of more than one variable.
A modification of the Riemann-Liouville definition of fractional integrals, given by

$$ \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_{0}^{x} \left(x^2 - t^2\right)^{\alpha-1} t^{2\eta+1}\varphi(t)\,dt, \qquad \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_{0}^{x} \left(x^2 - t^2\right)^{\alpha-1} t^{1-2\alpha-2\eta}\varphi(t)\,dt, \tag{26} $$

were introduced by Erdelyi et al. in [30-32], which became useful in various applications. While these ideas are tightly connected to fractional differentiation of the functions x^2 and $\sqrt{x}$, already done by Liouville 1832, the fact that Erdelyi and Kober used the Mellin's transform for their results is noteworthy.

Among the most significant modern contributions to fractional calculus are those made by the results of M. Caputo in 1967,[33]. One of the main drawbacks of Riemann-Liouville definition of fractional derivative is that fractional differential equations with this kind of differential operator require a rather "strange" set of initial conditions. In particular, values of certain fractional integrals and derivatives need to be specified at the initial time instant in order for the solution of the fractional differential equation to be found. Caputo [33,34] reformulated the more "*classic*" definition of the Riemann-Liouville fractional derivative in order to use classical initial conditions, the same one needed by integer order differential equations [34]. Given a function *f* with an $(n-1)$ absolutely continuous integer order derivatives, Caputo defined a fractional derivative by the following expression

$$ D_{*}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^{n} f(s)\,ds, \tag{27} $$

Derivative (27) is strongly connected to the Riemann-Liouville fractional derivative and is today frequently used in applications. It is interesting to note that Rabotnov [35] introduced the same differential operator into the Russian viscoelastic literature a year before Caputo's paper was published. Regardless of this fact, the proposed operator is in the present-day literature commonly named after Caputo.

By the second half of the twentieth century, the field of fractional calculus had grown to such extent that in 1974 the first conference "*The First Conference on Fractional Calculus and its Applications*" concerned solely with the theory and applications of fractional calculus was held in New Haven. In the same year, the first book on fractional calculus by Oldham and Spanier [3] was published after a joint collaboration started in 1968. A number of additional books have appeared since then, for example McBride (1979) [36], Nishimoto (1991) [37], Miller and Ross (1993), [4], Samko et al. (1993),[38], Kiryakova (1994) [39], Rubin (1996) [40], Carpinteri and Mainardi (1997),[41], Davison and C. Essex (1998), [42],Podlubny (1999) [43], R. Hilfer (2000) [44], Kilbas et.al (2006),[5], Das (2007)[45], J. Sabatier *et. al* (2007) [46], and others. In 1998 the first issue of the mathematical journal *"Fractional calculus & applied analysis"* was printed. This journal is solely concerned with topics on the theory of fractional calculus and its applications. Finally, in 2004 the first conference *"Fractional differentiation and its applications*" was held in Bordeaux, and it is organized every second year since 2004,[47].

1. 2 Basic Definitions of Fractional Order Differintegrals

There are many different forms of fractional operators in use today. Riemann-Liouville, Grunwald-Letnikov, Caputo, Weyl and Erdely-Kober derivatives and integrals are the ones mentioned in the previous historical survey. In addition, most of these operators can be defined in two distinct forms, as the left and as right fractional operators. The two most frequently used definitions for the general fractional differintegral are: the Grunwald-Letnikov (GL) and the Riemann-Liouville (RL) definitions,[3-5],[43]. Also, the Caputo derivative, as a variation of the Riemann-Liouville differential operator, is used frequently. A short account of these most frequently used operators is given next. Grunwald and Letnikov defined fractional derivative in the following way

$$
\begin{aligned}
{}_{GL}D_x^{\alpha} f(x) &= \lim_{h\to 0} \frac{\left(\Delta_h^{\alpha} f(x)\right)}{h^{\alpha}}, \\
\Delta_h^{\alpha} f(x) &= \sum_{0\le |j|<\infty} (-1)^{|j|} \binom{\alpha}{j} f(x-jh),\ h>0,
\end{aligned}
\tag{28}
$$

known as the *left Grunwald-Letnikov* (GL) *derivative*. This derivative can be seen as a limit of the fractional order backward difference. The right sided derivative is defined accordingly

$$
\begin{aligned}
{}_{GL}D_x^{\alpha} f(x) &= \lim_{h\to 0} \frac{\left(\Delta_{-h}^{\alpha} f(x)\right)}{h^{\alpha}}, \\
\Delta_{-h}^{\alpha} f(x) &= \sum_{0\le |j|<\infty} (-1)^{|j|} \binom{\alpha}{j} f(x+jh),\ h<0,
\end{aligned}
\tag{29}
$$

Definitions (28) and (29) are valid for both $\alpha > 0$ (fractional derivative) and for $\alpha < 0$ (fractional integral) and, commonly, these two notions are grouped into one single operator called GL *differintegral*. The GL derivative and RL derivative are equivalent if the functions they act upon are sufficiently smooth. The generalized binomial coefficients, calculation for $\alpha \in R$ and $k \in \mathbb{N}_0$, is the following

$$
\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, \quad \binom{\alpha}{0} = 1 \tag{30}
$$

Let us consider $n = t - a/h,$ where a is a real constant. This constant can be interpreted as the lower terminal (an analogue of the lower integration limit, necessary even for the derivative operator due to its non-local properties). The GL differentigral can be expressed as a limit

$$
{}_{GL}D_{a,t}^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{31}
$$

where [*x*] means the integer part of *x*, *a* and *t* are the bounds of the operation for ${}_{GL}D_{a,t}^{\alpha} f(t)$. For the numerical calculation of fractional-order derivatives we can use the following relation (32) derived from the GL definition (31). The relation to the explicit numerical approximation of the α-th derivative at the points *kh,* (*k*=1,2,...) has the following form, [43]

$$
{}_{((x-L)}D_x^{\pm\alpha} f(x) \approx h^{\mp\alpha} \sum_{j=0}^{N(x)} b_j^{(\pm\alpha)} f(x-jh) \tag{32}
$$

where L is the "memory length", *h* is the step size of the calculation,

$$
N(t) = min\left\{\left[\frac{x}{h}\right], \left[\frac{L}{h}\right]\right\}, \tag{33}
$$

$[x]$ is the integer part of x and $b_j^{(\pm\alpha)}$ is the binomial coefficient given by

$$b_0^{(\pm\alpha)} = 1, \quad b_j^{(\pm\alpha)} = \left(1 - \frac{1 \pm \alpha}{j}\right) b_{j-1}^{(\pm\alpha)} \tag{34}$$

This approach is based on the fact that (for a wide class of functions and assuming all initial conditions are zero) the three most commonly used definitions - GL, RL, and Caputo's - are equivalent,[48].

For expression of the Riemann-Liouville definition, we will consider the Riemann-Liouville *n*-fold integral for $n \in N, n > 0$ defined as (this expression is usually referred to as the Cauchy repeated integration formula)

$$\underbrace{\int_a^t \int_a^{t_n} \int_a^{t_{n-1}} \cdots \int_a^{t_3} \int_a^{t_2}}_{n-fold} f(t_1) dt_1 dt_2 \ldots dt_{n-1} dt_n = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{35}$$

The fractional Riemann-Liouville integral of the order α for the function $f(t)$ for $\alpha, a \in R$ can be expressed as follows

$${}_{RL}I_a^{\alpha} f(t) \equiv {}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{36}$$

For the case of $0 < \alpha < 1$, $t > 0$, and $f(t)$ being a causal function of t, the fractional integral is presented as

$${}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \qquad 0 < \alpha < 1, \quad t > 0 \tag{37}$$

Moreover, the left Riemann-Liouville fractional integral and the right Riemann-Liouville fractional integral are defined [5],[38],[43]respectively as

$${}_{RL}I_a^{\alpha} f(t) \equiv {}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{38}$$

$${}_{RL}I_b^{\alpha} f(t) \equiv {}_{RL}D_{t,b}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau, \tag{39}$$

where $\alpha > 0$, $n-1 < \alpha < n$. Both Gamma function and Riemann-Liouville fractional integral can be defined for an arbitrary complex order α with positive real order, as well as for purely imaginary order α. However, since the target application area of the present book are stability issues, process control and signal processing, as well as modeling, the operations of only real order are considered. Furthermore, the *left Riemann-Liouville fractional derivative* is defined as

$${}_{RL}D_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{40}$$

and the *right Riemann-Liouville fractional derivative* is defined as

$${}_{RL}D_{t,b}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau, \tag{41}$$

where $n-1 \le \alpha < n$, *a, b* are the terminal points of the interval $[a,b]$, which can also be $-\infty, \infty$. In the very important case of $\alpha \in (0,1)$ where the above definition of the *left Riemann-Liouville fractional derivative* is reduced to

$$ {}_{RL}D^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} d\tau \,. \tag{42} $$

A very important fact is that for integer values of order α the Riemann-Liouville derivative coincides with the classical, integer order one. In particular [48]

$$ \lim_{\alpha \to (n-1)^+} {}_{RL}D^{\alpha}_{a,t} f(t) = \frac{d^{n-1} f(t)}{dt^{n-1}} \tag{43} $$

and

$$ \lim_{\alpha \to n^-} {}_{RL}D^{\alpha}_{a,t} f(t) = \frac{d^{n} f(t)}{dt^{n}} \tag{44} $$

A very interesting property of the fractional derivative is that the fractional derivative of a constant is not equal to zero. The RL fractional derivative of a constant C takes the form

$$ {}_{RL}D^{\alpha}_{a,t} C = C \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 \tag{45} $$

However, definitions of the fractional differentiation of Riemann-Liouville type create a conflict between the well-established and polished mathematical theory and proper needs, such as the initial problem of the fractional differential equation, and the nonzero problem related to the Riemann-Liouville derivative of a constant. A solution to this conflict was proposed by Caputo, see [33,34]. The *left Caputo fractional derivative* is

$$ {}_{C}D^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{46} $$

and the *right Caputo fractional derivative* is

$$ {}_{C}D^{\alpha}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{47} $$

where $f^{(n)}(\tau) = d^n f(\tau)/d\tau^n$ and $n-1 \le \alpha < n \in \mathbb{Z}^+$. It is obvious from the definition (47) that the Caputo fractional derivative of a constant is zero. Regarding continuity with respect to the differentiation order, Caputo derivative satisfies the following limits

$$ \lim_{\alpha \to (n-1)^+} {}_{C}D^{\alpha}_{a,t} x(t) = \frac{d^{n-1} x(t)}{dt^{n-1}} - D^{(n-1)} x(a) \tag{48} $$

and

$$ \lim_{\alpha \to n^-} {}_{C}D^{\alpha}_{a,t} x(t) = \frac{d^{n} x(t)}{dt^{n}} \,. \tag{49} $$

Obviously, Riemann-Liouville operator ${}_{RL}D^{n}_{a}$, $n \in (-\infty, +\infty)$, varies continuously with n. This is not the case with the Caputo derivative. Obviously, Caputo derivative is stricter than Riemann-Liouville derivative; one reason is that the *n*-th order derivative is required to exist. On the other hand, the initial conditions of fractional differential equations with Caputo derivative have a clear physical meaning and Caputo derivative is extensively used in engineering applications. The left and right Riemann-Liouville and Caputo fractional derivatives are interrelated by the following expressions

$$ {}_{RL}D^{\alpha}_{a,t} f(t) = {}_{C}D^{\alpha}_{a,t} f(t) + \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha} \quad , \tag{50} $$

$$ {}_{RL}D^{\alpha}_{t,b}f(t) = {}_{C}D^{\alpha}_{t,b}f(t) + \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha} . \tag{51} $$

1. 3 Basic Properties of Fractional Order Differintegrals

As stated previously, for a wide class of functions, Grunwald-Letnikov definition of the fractional derivative operator coincides with the Riemann-Liouville definition. Thus, in the present section only Riemann-Liouville and Caputo derivatives will be considered. Also, left-side operators are used primarily in the following chapters. Thus, all of the properties presented next will be accounted for this kind of fractional operators only. Similar properties can be formulated and proven for the right-sided operators accordingly. The reader is referred to the available literature [3-5],[43].

Similar to the classical, integer-order integral, the Riemann-Liouville fractional integral satisfies the semi-group property,[38] i.e. for any positive orders α and β

$$ {}_{RL}I^{\alpha}_{t,a}\,{}_{RL}I^{\beta}_{t,a}f(t) = {}_{RL}I^{\beta}_{t,a}\,{}_{RL}I^{\alpha}_{t,a}f(t) = {}_{RL}I^{\alpha+\beta}_{t,a}f(t) . \tag{52} $$

Interestingly, the same does also hold for integer order derivatives, but not for fractional order ones. Let us introduce the following notation

$$ f^{(n-j)}_{n-\alpha}(t) = \left(\frac{d}{dt}\right)^{n-j} {}_{RL}I^{n-\alpha}_{a,t}f(t) . \tag{53} $$

A combination of Riemann-Liouville derivatives, for example, results in the following expression

$$ {}_{RL}D^{\alpha}_{a,t}\,{}_{RL}D^{\beta}_{a,t}f(t) = {}_{RL}D^{\alpha+\beta}_{a,t}f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\beta}(a)}{\Gamma(1-j-\alpha)}(t-a)^{-j-\alpha} , \tag{54} $$

with *n* being the smallest integer bigger then β. Thus, in general,

$$ {}_{RL}D^{\alpha}_{a,t}\,{}_{RL}D^{\beta}_{a,t}f(t) \neq {}_{RL}D^{\beta}_{a,t}\,{}_{RL}D^{\alpha}_{a,t}f(t) \neq {}_{RL}D^{\alpha+\beta}_{a,t}f(t) . \tag{55} $$

A similar result can be obtained for the Caputo derivative. *Fractional derivatives do not commute*!

It is a well-known fact that the classical derivative is the left inverse of the classical integral. The similar relation holds for the Riemann-Liouville derivative and integral

$$ {}_{RL}D^{\alpha}_{a,t}\,{}_{RL}I^{\alpha}_{a,t}f(t) = f(t) . \tag{56} $$

The opposite, however, is not true (in both the fractional and integer order case)

$$ {}_{RL}I^{\alpha}_{a,t}\,{}_{RL}D^{\alpha}_{a,t}f(t) = {}_{RL}D^{\alpha+\beta}_{a,t}f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j} . \tag{57} $$

Utilizing expression (50), similar expressions can be obtained relating the Riemann-Liouville integral and derivative of Caputo type. In particular, assuming that the integrand is continuous or, at least, essentially bounded function, Caputo derivative is also the left inverse of the fractional integral.

It is rather important to notice that the Caputo and the Riemann-Liouville formulations coincide when the initial conditions are zero [43]. Besides, the RL derivative is meaningful under weaker smoothness requirements. In fact, assuming that all initial conditions are zero, a number of relations between the fractional order operators is greatly simplified. In such a case, both fractional integral and fractional derivatives possess the semi-group property; the fractional derivative is both left and right inverse to the fractional integral of the same order; and the operations of fractional integration and differentiation can exchange places freely. In the symbolic notation, for any $0 < \alpha < \beta$

$$ {}_{RL}D^{\alpha}_{a,t}\,{}_{RL}D^{\beta}_{a,t}f = {}_{RL}D^{\beta}_{a,t}\,{}_{RL}D^{\alpha}_{a,t}f = {}_{RL}D^{\alpha+\beta}_{a,t}f , \tag{58} $$

$$ {}_{RL}I^{\alpha}_{a,t}\,{}_{RL}D^{\alpha}_{a,t}f = {}_{RL}D^{\alpha}_{a,t}\,{}_{RL}I^{\alpha}_{a,t}f = f(t) , \tag{59} $$

$$ {}_C D_{a,t}^{\alpha}\, {}_C D_{a,t}^{\beta} f = {}_C D_{a,t}^{\beta}\, {}_C D_{a,t}^{\alpha} f = {}_C D_{a,t}^{\alpha+\beta} f \,, \tag{60} $$

$$ {}_{RL} I_{a,t}^{\alpha}\, {}_C D_{a,t}^{\alpha} f = {}_C D_{a,t}^{\alpha}\, {}_{RL} I_{a,t}^{\alpha} f = f(t) \,, \tag{61} $$

Laplace transform is one of the major formal tools of science and engineering, especially when modeling dynamical systems. Also, Laplace transform is also usually used for solving fractional integro-differential equations involved in various engineering problems. The Laplace transform $\mathcal{L}\{.\}$ of the RL fractional derivative is

$$ \mathcal{L}\{ {}_{RL} D_{0,t}^{\alpha} f(t)\} = \int_0^{\infty} e^{-st}\, {}_{RL} D_{0,t}^{\alpha} f(t) dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{k}\, {}_{RL} D_{0,t}^{\alpha-k-1} f(t)_{|t=0} \tag{62} $$

Laplace transform of the Riemann-Liouville fractional integral (38) of $f(t)$ is

$$ \mathcal{L}\{ {}_{RL} I_0^{\alpha} f(t)\} = \frac{1}{s^{\alpha}} F(s) \,, \tag{63} $$

The terms appearing in the sum on the right hand side of the expression (62) involve the initial conditions and these conditions must be specified when solving fractional differential equations. Laplace transform of Caputo fractional derivative is

$$ \int_0^{\infty} e^{-st}\, {}_C D_{0,t}^{\alpha} f(t) dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \;\; n-1 < \alpha < n \tag{64} $$

which implies that all the initial conditions required by a fractional differential equation are presented by a set of only classical integer-order derivatives. Note also that the assumption of zero initial conditions is perfectly sensible when implementing fractional order controllers and filters. However, when attempting to simulate a fractional order system, the effect of initial conditions must be taken into consideration. In such a case, also, the difference between various definitions of fractional operators cannot be neglected. Besides that, the geometric and physical interpretations of fractional integration and fractional differentiation can be found in Podlubny's work,[43].Assuming that all initial conditions are equal to zero the fractional differintegral can be exactly represented by its transfer function

$$ G(s) = \frac{1}{s^{\alpha}} \tag{65} $$

which corresponds to the fractional derivative for negative values of the exponent α and to the fractional integral for the positive ones. By substituting $s = j\omega$ into (65) one obtains the frequency characteristic of fractional operators. Thus, the important difference between integer-order and fractional-order systems is revealed,[49,50]. A well-known fact is that the slope of the amplitude characteristics of the integer order systems is always an integer multiple of 20 dB/decade. This is not true for fractional order systems which can, in general, have amplitude characteristics of arbitrary slope. Similarly, an integer order system can have constant phase only if it is a multiple of pi/2, while the fractional order systems can have arbitrary constant phase. Thus, sometimes,fractional systems are referred to as ideal Bode systems. Amplitude and phase characteristics of fractional differintegrals of different order are shown in the following Figures 1,2.

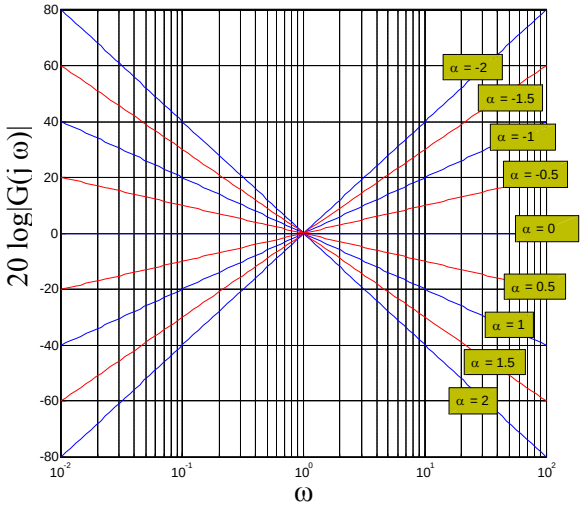

Fig. 1: Logarithmic amplitude characteristics of few fractional differintegrator (65)

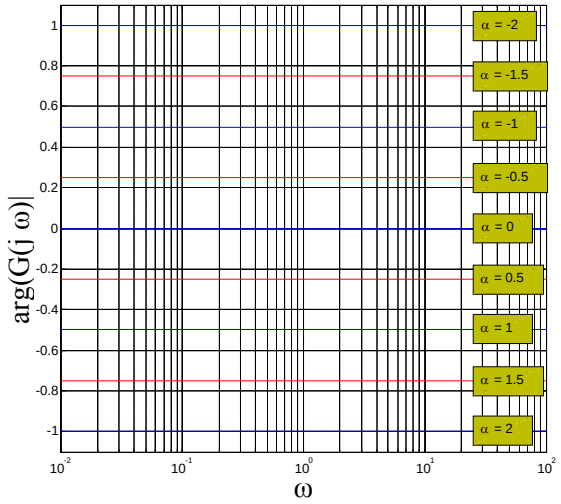

Fig. 2: Phase characteristics of few fractional differintegrators (65)

In the field of control theory, many aspects of linear systems have been investigated, in particular, different forms of stability and robustness criteria are developed. An in-depth generalization of various aspects of control theory to fractional order systems has been presented in [49-51].

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Part II

“Control and Stability Issues”

Direct and Indirect Method for Discretization of Linear Fractional Systems

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Abstract: - The problem of discrete-time (digital) implementation of fractional order systems, fractional differintegrators in particular, is addressed in the present chapter. Two novel methods have been presented and investigated in detail: a direct optimal and and indirect one. The direct optimal implementation is obtained by minimization of a custom frequency-domain discrepancy measure between the target fractional order system and its discrete-time rational approximation. The optimization process has been subjected to a number of constraints ensuring that the obtained approximations are stable, have stable inverses, and retain the desired steady state behavior. Both the optimality criterion and constraints are discussed in detail. Particle Swarm Optimization (PSO) algorithm has been utilized during the optimization procedure. This is a robust, swarm-based optimizer which is capable of solving hard, non-convex optimization problems with multiple local optima at the expense of a significant computational load. The indirect method consists of applying a recently proposed discretization method to a suitably chosen analogue, rational approximation of the target fractional systems. In comparison with the direct method, the indirect method is somewhat less accurate, but it is also significantly less expensive computationally. It has been demonstrated by means of a number of numerical examples that both presented methods outperform some of the recently proposed digital approximations of fractional systems.

Key-Words: - fractional order systems (FOS), rational approximations, discretization, particle swarm optimization (PSO)

2.1 Introduction

Dynamical systems described by fractional-order models, i.e. by (systems of) differential equations involving non-integer order differentiation and integrations, are referred to as *fractional order systems (FOS),* or simply *fractional systems*. Appearance of fractional order models leads naturally to the idea of *fractional order controllers (FOC)* [1-20]. Even if the model is of integer order, it is theoretically clear that introduction of fractional order controllers provides additional flexibility in the design process. Although, according to [21], pioneering studies in this area date back to late 1950's, it was not until recently that FOC raised considerable interest among engineers. A common example from control engineering is CRONE ("*Commande robuste d'ordre non entier*", which could be translated as: *non-integer robust control*), proposed in 1990's by Alain Oustaloup [22, 23]. Another example dating from the same period is Igor Podlubny's fractional generalization of PID controller, $PI^{\lambda}D^{\mu}$ [1], where both first-order integral and first- order differential controller actions have been replaced by their respective fractional counterparts. Fractional PID controller is briefly discussed in Section 3. The reader is also referred to [24,25-27]. Other types of both linear and nonlinear fractional controllers have been discussed in literature. For example, sliding mode (variable-structure) controllers have been discussed in [28].

The key operator of both FC and FOC is *fractional differintegrator*, a generalization of both classical differentiation and integration operators. Transfer function of such an operator is

$$G(s) = s^{\alpha} \tag{1}$$

where s is the Laplace variable and α can take arbitrary real values. For positive α, the differintegrator is a generalization of the classical integer order derivative, while for negative α, it is a generalization of the repeated, or n-fold, integral. Most linear fractional order systems can be implemented using fractional integrators as basic building blocks, just as integer order systems can be implemented using classical, first-order integrators. Thus, fractional differintegrator is often referred to as the *fundamental operator of fractional calculus* [25].

In principle, any fractional order system, including the fractional differintegrator, can be seen as a continuous time, linear, infinite dimensional filter [25, 29]. Such a filter cannot be directly implemented; therefore, it is of crucial importance to find suitable finite dimensional approximation of any FOC in general and fractional differintegrators in particular [21, 29-37]. There are many discretization schemes reported in literature which can be classified as either *direct or indirect*. The distinction is made based on whether an auxiliary continuous-time (s domain) approximation is constructed in the process. With direct methods, an intermediate continuous time approximation is not necessary, while with indirect methods such analogue approximation must be created.

Most of the direct methods start with a suitable discrete approximation of the first-order derivative or integral. Discretization scheme is then obtained by truncating some expansion of an appropriate non-integer power of the selected approximation. For example, a method based on power series expansion (PSE) of Euler operator is reported in [30], while continued fraction expansion (CFE) is applied to Tustin operator in [30]. Further direct schemes are reported in [21, 32-35]. Recently, Barbosa, Tenreiro-Machado and Silva reported novel methods obtained by least-squares fitting in time domain [36].

Indirect methods are constructed in two steps. In the first step, a finite dimensional, continuous time approximation of the target fractional order system is found. Oustaloup's rational approximation (ORA) reported in [36] and sub-optimum H_2 rational approximation reported in [16] are examples of such approximations. Other such approximations can be found in [29, 37] and are also discussed in [38]. Once a satisfactory continuous-time approximation has been found, the second step of each indirect method is to find its discrete-time equivalent. A number of discretization methods applicable to systems having rational transfer functions have been proposed in literature: approximations of Euler and Tustin, response invariant transformations (impulse-invariant and step invariant) and others [39]. Flexible first-order integrator, known as the tunable, or T-integrator, was proposed by Smith [40]. Similar tunable first-order discretization schemes were proposed by Le Bihan [40], Šekara *et al* [42-44] and others [38, 45-48]. Efficient implementation of discretization algorithms was discussed in [49]. Tustin operator, and especially its efficient implementation, is considered further in [50]. Some theoretical aspects of the discretization process are addressed in [51, 52].

This chapter addresses the problem of (direct and indirect) discretization of fractional order integrators and fractional order systems in general. Without loss of generality we focus primarily on the half-integration operator, although our methods and findings can be applied in a wider context. Since Riemann-Liouville, Caputo and Grunwald-Letnikov definitions differ practically only in the type of required initial conditions, the respective transfer functions of the fractional order differentiators and integrals are the same. Thus, the considerations of the present chapter are applicable to fractional order models expressed in terms of any of the before mentioned types of fractional operators.

A frequency domain error that enables an efficient and flexible quantitative measure of the discrepancy between the target system and its approximation will be introduced in the sequel. This measure is used throughout the chapter to access the quality of the obtained approximations. It is defined as a weighted sum of normalized amplitude and normalized phase error in a predefined frequency range. In the sequel, this error measure is denoted as J_β, where $\beta \in [0,1]$ is a design parameter affecting relative impact of the two factors (amplitude and phase error). This error measure can, in principle, be used in conjunction with any tunable scheme for frequency response approximation, including discretization methods and methods for model order reduction. Based on J_β , a direct optimal discretization scheme is described. It consists of finding a discrete transfer function of predefined order and adhering to certain constraints that give the smallest value of J_β. The constraints imposed on the optimizer are that it is stable, has stable inverse and (in the case when approximating fractional integrator) has infinite static gain. In addition to direct optimal approximation, an indirect method is also proposed. The indirect method is based on discretization polynomials reported in [39, 42, 45, 53, 54], and discussed further in [55]. Alain Oustaloup's ORA was used as the auxiliary approximation in continuous time [37]. However, any sufficiently good continuous-time rational approximation can be used.

Some other analogue realizations of fractional order systems have been reported in [29]. The advantage of the indirect method is that it performs formidably although it is optimization-free and therefore computationally inexpensive.

With both of the proposed methods, special care is taken to ensure proper behavior at low frequencies. This issue,commonly neglected in literature, is crucial for the correct steady-state behavior of rational approximations. The importance of low-frequency behavior will be demonstrated in the sequel by a number of numerical examples (see Section 5). The authors have discussed this and other related issues in [56].

The outline of the chapter is as follows. The problem is stated in Section 2, where the classical example of heat conduction is discussed briefly. Section 2 also gives concise statement of the problem addressed in the Chapter. This example gives rise to a fractional order description in terms of semi-derivatives in a natural and straightforward way. The problem is further motivated by stating the role of the fractional differintegrator in design of fractional control laws, Fractional PID controller in particular. The direct optimal discrete implementation procedure is addressed in Section 3, while the indirect procedure is presented in Section 4. Several examples illustrating the effectiveness of the proposed procedures are discussed in Section 5. The concluding remarks and some guidelines for further research are presented in Section 6. The text also incorporates an Appendix B, where a number of first order discretization schemes applicable to continuous-time systems with rational transfer functions have been outlined.

2.2 Motivation and Formulation of the Problem

Due to the popularity of the fractional calculus in recent years, a vast number of studies emerged showing that fractional order models are more adequate tool in describing many physical phenomena than the classical models formulated in terms of derivatives and integrals of integer order. A detailed account of such examples is beyond the scope of this book, and certainly beyond the scope of the present chapter. The interested reader is referred to the available literature, for example [9, 16, 57, 58]. It has been pointed out in the introductory section that many fractional order systems can be implemented using differintegrators as building blocks. One of our aims in this preliminary section is to demonstrate that differintegrals and related transfer functions emerge naturally even in the context of well-established mathematical models of physical phenomena. This is especially true for the semi-integrator, which is the target system of the present chapter. In addition, we would also like to stress that differintegrals appear as building blocks of a variety of fractional control strategies. One of such controllers, namely Fractional PID regulator, is briefly addressed at the end of this section.

Consider, first, a process described by classical *diffusion equation* (also reffered to as the *heat equation*), which is ubiquitous in science and engineering since it simultaneously describes a number of transfer phenomena, including heat-transfer and a number of other diffusion-like processes. These diffusion-like processes include diffusion of mass (mechanical diffusion), diffusion of momentum (viscosity), diffusion of electrical potential (in long lines, when inductivity is negligible), and many others. One-dimensional diffusion equation is a partial differential equation of the form

$$\tau \frac{\partial^2 \rho}{\partial z^2} = \frac{\partial \rho}{\partial t}, \ \tau > 0 \tag{2}$$

describing the process of transport (diffusion) of a quantity ρ along the z axis in time t. For simplicity, let us address only the diffusion within a semi-infinite medium, where both space and time variable take arbitrary positive values. Let us assume also that the process can be controlled by acting on the cross-section $z = 0$, and that the process output is taken (measured) at the cross-section $z = L$. The dynamics of the process is influenced by the diffusion time constant $\tau = \tau(z,t)$, which is, in general, a function of both space and time. However, in a variety of practically interesting cases this coefficient can be approximated by a constant factor.

Without loss of generality, assume that (2) describes a heat conduction process schematically shown in Fig. 1. Let us obtain its transfer function. In this particular case, $\rho = \rho(z,t)$, is the temperature of the crossection defined by space coordinate z evaluated at time instant t. Let $\tilde{\rho} = \tilde{\rho}(z,s)$ denote the Laplace transform of ρ, where the Laplace transform is taken with respect to the time variable t and the space variable z is considered as a parameter,

$$\tilde{\rho}(z,s) = \int_0^\infty \rho(z,t) e^{-st} dt \, . \tag{3}$$

By applying the Laplace transform to equation (2), one obtains general solution

$$\tilde{\rho}(s,z) = C_1(s)e^{-z\sqrt{s/\tau}} + C_2(s)e^{z\sqrt{s/\tau}} \quad . \tag{4}$$

Since any heat conduction process is stable, the Laplace transform of the temperature in any cross-section must be bounded, i.e.

$$\lim_{z\to\infty} \tilde{\rho}(s,z) = const. \tag{5}$$

has to be satisfied, thus C_2=0 and equation (4) takes the form

$$\rho(s,z) = C_1(s)e^{-z\sqrt{s/\tau}} . \tag{6}$$

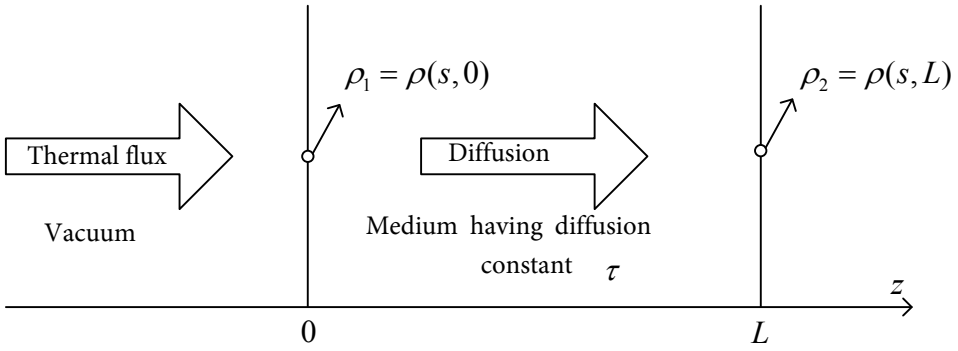

Fig. 1. A sketch of the process of heat conduction by diffusion.

Integration "constant" C_1 as well as the conduction function is determined from the known (or given) boundary conditions. In view of this, the most frequent cases in practice are:

Case 1. *Heat conduction without any convective exchange of heat with the environment and fixed temperature at the "left" boundary*. In this particular case, the temperature of the cross-section $z = 0$ could be controlled directly, and considered as the input of the process, while the dependent temperature of the cross-section $z = L$ could be considered as the output. The left boundary condition for this case is $\tilde{\rho}(s,0) = C_1(s)$, and the transfer function takes the form

$$G_a(s) = \frac{\tilde{\rho}_2(s,L)}{\tilde{\rho}_1(s,0)} = e^{-L\sqrt{s/\tau}} = e^{-\sqrt{Ts}}, \; T = L^2/\tau . \tag{7}$$

Case 2. *Heat conduction without any convective exchange of heat with the environment and fixed thermal flux at the "left" boundary*. The process is influenced by gradient of quantity ρ at z = 0 (this is the boundary surface of the medium of Fig. 1), the input quantity of the process being thermal flux through the boundary surface (again without any convective exchange with the environment)

$$\psi = -\lambda \left.\frac{d\tilde{\rho}(s,z)}{dz}\right|_{z=0} \tag{8}$$

and the process (output) quantity is $\rho_2 = \rho(s, \mathrm{L})$, and the transfer function is

$$G_b(s) = \frac{\tilde{\rho}_2}{\psi} = \frac{K}{\sqrt{s}} e^{-\sqrt{Ts}}, \; T = L^2/\tau, \; K = \sqrt{\tau}/\lambda . \tag{9}$$

Case 3. *Heat conduction without any convective exchange of heat with the environment*. The last characteristic case is when the convection is no longer neglected. Now, the process is influenced by a linear combination of the thermal flux and temperature at the "left" boundary

$$u = -\lambda \left.\frac{d\rho(s,z)}{dz}\right|_{z=0} + \eta\,\rho(s,z), \text{ with output } \rho_2 = \rho(s,L), \tag{10}$$

and the transfer function is

$$G_c(s)=\frac{\tilde{\rho}_2}{u}=\frac{K}{1+\sqrt{T_1 s}}e^{-\sqrt{T s}},\ K=1/\eta,\ T_1=\frac{\lambda^2}{\eta^2\tau},\quad T=L^2/\tau. \tag{11}$$

In the examples above, the semiderivative operator has appeared in a number of contexts. It should be mentioned that other forms of fractional order transfer functions emerge during investigations of different transfer phenomena. In the analysis of axial diffusion, i.e. diffusion from the axis of the cylinder towards its lateral surface or vice versa, one meets transfer functions originating from the Laplace transforms of Bessel functions, which have the form

$$G(s)=\frac{K}{\sqrt{1+sT}}. \tag{12}$$

From this example, transfer functions given by equations (7), (9), (11), and (12) belong to the fractional systems having transfer functions which belong to the class of irrational functions [58].

Since these transfer functions describe adequately physical processes, a logical question arises whether it is possible to formulate fractional control laws and what would be their contribution to process control. Among many modern control strategies utilizing fractional calculus, Podlubny's Fractional PID [1] regulator is emphasized here. Classical PID is arguably the most utilized control strategy in use today. By replacing classical integral and differential actions by their respective fractional analogues, the flexibility and applicability of the PID regulator can be greatly increased. Transfer function of the fractional PID is of the form

$$\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}(s)=k+k_{\mathrm{i}}s^{-\lambda}+k_{\mathrm{d}}s^{\mu},\ \lambda,\mu\in[0,1], \tag{13}$$

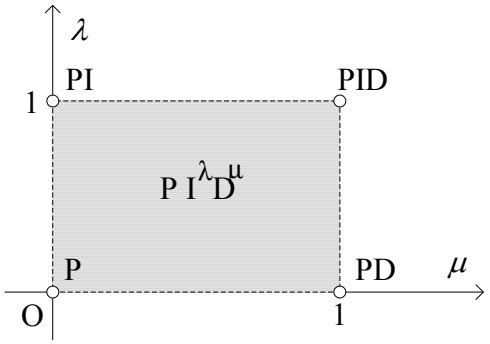

Fig. 2. Parameter plane of the fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ regulator

The reader should notice that the implementation of Fractional PID requires direct implementation of fractional integrator and differentiator. Similar is also true for other types of fractional regulators, as it can be seen from [22, 23, 25, 26, 30]. Such regulators are typically implemented as high order FIR or IIR filters [29-37], Realization of fractional control laws involving an adequate discretization is possible thanks to the fast modern computers. It is known that in the regulator design two approaches are possible, direct design in the discrete domain and the other approach is design in the continuous domain first and then transition to the discrete domain. Obviously, discretization is required by both approaches. However, the discretization procedure is not straightforward when fractional systems are in question, a problem which has been causing a considerable interest over the past years.

In the following sections some novel techniques for designing rational discrete approximations of semi-integral operator are described in detailed. The presented methods approximate the fractional order (irrational) continuous-time transfer function

$$G(s)=\frac{1}{\sqrt{s}} \tag{14}$$

with a discrete-time rational one,

$$H(z)=\frac{b_m s^m+\ldots+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_0} \tag{15}$$

where $n \geq m$. The quality of the obtained approximations will be measured in frequency domain, i.e. the similarity of $G(j\omega)$ and $H(e^{j\omega T_s})$ will be investigated within a given frequency range $\omega \in [\omega_{\min}, \omega_{\max}]$. T_s stands for the sampling interval.

2. 3 The Direct Optimal Method for Discretization of Fractional Integrators

Discretization of fractional systems can be accomplished by applying direct methods which make use of the special techniques for converting fractional continuous-time transfer function $G(s)$ to a rational discrete-time approximation $G(z)$, or by applying indirect methods which use adequate approximations of transfer function $G(s)$ of a fractional system by a rational transfer function $H(s)$ to which it is possible to apply standard discretization methods in order to obtain final solution $H(z)$, i.e. $G(z)$.

2.3.1 The optimality criterion

In order to apply an efficient direct method of discretization, let us define the optimality criterion applied in [56]. Let $a_t(\omega)$ and $\varphi_t(\omega)$ denote amplitude and phase response of the target system. Similarly, let $a_a(\omega)$ and $\varphi_a(\omega)$ denote amplitude and phase response of the approximation to be evaluated. Amplitude responses are measured in decibels, while phase responses are measured in degrees or radians. At each frequency point, ω, the approximation error is defined as

$$\varepsilon_\beta(\omega) = \frac{(1-\beta)a_s^2[a_t(\omega) - a_a(\omega)]^2 + \beta\varphi_s^2[\varphi_t(\omega) - \varphi_a(\omega)]^2}{2}. \tag{16}$$

Design parameter β controls relative impact of the two terms. For β equal to 0 only the amplitude error is measured, while for β equal to 1 only the phase error is of importance. Clearly, for intermediate values of β both terms are considered, yet with different impact on the error value. Scaling factors a_s and φ_s are introduced primarily for dimensional purposes. In addition, they can help reduce amplitude and phase errors to approximately the same range of values, which can enable a more intuitive selection of appropriate value for β. The overall approximation error can be obtained by integrating (16) over the entire frequency range

$$J_\beta = \int_0^\infty \theta(\omega)\varepsilon_\beta(\omega)d\omega \tag{17}$$

where θ(ω) denotes relative importance of each frequency. Introduction of θ is particularly useful in cases where especially good fit is desired at certain frequencies. Approximation error at such frequencies should contribute more to the overall approximation error than equal discrepancy at other frequencies. In practice, frequency responses are only evaluated in some finite number of points. Denote this set of relevant frequencies by $\Omega \in \{\omega_i, 1 \leq i \leq N\}$. The overall approximation error (17) should be replaced by computationally far more convenient expression

$$J_\beta = \sum_{i=1}^{N} \theta_i \varepsilon_\beta(\omega_i). \tag{18}$$

Throughout the present section, Ω is chosen to be a set of frequencies distributed homogenously on logarithmic scale in a predefined frequency range. All of these frequencies are equally valued. Each weight θ_i is set to the same value, equal to the reciprocal of the number of relevant frequencies N. The final form of the discrepancy measure is

$$J_\beta = \frac{1}{N}\sum_{i=1}^{N} \varepsilon_\beta(\omega_i) \tag{19}$$

This can be interpreted as combined amplitude-phase error averaged over all relevant frequencies. Expression (19) is to be used as optimality criterion throughout this text. In specific applications, its more general form (18) can be more appropriate. All of the methods presented in the chapter can be used unaltered even in such cases.

2.3.2 Optimal discrete approximations of the fractional integrators

The method of obtaining discrete approximations of fractional semi-integrator by direct minimization of optimality criteria (19) over all discrete systems of a predefined order is presented next. The procedure is applicable to fractional differintegrators of arbitrary order. During the optimization process, a single pole of the discrete approximation is fixed at one. When approximating fractional differentiators, a single zero should be fixed at one. Magnitudes of all other poles and zeros are constrained to be not greater than one. These constraints ensure that all of the obtained discrete equivalents are stable, have stable inverses and retain correct steady state behavior. The key parameters that should be determined prior to the optimization process are:

i) order of the approximation (n),
ii) set of relevant frequencies used to calculate optimality criteria (Ω),
iii) appropriate scaling factors and relative importance of the amplitude and phase errors, i.e. the appropriate values for a_s, φ_s and β, and
iv) sampling time (T).

Frequency responses of optimal discrete approximations of orders 3, 5, 7 and 9 are compared to the frequency response of the semi-integrator of Fig. 3. Relevant frequencies are chosen to be distributed homogenously on logarithmic scale from $\omega_L = 0.001$ rad/s^{-1} up to the Nyquist frequency, $\omega_N = \pi/T_s$. The amplitude scale factors are set to $a_s = 1/\left|a_t(\omega_L) - a_t(\omega_N)\right|$ dB, a_t being the frequency response of the semi-integrator, while the phase scale is set to $\varphi_s = 1/\pi$ rad. Both amplitude and phase errors are valued equally, $\beta = 0.5$. The sampling time is chosen to be $T = 10$ ms. Frequency response was sampled at $N = 1000$ points.

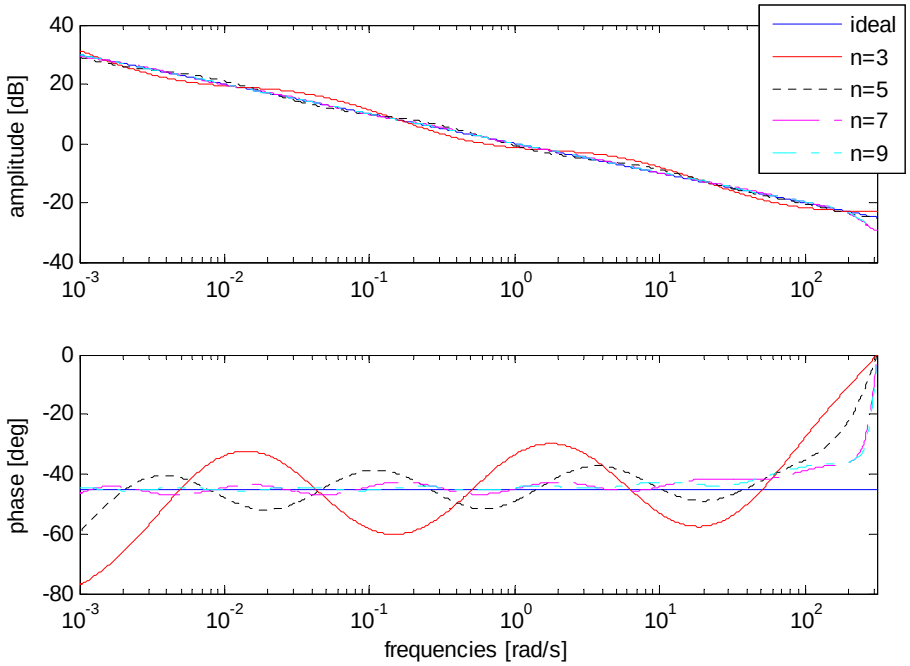

Fig. 3. Frequency responses of the optimal direct approximations of the semi-integrator of various order ($\beta = 0.5$). Values of the optimality criteria J_β (19) for different values of approximation order are: $J_\beta = 2.57 \cdot 10^{-3}$ $(n = 3)$, $J_\beta = 7.5 \cdot 10^{-4}$ $(n = 5)$, $J_\beta = 2.91 \cdot 10^{-4}$ $(n = 7)$, $J_\beta = 2.68 \cdot 10^{-4}$ $(n = 9)$

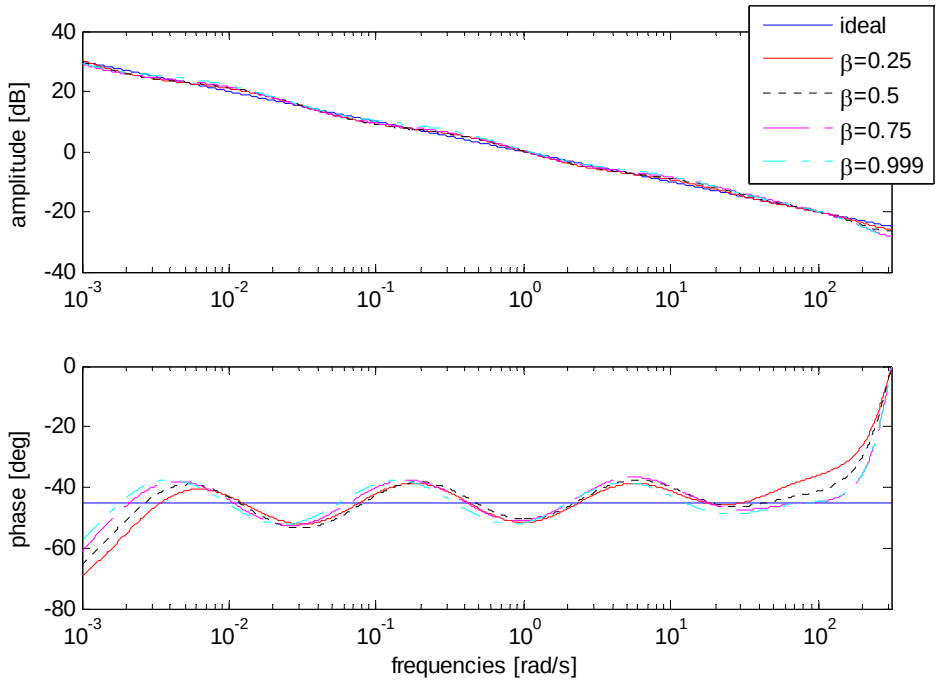

Fig. 4. Frequency responses of the fifth order optimal direct approximations of the semi-integrator for various values of β.

Frequency responses of the optimal fifth order approximations (n =5) obtained for various values of β are presented in Fig. 4. Unless additional constraints are introduced, values of parameter β should always be chosen from open interval (0,1). If only the phase error is considered (β =1), the optimization is insensitive to gain, and therefore incorrect. One could decide to fix gain at certain frequency prior to phase error minimization, but the problem of proper choice of such a frequency remains open. On the other hand, numerous experiments conducted by the authors testify that if only the amplitude error is considered (β =0), the optimizer attempts minor corrections of amplitude response that, in turn, greatly degrade quality of phase fit. The appropriate value of β is dependent on user's goals, as well as on the previous selection of scale factors. In general, both amplitude and phase errors should contribute comparably to the value of optimality criteria. However, the phase response is in general harder to approximate and larger values of β are sometimes more adequate. Throughout this text, β is chosen to be 0.5. The relevant frequency range is of utmost importance. When optimizing over a broad frequency range, the effort of the optimizer is spread, resulting in good approximation on average. To the contrary, when a narrower band of relevant frequencies is considered, the optimizer produces approximations that are more accurate on that small region. A comparison between the seventh order narrow band ($\omega \in (0.05,20)$) and broad band ($\omega \in [0.001,\omega_N]$) approximations is made in Fig. 5. All other optimization parameters are the same as above. The broad band approximation is, as expected, valid over a wider frequency range, especially at low frequencies. However, if only the two central decades are relevant, it can be seen that maximal amplitude and phase errors of the narrow band approximation are about 0.0283 dB and 0.253 degrees, which is roughly 10 times less in comparison to maximal errors of the broad band approximation over the same frequency range. Clearly, for fixed approximation order there is a tradeoff between approximation accuracy and bandwidth.

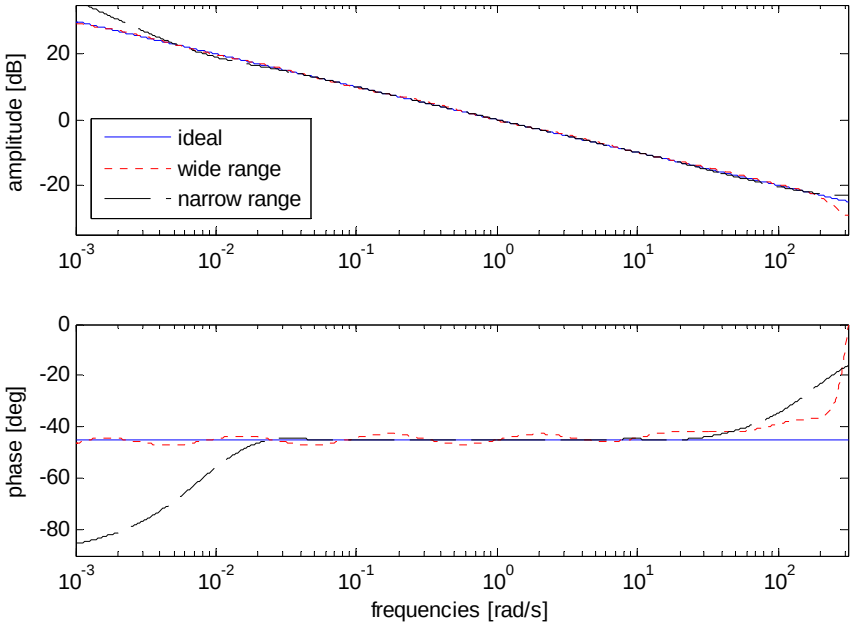

Fig. 5. Frequency responses of the seventh order broad band and narrow band optimal approximations obtained for β=0.5.

Poles, zeros, and gains of optimal discrete approximations of different order, obtained for β =0.5, are presented in Table 1 for further reference. Note that poles and zeros are interlaced along the segment of the real axis inside the unit circle. This phenomenon, known in literature [30,35], is not enforced in any way, but occurs as a natural result of the optimization process.

Table1. Zeros (z), poles(p), and gain (k) of optimal direct approximations to the semi-integrator obtained for the frequency range $(0.001, 2\pi/T)$, with sampling time T=0.01s and β=0.5.

	n=5	n=7	n=9
z1	0.999986305429608	0.999994103939234	0.999996826768815
z2	0.999530506330752	0.999915583664228	0.999967747286457
z3	0.985129852403502	0.999091068030843	0.999796345479915
z4	0.663247873036833	0.989506466787377	0.998645266695162
z5	-0.047565565066957	0.896856212527238	0.990956987187242
z6		0.407958744308062	0.940943134980605
z7		-0.546921874442973	0.605320223754678
z8			0.027743161825917
z9			-0.615695445938038
p1	1.000000000000000	1.000000000000000	1.000000000000000
p2	0.999916770576491	0.999974469462250	0.999987680125186
p3	0.997377207355588	0.999727493208832	0.999919393090079
p4	0.915902973892208	0.996874407191277	0.999477334928292
p5	0.298764110716001	0.965651963636666	0.996459218888345
p6		0.718144401760234	0.976917451551429
p7		-0.131263646081857	0.839993326475073
p8			0.222315405191216
p9			-0.288175605056004
K	0.089295640868600	0.084651005226214	0.084824546079785

In Figs. 6 and 7 the proposed schemes are compared to those recently presented by other authors. In particular, the comparison is made with respect to the least-squares approximation of the Tustin operator (BTMS) reported by Barbosa, Tenreiro, Machado, and Silva [36], as well as with respect to the recursive (VCP-TR) and CFE (VCP-TCFE) approximations of the Tustin operator reported by Vinagre, Chen and Petras [31]. These approximations are chosen because they perform similar or better than other widely used methods, and shall henceforth be denoted by previously introduced abbreviations. Approximations of order 5 are presented in Fig. 6, while the 7-th order approximations are presented in Fig.7. Notice the different behavior in the low frequency range. Due to a pole fixed at 1, the approximations introduced in this text behave (up to a multiplying constant) as pure integrators at low frequencies. This behavior is crucial in retaining the static gain of more complex systems, since it ensures infinite gain of the approximation at zero frequency. However, this is not the only reason for comparatively good performance of the proposed approximations, as will be demonstrated later (see Example 2 in Section 5).

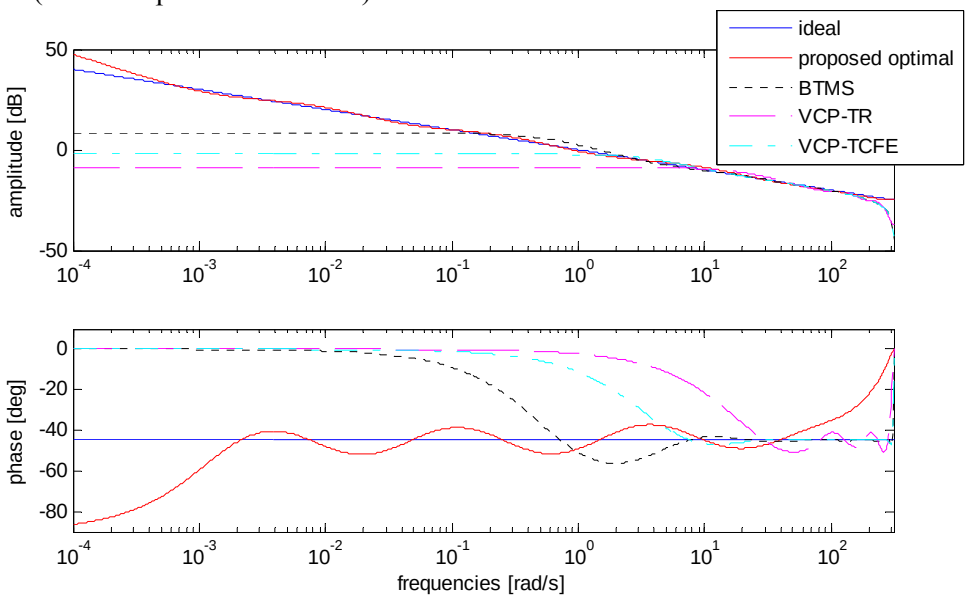

Fig. 6. Frequency domain comparison of the fifth order direct optimal approximation (β =0.5) with the fifth order approximations reported by other authors. The corresponding values of J_β criterion are: $J_\beta = 7.5\cdot10^{-4}$ (optimal), $J_\beta = 2.75\cdot10^{-2}$ (BTMS), $J_\beta = 1.28\cdot10^{-1}$ (VCP-TR), $J_\beta = 7.6\cdot10^{-2}$ (VCP-TCFE).

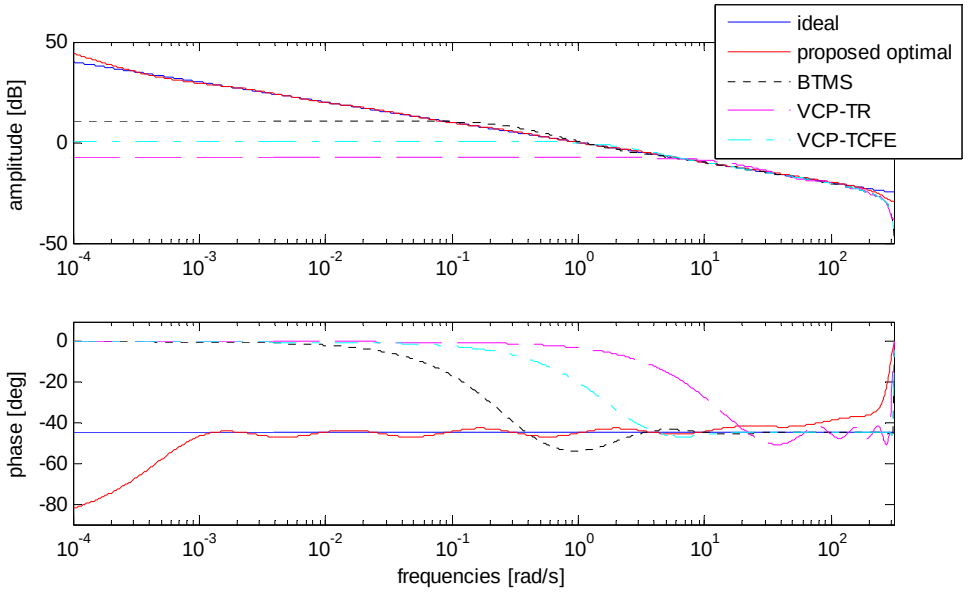

Fig. 7. Frequency domain comparison of the seventh order direct optimal approximation (β =0.5) with the seventh order approximations reported by other authors. The corresponding values of J_β criterion are: $J_\beta = 2.91\cdot10^{-4}$ (optimal), $J_\beta = 2.1\cdot10^{-2}$ (BTMS), $J_\beta = 1.17\cdot10^{-1}$ (VCP-TR), $J_\beta = 6\cdot10^{-2}$ (VCP-TCFE),

2.3.3 Remarks on the optimization procedure

Optimality criterion (19) is highly nonlinear, non-convex and multimodal. Standard, local optimization procedures such as line-search or trust-region methods will, in general, fail to provide adequate solutions. Utilization of a robust, global optimization scheme is therefore necessary. In recent years many global optimizers have been proposed, most of which are evolutionary and population based. At the expense of higher computational load, such algorithms are capable of finding near optimal solutions to hard optimization problems that cannot be tackled by conventional local optimizers.

Among global methods the *Particle Swarm Optimization (PSO)* algorithm is known to be very effective [59]; it is a general-purpose optimizer which is relatively efficient in both computational and storage requirements, it is not difficult to implement and it can be fine-tuned for specific problems. A theoretical and empirical analysis of the PSO algorithm, with special emphasis on convergence and parameter selection is presented in [60]. All of the solutions and examples presented in this text are obtained by the PSO algorithm with time-varying parameters, described in [54]. A MATLAB implementation of this algorithm is freely available [61, 62]. In the current work, the PSO algorithm with 32 particles in the swarm and 200 iterations during each run was used. Occasionally, re-initialization of the swarm and 200 additional iterations were used to fine-tune the obtained solution.

In the present work the design variables of the optimization procedure were chosen to be poles, zeros and gain of the discrete-time rational approximations. Prior to initialization process, it is necessary to specify the order of the desired approximation. Let us denote the approximation order by n. Thus, there are $2n+1$ design variables, n zeros, n poles and gain. When approximating fractional integrators, however, a single pole must always be equal to 1 in order to preserve steady state behavior. Thus, the actual number of free parameters is $2n$. As mentioned earlier, the approximations must be stable, with stable inverses; since fractional integrals have non-oscillatory impulse responses, all approximation poles and zeros were constrained to be real with absolute values less than 1. These constraints have been enforced during the optimization process by means of penalty functions. The actual optimality criterion used in the optimization process was obtained from the criterion (4) by adding a penalty function of the form

$$P = mC, \tag{20}$$

where m is the number of additional poles and zeros having absolute value greater or equal to one and C is some large number (C=1000 was used in the examples presented in this text).

The general scheme for obtaining a direct optimal approximation using MATLAB software is the following. At first, it is necessary to write an m-function implementing the optimality criterion (15) (or preferably its more general form (18)). The input arguments to such a function should be the poles and zeros of the discrete approximation and the parameters of the optimality criterion itself: the scaling factors and a vector of the frequency points of interest. In general, it can also take a vector of the weighting factors. The single output argument should be the value of the optimality criteria (15) with penalty function (20) added. The implementation is rather straightforward; however, the authors will readily provide such a function upon request.

2.4 Indirect Method for Discretization of Fractional Integrators

The indirect discretization method proposed in the present work consists of the application of discretization polynomials presented in [38, 42, 45, 53-55] to suitably chosen continuous-time rational approximation of the target differintegrator. Due to its flexibility, accuracy, and clear geometrical and physical interpretation, the Oustaloup's rational approximation (ORA) is utilized. However, different continuous-time approximations could be used as well. By construction, ORA of some sufficiently high order is capable of approximating fractional differintegrators with arbitrary accuracy within an arbitrary frequency band. By utilization of the discretization polynomials, the amplitude and phase characteristics of the continuous-time approximations are preserved with high accuracy, except at close vicinity of the Nyquist frequency. First few discretization polynomials are presented in Table 2; a detailed account of their properties can be found in [42].

The procedures obtained in this way are computationally far less expensive than the optimal procedures of the previous section. Despite of this, their performance is not significantly worse. In fact, the proposed

procedures can be seen as sub-optimal, and thus suitable as initializes for the optimization procedures of the previous Section.

Table 2. Discretization polynomials reported in [42], of $p = 1 - z^{-1}$.

n	s^{-n}	$f_n(p)$ - transformation polynomial	$f_n(z)$ - approximation function
1	$\frac{1}{s}$	$T(p^{-1} - \frac{1}{2})$	$T\frac{z}{z-1} - \frac{T}{2}$
2	$\frac{1}{s^2}$	$T^2(p^{-2} - p^{-1} + \frac{1}{12})$	$T^2\frac{z}{(z-1)^2} + \frac{T^2}{12}$
3	$\frac{1}{s^3}$	$T^3(p^{-3} - \frac{3}{2}p^{-2} + \frac{1}{2}p^{-1})$	$\frac{T^3}{2}\frac{z(z+1)}{(z-1)^3}$
4	$\frac{1}{s^4}$	$T^4(p^{-4} - 2p^{-3} + \frac{7}{6}p^{-2} - \frac{1}{6}p^{-1} - \frac{1}{720})$	$\frac{T^4}{6}\frac{z(z^2+4z+1)}{(z-1)^4} - \frac{T^4}{720}$
5	$\frac{1}{s^5}$	$T^5(p^{-5} - \frac{5}{2}p^{-4} + \frac{25}{12}p^{-3} - \frac{5}{8}p^{-2} + \frac{1}{24}p^{-1})$	$\frac{T^5}{24}\frac{z(z+1)(z^2+10z+1)}{(z-1)^5}$
6	$\frac{1}{s^6}$	$T^6(p^{-6} - 3p^{-5} + \frac{13}{4}p^{-4} - \frac{3}{2}p^{-3} + \frac{31}{120}p^{-2} - \frac{1}{120}p^{-1} + \frac{4}{3\cdot 8!})$	$\frac{T^6}{5!}\frac{z(z^4+26z^3+66z^2+26z+1)}{(z-1)^6} + \frac{4T^6}{3\cdot 8!}$
7	$\frac{1}{s^7}$	$T^7(p^{-7} - \frac{7}{2}p^{-6} + \frac{14}{3}p^{-5} - \frac{35}{12}p^{-4} + \frac{301}{360}p^{-3} - \frac{7}{80}p^{-2} + \frac{1}{720}p^{-1})$	$\frac{T^7}{6!}\frac{z(z+1)(z^4+56z^3+246z^2+56z+1)}{(z-1)^7}$
⋮		⋮	⋮
k	$\frac{1}{s^k}$	$T^k(a_k^k\ p^{-k} + \cdots + a_1^k\ p^{-1} + a_0^k)$	$f_k(z)$

It should be mentioned that the calculation of approximation functions $f_n(z)$ in [37,41,44] can efficiently be specified by applying Pade approximation of the order (n,n) to function $(T/\ln(z))^n$ at point z=1. By direct application of MAPLE software and Pade approximation $f_n(z)$. E.g. for n=1 and n=8

```
> restart:
> with(numapprox):
> f:=n->factor(pade((T/ln(z))^n,z=1,[n,n])):
```

```
> f(1);
```

$$\frac{T(z+1)}{2(z-1)}$$

```
> f(8);
```

$$-\frac{T^8(1 - 579770\,z^4 - 28772\,z^2 - 285896\,z^3 - 285896\,z^5 - 28772\,z^6 - 248\,z^7 + z^8 - 248\,z)}{1209600\,(z-1)^8}$$

The main property of these approximation functions of an analogue integrator $1/s^n$, compared to any other approximation, is that the phase deviation along the whole of ω axis is equal to 0

$$\left|\arg(1/(\mathrm{j}\omega)^n) - \arg(f_n(\exp(\mathrm{j}\omega T))\right| \equiv 0,\ n \in \mathbb{N}, \omega \in \mathbb{R},\ \mathrm{j} = \sqrt{-1}, \tag{21}$$

while deviation of the absolute value of the module 0 at ω=0 is guaranteed by Pade approximation

$$e_r = \left|1/(\mathrm{j}\omega)^n - f_n(\exp(\mathrm{j}\omega T))\right|_{\omega=0} = 0,\ n \in \mathbb{N} \tag{22}$$

at the point of expansion z=1, corresponding to point s=0 based on relation z=exp(sT).

For illustration, Fig. 8 shows absolute deviation between the analogue integrator $1/s^n$ and the approximation functions for n=2,3,4,5, and 6.

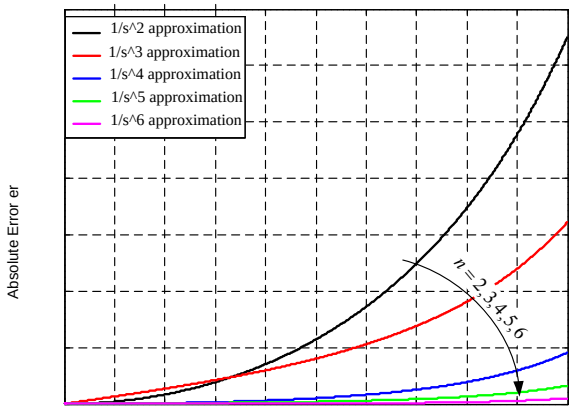

Fig. 8. Absolute deviation between the analogue integrator $1/s^n$ and the approximation functions for n=2, 3, 4, 5, and 6

It can be noted that the absolute error e_r dramatically decreases as the degree n of the analogue integrator increases, which is very important for approximations of fractional functions described by a high order for the purpose of adequate discretization.

A large number of other s-z approximations of the first order, presented in Appendix B, are special cases of a T-integrator [40]

$$X_n = X_{n-1} + T\gamma[P(dX/dt)_n + (1-P)(dX/dt)_{n-1}] \tag{23}$$

i.e.

$$1/s \approx P\frac{Tz}{z-1}\gamma + (1-P)\frac{T}{z-1}\gamma\ ,\ P \in [0,2] \tag{24}$$

where parameters P and γ are adjustable and parameter T is integration step length. It should be emphasized that in [40], published in 1987, it is explicitly stated that specail cases of a T-integrator, depending upon parameters P and γ, are reduced to the conventional integrators presented in Table 3.

Table 3. The conventional integrators which are special cases of a T-integrator

γ	P	Name Integrator
1	0	Euler integrator
1	1/2	Trapezoidal integrator
1	1	Rectangular integrator
1	3/2	Adams-Bashforth Corrector

Consider, for example, a semi-integrator $s^{-0.5}$. It is necessary to ensure proper low frequency behavior, therefore the transfer function should be rewritten as $s^{0.5}/s$ and ORA should be applied to semi-derivative $s^{0.5}$ only. One should decide the frequency range of interest, as well as the desired accuracy. This will, in turn, determine the order of the approximation, as well as the actual values of the coefficients. Suppose that the obtained continuous-time rational approximation to the semi-integrator is

$$H_{\mathrm{ORA}}(s) = K\frac{s^2 + b_1 s + b_0}{s(s^2 + a_1 s + a_0)} = K\frac{s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 + a_1 s^{-1} + a_0 s^{-2}} \tag{25}$$

Each s^{-n} in (19) should be substituted by the approximation function $f_n(z)$ from Table 2. Finally, the discrete approximation of the semi-integrator is

$$H(z) = K\frac{f_1(z) + b_1 f_2(z) + b_0 f_3(z)}{1 + a_1 f_1(z) + a_0 f_2(z)}. \tag{26}$$

It is interesting to note that flexibility of this discretization procedure depends on adaptability of the utilized continuous-time approximation scheme; the polynomials serve only to accurately translate the continuous-time design to the discrete-time domain. It has been demonstrated that application of discretization polynomials yields superior performance with respect to the following time-domain measure of discrepancy [42]

$$J = \sum_{k=0}^{M-1}\left(\int_{kT}^{(k+1)T}(g(t) - g_{\mathrm{ap}}(kT))dt\right)^2, \tag{27}$$

where $g(t)$ and $g_{\mathrm{ap}}(t)$ are, respectively, the exact and approximate step response of the system under consideration, T is the sampling time and M is length (in samples) of the time interval under consideration. This criterion is used in the sequel of the current work as a time-domain discrepancy measure. Fig. 9 shows comparatively frequency responses of the semi-integrator and the proposed indirect discrete approximation of order 5. The figure also shows frequency response of the 5-th order continuous-time approximation obtained after application of 4-th order ORA. The results presented in this figure should be compared to those presented in Fig. 4.

The proposed method is computationally inexpensive, since it requires only a direct substitution and is therefore optimization-free. Nevertheless, numerous numerical experiments conducted by the authors testify that its performance is excellent, even when measured in terms of optimality criterion (15). The proposed indirect method can, in fact, be considered as suboptimal solution to the direct optimization problem presented earlier, i.e. it can be used as an initial guess for the computationally more involved, yet more accurate direct method.

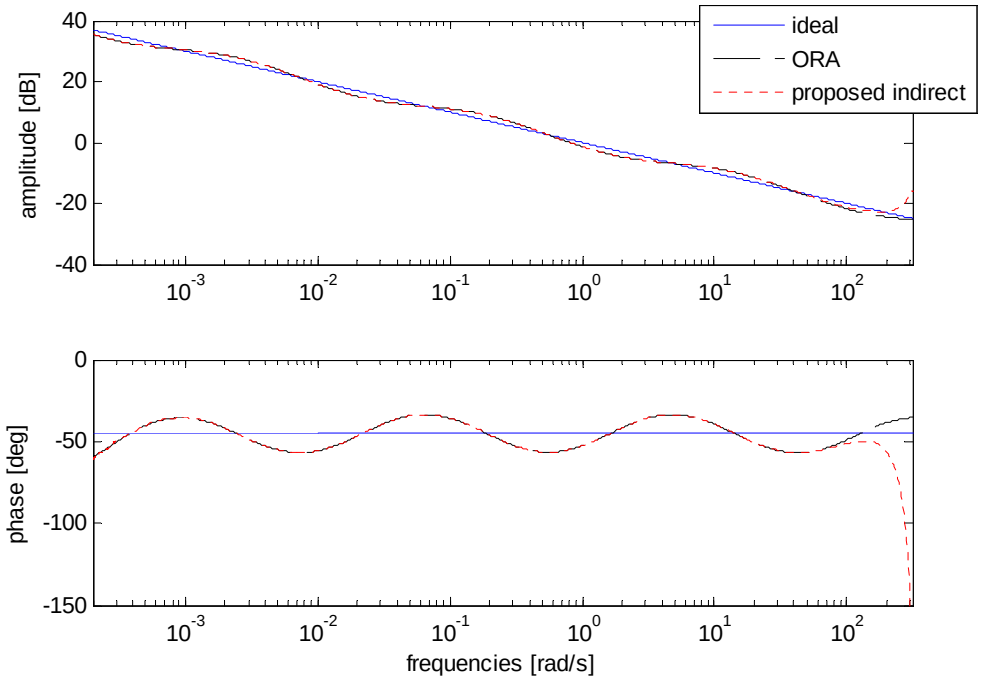

Fig. 9. Frequency domain comparison of the fifth order ORA incorporating corrected low frequency behavior (ORA for semi-derivative divided by s) and its discretization obtained by discretization polynomials. The corresponding values of J_β criterion are: $J_\beta = 1.03\cdot 10^{-3}$ (ORA), $J_\beta = 2.13\cdot 10^{-3}$ (the proposed indirect).

2. 5 Examples

Example 1. As the first example, let us calculate the semi-integral of the unit step-function $h(t)$. The step response of the ideal semi-integrator is compared to step responses of its various approximations of the seventh order in Figs. 10 and 11. Slow, non-exponential growth of the ideal response is approximated well by both direct and indirect method presented in the present text.

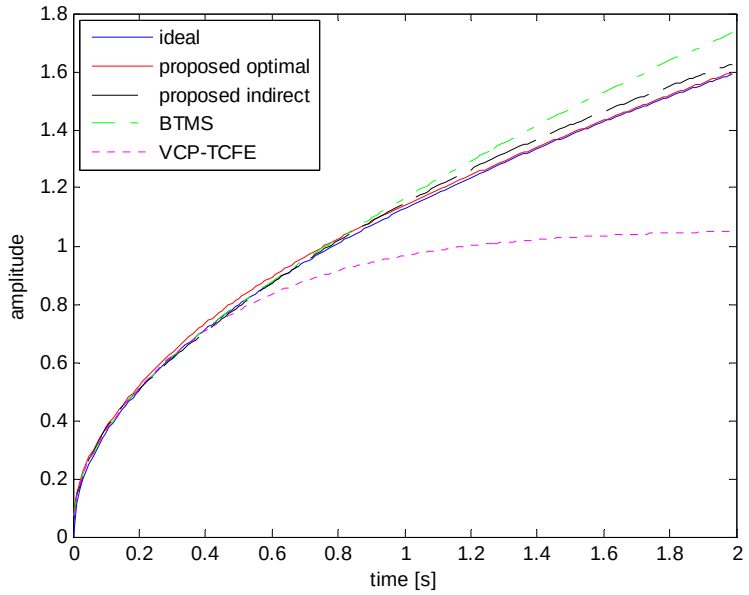

Fig. 10. Comparison of the initial behavior of the step response of the semi-integrator to responses of its various approximations. The quality of the responses can be numerically compared by using time domain criterion (27): $I = 4.54 \cdot 10^{-6}$ (direct optimal), $I = 1.19 \cdot 10^{-5}$ (indirect), $I = 9.2 \cdot 10^{-5}$ (BTMS), $I = 1.9 \cdot 10^{-3}$ (VCP-TCFE)

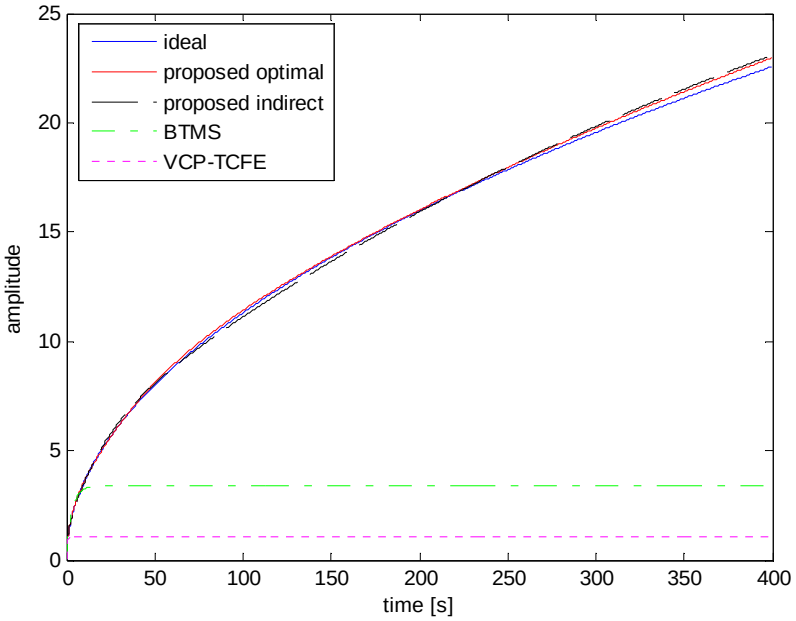

Fig. 11. Comparison of the step response of the semi-integrator to responses of its various approximations.

Example 2. Consider a system described by a fractional order transfer function

$$G_1(s) = \frac{1}{\sqrt{s}+1} = \frac{s^{-0.5}}{1+s^{-0.5}}\,. \tag{28}$$

The system is characterized by a rapid initial response followed by a slow steady state approach. The steady state gain of the system is one. The discrete approximation of the system will be constructed as

$$\tilde{G}_1(z) = \frac{H(z)}{1+H(z)}\,, \tag{29}$$

where $H(z)$ is some discrete approximation of the semi-integrator. Frequency responses of the original system (28) and its approximations obtained by substitution of various discrete approximations of semi-integrator of the fifth order are compared in Fig. 12. The direct optimal method proposed in the current work produces optimal approximations to fractional differintegrators, not to arbitrary fractional order systems. Criterion (19) is, however, a good measure of discrepancy between any system and its approximation. Values of this criterion calculated for different approximations are given in the caption of Fig. 12. Step responses are compared in Figs. 13 and 14.

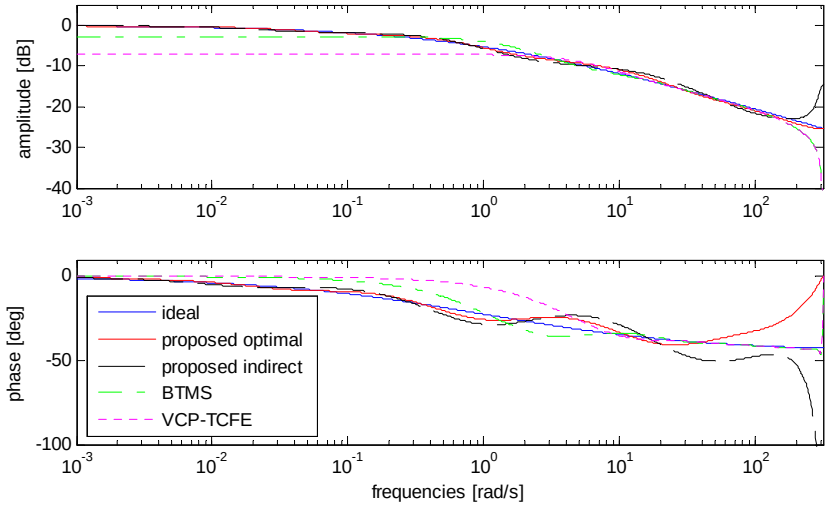

Fig. 12. Comparison of the frequency responses of the fractional system in Example 2 to responses of its various discrete approximations. The corresponding values of optimality criterion (19) are: $J_\beta = 6.47\cdot10^{-4}$ (optimal), $J_\beta = 3.15\cdot10^{-4}$ (indirect), $J_\beta = 5.65\cdot10^{-2}$ (BTMS), $J_\beta = 3.17\cdot10^{-2}$ (VCP-TCFE).

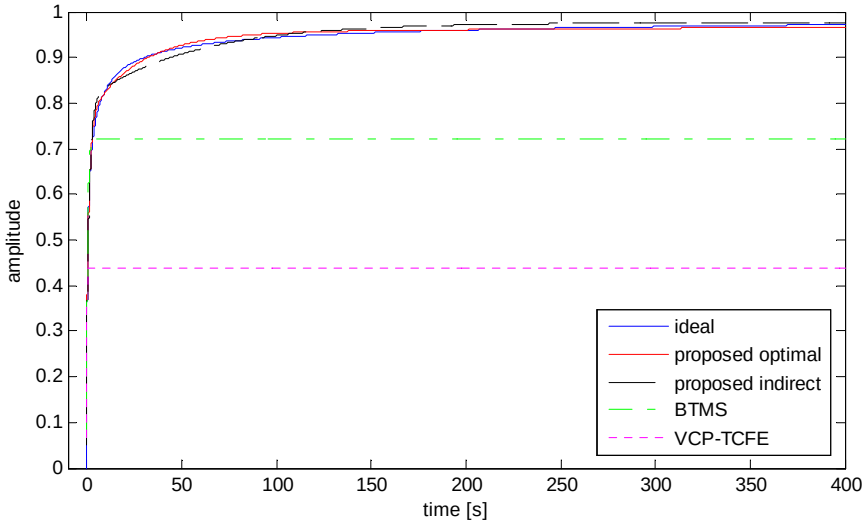

Fig. 13. Comparison of the step responses of the fractional system in Example 2 to responses of its various discrete approximations of the 5th order.

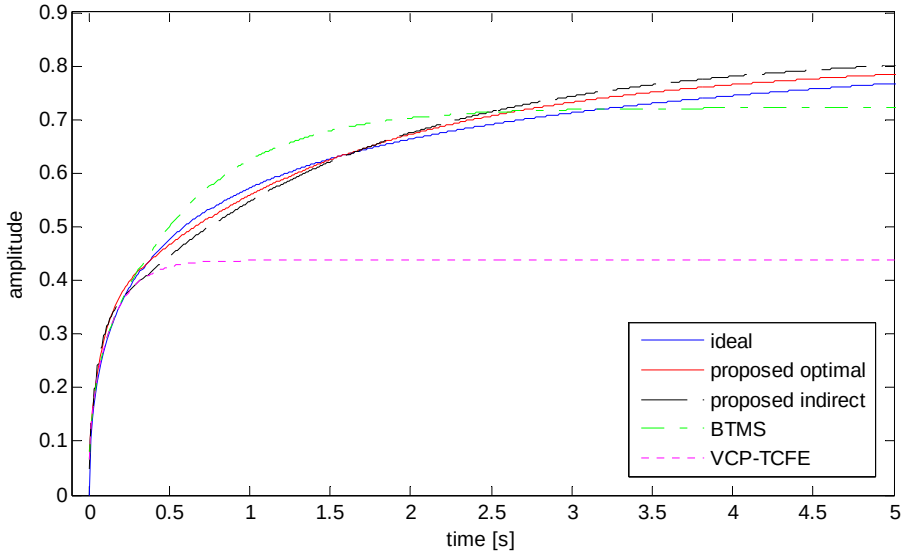

Fig. 14. Comparison of the initial behavior of the step response of the system in Example 2 to responses of its various discrete approximations. The quality of the responses can be numerically compared by using time-domain criterion (27): $I = 1.33 \cdot 10^{-4}$ (direct optimal), $I = 4.36 \cdot 10^{-4}$ (indirect), $I = 2.1 \cdot 10^{-1}$ (BTMS), $I = 1.04$ (VCP-TCFE)

Fig. 13 is particularly important because it clearly demonstrates the consequence of the different low-frequency behavior of different approximations. Contrary to other presented methods, both novel discretizations, direct optimal and indirect, preserve the steady state behavior of the original system. In fact, any rational approximation of the semi-integrator with a bounded gain at zero frequency will fail to preserve steady state behavior of the target system. It is interesting to note that it is possible to fix the low-frequency behavior of the approximations proposed by other authors. In particular, one would obtain the approximation to a fractional integrator s^{-r} by approximating s^{1-r} and then multiplying the resulting approximation by the transfer function of the discrete integrator (Tustin's approximation, for example). However, the obtained approximations would still, in general, fail in closely replicating the fractional order dynamics of the target system in the wider range of frequencies. This is illustrated in Fig. 15 where the 5th order approximations proposed in the current work are compared with the corresponding "corrected" approximations proposed by other authors.

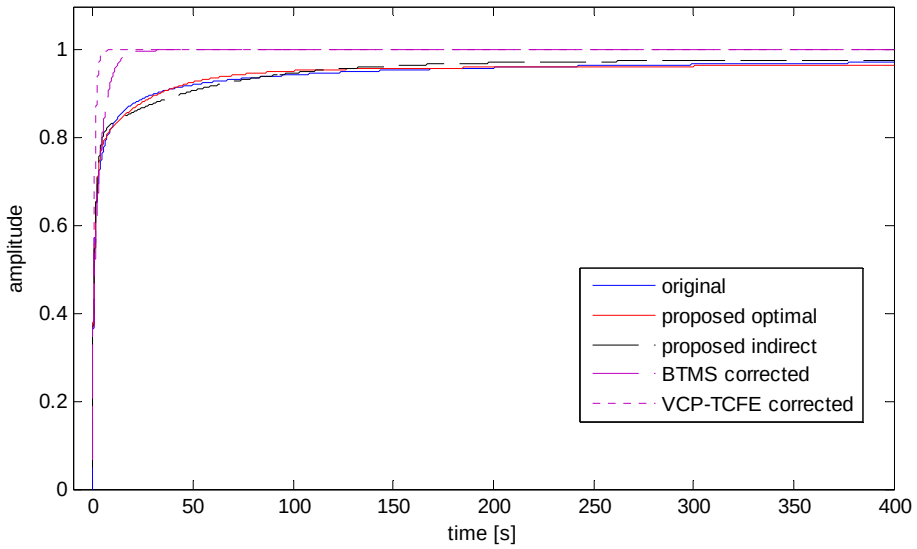

Fig. 15. Comparison of the step response of the fractional system of Example 2 with the step responses of its various approximations. The figure illustrates performance of the approximations proposed by some other authors that incorporate corrected low-frequency behavior. Compare with Fig. 13.

Example 3. Let us calculate semi-integral of the causal cosine function

$$c(t) = \cos(t)h(t) \tag{30}$$

or, in other words, let us calculate the response of the semi-integrator of the cosine function. Approximations of the seventh order were used. The same example was addressed in [35]. The results presented in Fig. 16 are well expected, since they could be deduced by careful examination of the frequency responses presented in Fig 7.

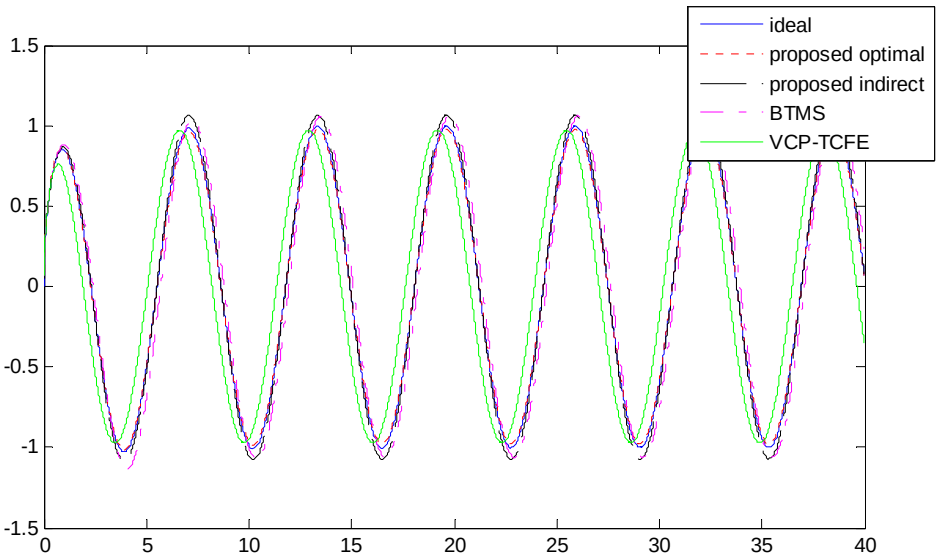

Fig. 16. Comparison of the responses of the semi-integrator and its various approximations to the causal cosine function. The quality of the responses can be numerically compared by using time domain criterion (27): $I = 9.97 \cdot 10^{-5}$ (direct optimal), $I = 9.22 \cdot 10^{-4}$ (indirect), $I = 5.7 \cdot 10^{-3}$ (BTMS), $I = 3.6 \cdot 10^{-2}$ (VCP-TCFE).

Example 4. Consider a fractional oscillator

$$G_2(s) = \frac{1}{s - \sqrt{2}s + 1} = \frac{s^{-1}}{1 - \sqrt{2}s^{-0.5} + s^{-1}}. \tag{31}$$

The system will be approximated by a rational discrete transfer function

$$\tilde{G}_2(z) = \frac{H_T(z)}{1 + \sqrt{2}H(z) + H_T(z)}, \tag{32}$$

with H_T being the Tustin discrete integration operator

$$H_T(z) = \frac{T}{2}\frac{z+1}{z-1}. \tag{33}$$

As before, H is some discrete approximation of the semi-integrator. This system is extremely difficult to approximate. A shorter sampling time T=2ms is therefore selected. Both optimal and indirect approximations were constructed over a narrower frequency range $\omega \in (0.05, 20)$. The comparison of the frequency response of the original system (31) and various approximations (32) with different selections of H is presented in Fig. 17. It is clearly noticeable from the given figure that, contrary to approximations reported by Barbosa et al [35] and Vinagre et al. [30], distinct frequency domain behavior of the oscillator was correctly replicated by the approximations presented in this text.

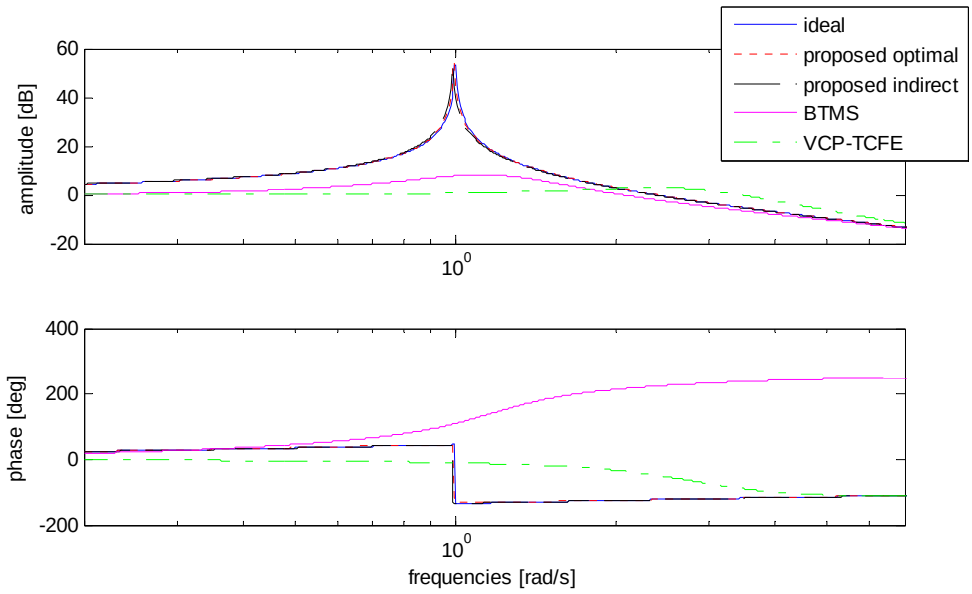

Fig. 15. Comparison of the frequency response of the fractional oscillator (31) to responses of its various discrete approximations of type (32). The corresponding values of criterion (19) are: $J_\beta = 5.72 \cdot 10^{-4}$ (optimal), $J_\beta = 1.76 \cdot 10^{-3}$ (indirect), $J_\beta = 5.23 \cdot 10^{-1}$ (BTMS), $J_\beta = 1.27 \cdot 10^{-1}$ (VCP-TCFE).

Step response of the original system was compared to the step responses of various approximations in Fig. 18. Application of the approximation presented in [35] (BTMS) results in a highly unstable system.

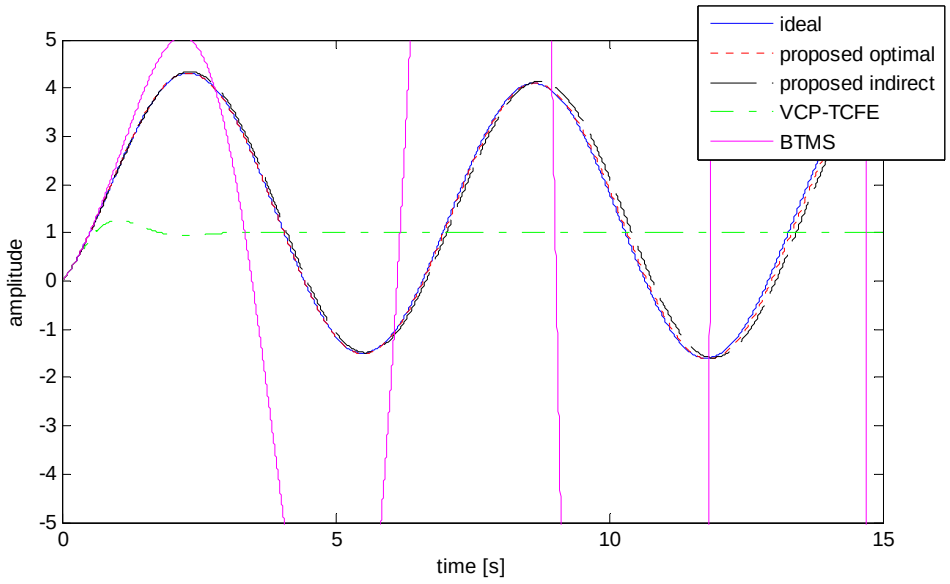

Fig. 16. Comparison of the step response of the fractional oscillator (31) to step responses of its various discrete approximations of type (32).

Due to locations of the poles of system (31), one must be extra careful when performing substitutions of various semi-integrator approximations in (32). Small numerical errors may very well flip the poles of the resulting approximation to the right-hand side of the complex plane, leading to unstable behavior.

2. 6 Conclusions

Two novel methods for discretization of fractional differintegrators have been presented: direct optimal and indirect. Both methods produce approximations of fractional differintegrators, which are then used to create approximations to more complex fractional order systems. The methods are highly flexible and can be used for construction of both general purpose and application-specific, specialized approximations. Special care has been taken to preserving steady state behavior of the approximated systems. In particular, for a fractional integrator of order $0<\alpha<1$ all proposed discrete rational approximations have a pole at 1, thus ensuring infinite static gain.

The effectiveness of the direct optimal approximation originates from the flexibility of the optimality criterion (19). Proper selection of the approximation order, parameter β, the relevant frequency range (Ω) and weigh factors θ_i can result in approximations having desired properties and tradeoffs. Choice of suitable, robust optimizer is not trivial. Utilization of the PSO procedure has resulted in good performance at the cost of increased computational load. Flexibility of the indirect discretization method is due to the flexibility of the ORA itself, as indeed it offers the possibility of using any desirable rational continuous-time approximation. Utilization of discretization polynomials ensures that the desired features of continuous-time approximation are properly translated into discrete domain.

The examples presented in Section 5 clearly demonstrate the effectiveness of the proposed methods. Note in particular the values of the frequency-domain (19) and time-domain (27) discrepancy measures of different approximations. In the current text the primary fractional order system under consideration was the semi-integrator. It is clear, however, that the proposed procedures are fully applicable to fractional differintegrators of arbitrary order.

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Finite-Time Stability of Fractional Order Time-Delay Systems

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Abstract:- In this study, some basic results of the stability criteria of fractional order systems with time-delay as well as free-delay are presented. Particularly, they are obtained and presented sufficient conditions for finite-time stability for (non)linear (non)homogeneous as well as perturbed fractional order time-delay systems. Several stability criteria for this class of fractional order systems are proposed using a recently suggested generalized Gronwall inequality as well as "*classical"* Bellman-Gronwall inequality. Some conclusions for stability are similar to those of classical integer-order differential equations. Lastly, numerical examples are given to illustrate the validity of the proposed procedure.

Key-Words: stability, fractional calculus, time-delay, fractional order system, finite-time stability

3. 1 Introduction

The question of stability is of main interest in control theory. In addition, the problem of investigation of time-delay systems has been exploited over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.,[1]. Delays are inherent in many physical and engineering systems. In particular, pure delays are often used to ideally represent the effects of transmission, transportation, and inertial phenomena. This is because these systems have only limited time to receive information and react accordingly. Such a system cannot be described by purely differential equations, but has to be treated with differential difference equations or the so called differential equations with difference variables. Delay differential equations (DDEs) constitute basic mathematical models for real phenomena, for instance, in engineering, mechanics, and economics, [2]. The basic theory concerning the stability of systems described by equations of this type were developed by Pontryagin(1942), Chebotarev(1940) and Myshkis (1949). Later, Krasovskii extended the Lyapunov's theory to time-delay systems in 1956 as well as Razumikhin who proposed a method to avoid functional in Lyapunov stability analysis; for a more comprehensive historic overview, see [3]. Also, important works have been written by Bellman and Cooke in 1963, [4]. The presence of time-delays in a feedback control system leads to a closed-loop characteristic equation, which involves the exponential type transcendental terms. The exponential transcendentality brings infinitely many isolated roots, and hence it makes the stability analysis of time-delay systems a challenging task. It is well recognized that there is no simple and universally applicable practical algebraic criterion, like the Routh–Hurwitz criterion for stability of delay-free systems, for assessing the stability of linear time-invariant time delayed (LTI-TD) systems. On the other side, the existence of pure time-delay, regardless of its presence in the control or/and state, may cause undesirable system transient response, or generally, even an instability. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov`s second method, or on using the idea of matrix measure,[5-8]. The analysis of time-delay systems can be classified such that the stability or stabilization criteria involve the delay element or not. In other words, delay independent criteria guarantee global asymptotic stability for any time-delay that may change from zero to infinity. As there is no upper limit to time-delay, often delay independent results can be regarded as conservative in practice, where unbounded time-delays are not so realistic. In practice, one is not only interested in system stability (e.g. in the Lyapunov sense), but also in bounds of system trajectories. A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space that are defined *a priori* in a given problem. Besides, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval. These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these

definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. Thus, the analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite-time or practical stability control is concerned. Motivated by "brief discussion" on practical stability in the monograph of LaSalle and Lefschet,[9] and Weiss and Infante,[10] have introduced various notations of stability over finite-time interval for continuous-time systems and constant set trajectory bounds. A more general type of stability ("practical stability with settling time", practical exponential stability, etc.) which includes many previous definitions of finite-stability was introduced and considered by Grujić,[11,12]. The concept of finite-time stability, called "final stability", was introduced by Lashirer and Story, [13] and further development of these results was due to Lam and Weiss,[14]. Recently, finite-time control/stabilization, and methods for stability evaluation of linear systems on finite-time horizont are proposed by Amato *et al.,* [15,16], respectively. Also, the analysis of linear time-delay systems in the context of finite and practical stability was introduced and considered in [17-19] as well as finite-time stability and stabilization [20].

Recently, there have been some advances in control theory of fractional (non-integer order) dynamical systems for stability questions such as robust stability, bounded input–bounded output stability, internal stability, finite-time stability, practical stability, root-locus, robust controllability, robust observability, etc. For example, regarding linear fractional differential systems of finite dimensions in state-space form, both internal and external stabilities are investigated by Matignon,[21]. Some properties and (robust) stability results for linear, continuous, (uncertain) fractional order state-space systems are presented and discussed,[21,22]. The frequency investigation method and utilization of the Nyquist frequency characteristics based on argument principle were described in paper [23]. However, we cannot directly use the algebraic tools as, for example, Routh-Hurwitz criteria for the fractional-order system, because we do not have a characteristic polynomial but pseudo-polynomial with rational power. It is possible in some special cases only,[24]. In paper [25], an effective numerical algorithm is proposed, based on Rouche's theorem for determining the location of poles, and zeros on the first Riemann sheet is presented. Buslowicz [26] considered the stability problem of LTI continuous-time systems of fractional commensurate order, where the new frequency domain methods based on the Mikhailov stability criterion for stability analysis are presented.

An analytical approach was suggested by Chen and Moore,[27], who considered the analytical stability bound of a class of fractional-delay using Lambert function *W*. Further, the analysis and stabilization of fractional (exponential) delay systems of retarded/neutral type are considered [28,29], as well as the BIBO stability [30]. Moreover, the analysis of robust BIBO-stability of LTI fractional order delay systems of retarded and neutral types, in the presence of real parametric uncertainties, are presented in [31].

Further, Hwang and Cheng, [32] proposed a numerical algorithm which uses methods that are based on the Cauchy integral theorem for testing the BIBO stability of a wide class of fractional-order delay systems with irrational and/or transcendental characteristic equations. Moreover, in paper [33], they presented Linear Matrix Inequality (LMI) stability conditions for fractional-order systems, where one may use the advantage of the LMI methods in control theory due to their connection with the Lyapunov method. Recently, in paper [34], it is studied the stability of fractional-order nonlinear time-delay systems for Caputo's derivative and they extended the Lyapunov- Krasovskii theorem for the fractional nonlinear systems. Also, the Razumikhin theorem for the fractional nonlinear time-delay systems for Riemann-Liouville and Caputo derivatives was extended in [35] because the Razumikhin stability theory is more widely used to prove the stability of time- delay systems, since the construction of Lyapunov-Krasovskii functional is more difficult than that of the Lyapunov-Razumikhin function. Also, in [36] the authors proposed and proved the Mittag-Leffler stability theorem in the presence of both the Riemann-Liouville or the Caputo fractional derivatives and delay. The obtained theorems contain particular cases of the fractional calculus versions as well as the time-delay ones.

While Lyapunov methods have been developed for stability analysis and control law synthesis of integer linear systems and have been extended to stability of fractional systems, only few studies deal with non-Lyapunov stability of fractional systems. Recently, for the first time, finite-time stability analysis of fractional time delay systems is presented and reported on papers [37-39]. Further, the papers [40- 42] extends some basic results from the area of finite-time and practical stability to nonlinear, perturbed, fractional order time-delay systems, where a robust stability test procedure is proposed in the presence of real parametric uncertainties.

Here, a Bellman-Gronwall`s approach is proposed, using "*classical"* Bellman-Gronwall inequality (see Appendix C.1) as well as a recently obtained generalized Gronwall inequality, reported in [43]. The problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of (non)linear (non)autonomous fractional order time-delay systems has been examined.

3. 2 Preliminaries On Integer Time-Delay Systems

A linear, multivariable time-delay system can be represented by differential equation:

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-\tau) \tag{1}$$

and with associated function of initial state:

$$x(t) = \psi_x(t), \qquad -\tau \le t \le 0, \tag{2}$$

Equation (1) is referred to as homogenous state equation. Also, a more general linear, multivariable time-delay system can be represented by the following differential equation:

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t) + B_1 u(t-\tau), \tag{3}$$

and with associated function of initial state and control:

$$\begin{aligned} x(t) &= \psi_x(t), \qquad -\tau \le t \le 0, \\ u(t) &= \psi_u(t), \end{aligned} \tag{4}$$

Equation (3) is referred to as nonhomogenous or the unforced state equation, $x(t)$ is state vector, $u(t)$ control vector, A_0, A_1, B_0 and B_1 are constant system matrices of appropriate dimensions, and τ is pure time delay, τ =*const.* (τ >0). Moreover, here, it is considered a class of non-linear system with time delay described by the state-space equation:

$$\frac{d\boldsymbol{x}(t)}{dt} = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-\tau) + B_0 \boldsymbol{u}(t) + B_1 \boldsymbol{u}(t-\tau) + \sum_{i=1}^{n} f_i\left(\boldsymbol{x}(t)\right) + \sum_{j=1}^{m} g_j\left(\boldsymbol{x}(t-\tau)\right), \tag{5}$$

with the initial functions (4) of the system. Vector functions $f_i, g_j,\ i = 1,n, j = 1,m$ present nonlinear parameter perturbations of the system in respect to $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t-\tau)$ respectively. Also, the next assumption is introduced that:

$$\begin{aligned} &\left\| f_i(\boldsymbol{x}(t)) \right\| \le c_i \left\| \boldsymbol{x}(t) \right\|,\ i = 1,n \quad t \in \left[0, \infty\right) \\ &\left\| g_j(\boldsymbol{x}(t-\tau)) \right\| \le c_j \left\| \boldsymbol{x}(t-\tau) \right\|,\ j = 1,m,\ t \in \left[0, \infty\right) \end{aligned} \tag{6}$$

where $c_i, c_j \in R^+$ are known real positive numbers. Moreover, a linear multivariable time-varying delay system can be represented by differential equation

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-\tau(t)) + B_0 u(t), \tag{7}$$

and with associated function of initial state

$$x(t) = \psi_x(t), \qquad -\tau_M \le t \le 0. \tag{8}$$

where $\tau(t)$ is an unknown time–varying parameter which satisfies

$$0 \le \tau(t) \le \tau_M, \quad \forall t \in J,\ J = \left[t_o, t_o + T\right], J \subset R \tag{9}$$

Moreover, here, it is considered a class of perturbed non-linear system with time delay described by the state space equation

$$\frac{dx(t)}{dt} = \left(A_0 + \Delta A_0\right) x(t) + \left(A_1 + \Delta A_1\right) x(t-\tau(t)) + B_0 u(t) + f_0\left(x(t), x(t-\tau(t))\right), \tag{10}$$

with the given initial functions of the system and vector function f_0. Vector function f_0 presents nonlinear parameter perturbations of the system in respect to $x(t)$ and $x(t-\tau(t))$, respectively, and matrices $\Delta A_0, \Delta A_1$ present perturbations of the system, too. Also, it is assumed that the next assumption is true.

$$\left\| f_0(x(t), x(t-\tau(t))) \right\| \le c_0 \left\| x(t) \right\| + c_1 \left\| x(t-\tau(t)) \right\|,\ t \in \left[0, \infty\right), \tag{11}$$

where $c_0, c_1 \in R^+$ are known real positive numbers. Dynamical behavior of the system (1),(3) or (5) with initial functions (2),or (4) is defined over time interval $J = \{t_o, t_o + T\}$, where quantity T may be either a positive real number or symbol $+\infty$, so finite-time stability and practical stability can be treated simultaneously. It is obvious that $J \in R$. Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded. Let index $''\varepsilon''$ stand for the set of all allowable states of the system and index $''\delta''$ for the set of all initial states of the system, such that the set $S_\delta \subseteq S_\varepsilon$. In general, one can write:

$$S_\rho = \left\{ \boldsymbol{x} : \|\boldsymbol{x}(t)\|_Q^2 < \rho \right\}, \ \rho \in [\delta, \varepsilon], \tag{12}$$

where Q will be assumed to be symmetric, positive definite, real matrix. S_{α_u} denotes a set of all allowable control actions. Let $|\boldsymbol{x}|_{(.)}$ be any vector norm (e.g., $. = 1, 2, \infty$) and $\|(.)\|$ the matrix norm induced by this vector. Matrix measure has been widely used in the literature when dealing with stability of time-delay systems. The matrix measure μ for any matrix $A \in C^{n \times n}$ is defined as follows:

$$\mu(A) = \lim_{\omega \to 0} \frac{\|I + \omega A\| - 1}{\omega} \tag{13}$$

The matrix measure defined in (8) can be subdefined in three different ways, depending on the norm utilized in its definitions,[44].

$$\mu_1(A) = \max_k \left(\mathrm{Re}(a_{kk}) + \sum_{\substack{i=1 \\ i \neq k}}^{n} |a_{ik}| \right), \tag{14}$$

$$\mu_2(A) = \frac{1}{2} \max_i \lambda_i \left(A^* + A \right), \quad \text{if } A \text{ is real matrix, then } \mu_2(A) = \mathrm{Re}(\lambda_{\max}(A)) \tag{15}$$

and

$$\mu_\infty(A) = \max_i \left(\mathrm{Re}(a_{ii}) + \sum_{\substack{k=1 \\ i \neq k}}^{n} |a_{ki}| \right) \tag{16}$$

Expression (4) can be written in its general form as:

$$\begin{aligned} \boldsymbol{x}(t_o + \theta) &= \psi_x(\theta), \quad -\tau \le \theta \le 0, \ \psi_x(\theta) \in C[-\tau, 0] \\ \boldsymbol{u}(t_o + \theta) &= \psi_u(\theta), \quad -\tau \le \theta \le 0, \ \psi_u(\theta) \in C[-\tau, 0] \end{aligned}, \tag{17}$$

where t_o is the initial time of observation of the system (1) and $C[\text{-}\tau, 0]$ is a Banach space of continuous functions over a time interval of length τ, mapping the interval $[t - \tau, t]$ into R^n with the norm defined in the following manner:

$$\|\psi\|_C = \max_{-\tau \le \theta \le 0} \|\psi(\theta)\|, \tag{18}$$

It is assumed that the usual smoothness condition is present, so there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data.

3.2.1 Some previous results related to integer time-delay systems

The existing methods developed so far for stability check are mainly for integer-order systems.

Definition 1. *The system given by (3) with* $\boldsymbol{u}(t-\tau)\equiv 0,\ \forall t$ *, satisfying the initial condition (4) is finite stable w.r.t* $\{\varsigma(t),\varepsilon,\alpha_u,\tau,J,\mu(A_0)\neq 0\}$, *if and only if:*

$$\psi_x \in S_\delta,\ \forall t\in[-\tau,0] \tag{19}$$

and

$$\boldsymbol{u}(t)\in S_{\alpha_u},\ \forall t\in J \tag{20}$$

imply:

$$\boldsymbol{x}(t;t_0,\boldsymbol{x_0})\in S_\varepsilon,\quad \forall t\in[0,T] \tag{21}$$

Illustration of the preceding definition is pictured in Fig. 1.

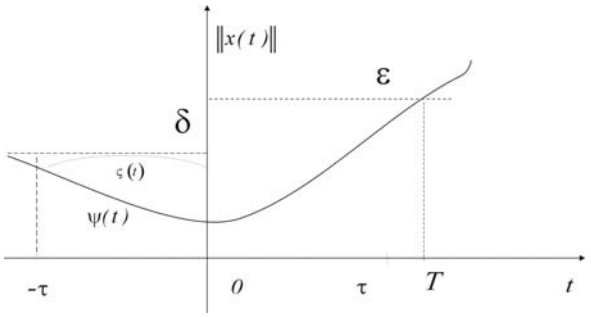

Fig.1 Finite-time stability concept illustration

Definition 2. *The system given by (3) satisfying the initial condition (4) is finite stable w.r.t* $\{\delta,\varepsilon,\alpha_\psi,\alpha_u,\tau,J,\mu(A_0)\neq 0\}$, *if and only if:*

$$\psi_x \in S_\delta,\ \forall t\in[-\tau,0] \tag{22}$$

$$\psi_u \in S_{\alpha_0},\ \forall t\in[-\tau,0] \tag{23}$$

and

$$\boldsymbol{u}(t)\in S_{\alpha_u},\ \forall t\in J \tag{24}$$

imply:

$$\boldsymbol{x}(t;t_0,\boldsymbol{x_0},\boldsymbol{u}(t))\in S_\varepsilon,\quad \forall t\in J \tag{25}$$

Theorem 1. *The system given by (3), with the initial function (4) is finite-time stable w.r.t* $\{\delta,\varepsilon,\alpha_\psi,\alpha_u,\tau,J,\mu(A_0)\neq 0\}$, *if the following condition is satisfied*,[43]:

$$\mu_2^{-1}(A_0)e^{\mu_2(A_0)t} < (\varepsilon/\delta)\sigma^{-1} \tag{26}$$

where:

$$\sigma = a_1\left(\mu_2(A_0)a_1^{-1}+\left(1-e^{-\mu_2(A_0)\tau}\right)c_1+\left(1-e^{-\mu_2(A_0)t}\right)c_2\right) \tag{27}$$

$$c_2=\gamma(b_0+b_1),c_1=1+b_1(\gamma+\gamma_\psi) \tag{28}$$

$$\gamma=\alpha_u/\varepsilon,\ \gamma_\psi=\alpha_\psi/\varepsilon,a_1=\|A_1\|,\ b_1=\|B_1\|/a_1,\ b_0=\|B_0\|/a_1 \tag{29}$$

Results that will be presented in the sequel enable one to check finite-time stability of the nonautonomous system to be considered (1),(3) or (5) and (2),(4) without finding the fundamental matrix or corresponding matrix measure.

Definition 3. *The system given by (3) satisfying the initial condition (4) is finite stable w.r.t* $\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\}$, $\delta<\varepsilon$ *if and only if:*

$$\|\psi_x\|_C<\delta,\ \|\psi_u\|_C<\alpha_0, \tag{30}$$

$$\|\boldsymbol{u}(t)\| < \alpha_u \quad , \qquad \forall t \in J \tag{31}$$

imply:

$$\|\boldsymbol{x}(t)\| < \varepsilon \quad , \qquad \forall t \in J \tag{32}$$

Theorem 2. *The nonautonomous system given by (3) satisfying the initial condition (5) is finite-time stable w.r.t.* $\{\delta, \varepsilon, \alpha_u, \alpha_0, t_o, J, \}$, $\delta < \varepsilon$, *if the following condition is satisfied,*[44]:

$$\left[1 + \sigma_{\max}^{A}\left(t - t_0\right)\right] \cdot e^{\sigma_{\max}^{A}(t-t_0)} + \gamma_1^*(t - t_0) + \gamma_0^* \tau \le \varepsilon / \delta, \qquad \forall t \in J. \tag{33}$$

where

$$\gamma_1^* = \gamma_1 / \delta,\ \gamma_0^* = \gamma_0 / \delta,\ \gamma_1 = \left(b_0 + b_1\right)\alpha_u,\ \gamma_0 = \left(\alpha_0 - \alpha_u\right)b_1, \tag{34}$$

3. 3 Preliminaries on Stability of Fractional Order Systems Including Time-Delays

In the field of fractional-order control systems, there are many challenging and unsolved problems related to stability theory such as robust stability, bounded input – bounded output stability, internal stability, root-locus, robust controllability, robust observability, etc. In engineering, the fractional order α is often less than 1, so we restrict $\alpha \in \left(0,1\right)$ as usual. Even if $\alpha > 1$, we can translate the fractional systems into systems with the same fractional order which lies in $\left(0,1\right)$ provided some suitable conditions are satisfied. Also, a fractional-order system can be described by a fractional differential equation of the form

$$\begin{aligned} a_n D^{\alpha_n} y\left(t\right) + a_{n-1} D^{\alpha_{n-1}} y\left(t\right) + \ldots + a_0 D^{\alpha_0} y\left(t\right) = \\ = b_m D^{\beta_m} u\left(t\right) + b_{m-1} D^{\beta_{m-1}} u\left(t\right) + \ldots + b_0 D^{\beta_0} u\left(t\right) \end{aligned} \tag{35}$$

where $D^{\delta} \equiv D_{0,t}^{\delta}$ denotes the Grunwald-Letnikov, the Riemann-Liouville or Caputo's fractional derivative [47] The corresponding transfer function of *incommensurate* real orders has the following form:

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}} = \frac{P\left(s^{\beta_k}\right)}{Q\left(s^{\alpha_k}\right)} \tag{36}$$

where a_k, $\left(k = 0,1,2,\ldots,n\right)$, b_k $(k = 0,1,2,\ldots,m)$ are constants, α_k, $\left(k = 0,1,2,\ldots,n\right)$, β_k $(k = 0,1,2,\ldots,m)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\alpha_n > \alpha_{n-1} > \ldots > \alpha_0$ and $\beta_m > \beta_{m-1} > \ldots > \beta_0$. The fractional-order linear time-invariant (LTI) system can also be represented by the following state-space model,[22]

$$\begin{aligned} D_{0,t}^{q} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{37}$$

where $x \in R^n, u \in R^r$ and $y \in R^p$ are the state, input and output vectors of the system and $A \in R^{n \times n}, B \in R^{n \times r}, C \in R^{p \times n}$, and $q = \left[q_1, q_2, \ldots, q_n\right]^T$ are fractional orders. If $q_1 = q_2 = \ldots = q_n = \alpha$, the system (37) is called a commensurate-order system, otherwise it is an incommensurate-order system.

To demonstrate the advantage of fractional calculus in characterizing the system behavior, the stability properties, here, let us consider the following illustrative example, [52].

Compare the following two systems with the initial condition $x(0)$ for $0 < \alpha < 1$:

$$\frac{d}{dt} x(t) = \nu t^{\nu - 1}, \qquad {}_C D_{0,t}^{\alpha} x(t) = \nu t^{\nu - 1},\ 0 < \alpha < 1. \tag{38}$$

The analytical solutions of the previous systems are $t^{\nu}+x(0)$ and $\frac{\nu\Gamma(\nu)t^{\nu+\alpha-1}}{\Gamma(\nu+\alpha)}+x(0)$, respectively. One can conclude, the integer-order system is unstable for any $\nu\in(0,1)$. However, the second given fractional dynamic system is stable as $0<\nu<\alpha-1$, which implies that fractional-order system can have additional attractive feature over the integer-order system. In 1996, Matignon [21] studied the following fractional differential system involving the Caputo derivative

$$ {}_{C}D_{0,t}^{\alpha}=\frac{d^{\alpha}x}{dt^{\alpha}}=Ax(t),\quad x(0)=x_0,\ \alpha\in(0,1) \tag{39}$$

where $x=(x_1,x_2,...,x_n)^T$ with initial value $x_0=(x_{10},x_{20},...,x_{n0})^T$, $A\in R^{n\times n}$.The stability of equilibrium of the system (39) was first defined and established by Matignon as follows.

Definition 4. *The autonomous fractional order system (39) is said to be*

(a) *stable if for any* x_0, *there exists* $\varepsilon>0$ *such that*

$$\|x\|\le\varepsilon \quad \text{for} \quad t\ge 0 \tag{40}$$

is asymptotically stable if $\lim_{t\to\infty}\|x(t)\|=0$ (41)

Also, Matignon [21] proposed the definition of the BIBO stability for the fractional differential system.

Definition 5. *An input/output linear fractional system (42)*

$$\begin{aligned}&\frac{d^{\alpha}x}{dt^{\alpha}}=Ax+Bu,\ x(0)=x_0\\&y=Cx\end{aligned} \tag{42}$$

$x\in R^n, y\in R^p$ is *externally stable* or *bounded-input bounded-output (BIBO) if* $\forall u\in L^{\infty}(R^+,R^m),\ y=h*u\in L^{\infty}(R^+,R^p)$ *which is equivalent to*: $h\in L^1(R^+,R^{p\times m})$.

Also, in [49] the authors give two definitions of the stability for differential systems with the Caputo derivative and Riemann-Liouville derivative, respectively. Besides, the asymptotical stability of higher-dimensional linear fractional differential systems with the Riemann-Liouville fractional order and Caputo fractional order were studied, where the asymptotical stability theorems were also derived.

Definition 6. *The zero solution of the following differential system with the α-th order Caputo derivative in which 0 < α < 1*

$$ {}_{C}D_{0,t}^{\alpha}X=AX \tag{43}$$

is said to be:

(*i*) *Stable, if* $\forall\varepsilon>0,\exists\delta>0$, *when* $\|X_0\|\le\delta$, *the solution* $X(t)$ *to (43) with the initial condition*

$$X(t)=X_0 \quad \text{satisfies} \quad \|X(t)\|\le\varepsilon \text{ for any } t\ge 0. \tag{44}$$

(*ii*) *Asymptotically stable, if the zero solution to (43) is stable, and it is locally attractive, i.e., there exists a* δ_0 *such that* $\|X_0\|\le\delta_0$ *implies that*

$$\lim_{t\to+\infty}\|X(t)\|=0 \tag{45}$$

Definition 7. *The zero solution of the following differential system with the α-th order Riemann- Liouville derivative in which* $0<\alpha<1$

$$ {}_{RL}D_{0,t}^{\alpha}X=AX \tag{46}$$

is said to be:

(*i*) Stable, if $\forall\varepsilon>0,\exists\delta>0$, when $\|X_0\|\le\delta$, the solution $X(t)$ to (46) with the initial condition $\left[{}_{RL}D_{0,t}^{\alpha-1}X(t)\right]_{t=0}=X_0$ satisfies

$$\|X(t)\|\le\varepsilon \text{ for any } t\ge 0. \tag{47}$$

(*ii) Asymptotically stable, if the zero solution to (46) is stable, and it is locally attractive, i.e., there exists a* δ_0 *such that* $\|X_0\| \le \delta_0$ *implies that*

$$\lim_{t\to+\infty} \|X(t)\| = 0 \tag{48}$$

Next, one may study the stability of fractional differential systems in two spatial dimensions, and then study the fractional differential systems with higher dimensions. Now, it is studied the fractional differential system with the Caputo derivative,

$${}_{*}D_{0,t}^{\alpha} X = AX, \ \ \alpha \in (0,1), \ \ A \in R^{n\times n} \tag{49}$$

where fractional derivative ${}_{*}D_{0,t}^{\alpha}(..) = {}_{C}D_{0,t}^{\alpha}(..)$ *or* ${}_{RL}D_{0,t}^{\alpha}(..)$. They studied the fractional differential system with the Caputo derivative, as follows:

$${}^{C}D_{0,t}^{\alpha} X = AX, \quad \alpha \in (0,1), \ \ A \in R^{n\times n} \tag{50}$$

Theorem 3. *If the real parts of all the eigenvalues of* ***A*** *are negative, then the zero solution to the system (50) is asymptotically stable.*

Also, for fractional differential system with the Riemann-Liouville derivative

$${}^{RL}D_{0,t}^{\alpha} X = AX, \quad \alpha \in (0,1), \ \ A \in R^{n\times n} \tag{51}$$

they stated the following theorem.

Theorem 4. *If the real parts of all the eigenvalues of* ***A*** *are negative, then the zero solution to the system (51) is asymptotically stable.*

A fractional-order linear time invariant system can be represented in the following pseudostate space form:

$$\begin{aligned} \frac{d^{\alpha}x(t)}{dt^{\alpha}} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{52}$$

where the notation d^{α}/dt^{α} indicates the Caputo fractional derivative of fractional commensurate order α, $x \in R^n, u \in R^m$ and $y \in R^p$ are pseudo-state, input, and output vectors of the system, respectively, and $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$. It is worth mentioning that the state-space form Eq. (52) is a pseudo-representation because the knowledge of vector x at time $t = t_0$ and input vector $u(t)$ for $t \ge t_0$ are not entirely sufficient to know the behavior of the system (52) for $t > t_0$. A fractional-order model is in fact infinite dimensional, therefore its true state vector should be also infinite dimensional.

Theorem 5[21]. *The following autonomous system,*

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t), \ x(t_0) = x_0, \quad 0 < \alpha \le 1 \tag{53}$$

$x \in R^n$*, and A as an* $n \times n$ *matrix, is asymptotically stable if and only if* $|arg(\lambda)| > \alpha\pi/2$ *is satisfied for all eigenvalues* (λ) *of matrix A. In this case, each component of the states decays toward 0, such as* $t^{-\alpha}$*. Also, this system is stable if and only if* $|arg(\lambda)| > \alpha\pi/2$ *is satisfied for all eigenvalues* (λ) *of matrix A with those critical eigenvalues satisfying* $|arg(\lambda)| = \alpha\pi/2$ *that have geometric multiplicity of one.*

Demonstration of this theorem is based on the computation of state vector of the system $\|x(t)\| < Nt^{-\alpha}, t > 0, \alpha > 0.$ response to non-zero initial conditions. However, this result remains valid whatever the definition used, given that for a linear system without delay an autonomous system with non-zero initial conditions can be transformed into a non-autonomous system with null initial condition. Also, the stable and

unstable regions for $0 < \alpha \le 1$ are shown in Fig. 2 and they denote the stable and unstable regions for $0 < \alpha \le 1$ by C_α^- and C_α^+, respectively.

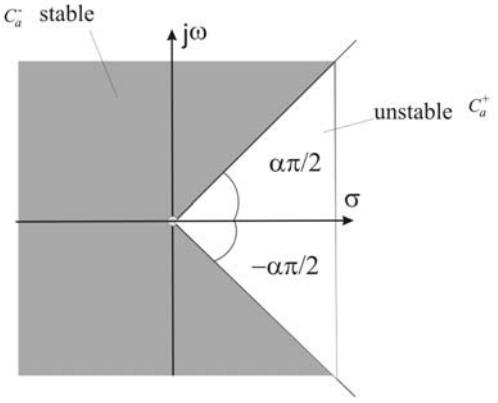

Fig. 2 *Stability region of fractional order linear time invariant system with order* $0 < \alpha \le 1$

For a minimal realization of (52), Matignon has also demonstrated the following theorem,[48].

Theorem 6. *In [48], consider a system given by the following linear pseudostate space form with inner dimension n*:

$$\begin{aligned} &\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad x(0) = x_0 \\ &y(t) = Cx(t) \end{aligned} \tag{54}$$

where $0 < \alpha \le 1$. *Also, assume that the triplet (A,B,C) is minimal. The system (54) is bounded-input bounded-output (BIBO) stable if and only if* $\left|\arg(\lambda) > \alpha\pi/2\right|$ *is satisfied for all eigenvalues* λ *of matrix A. When the system (88) is externally stable, each component of its impulse response behaves like* $t^{-1-\alpha}$ *at infinity.*

Exponential stability thus cannot be used to characterize asymptotic stability of fractional systems. A new definition is introduced.

Definition 8. $t^{-\gamma}$ *stability*

Trajectory $x(t) = 0$ of the system $d^\alpha x(t)/dt^\alpha = f(t, x(t))$ (unforced system) is $t^{-\gamma}$ asymptotically stable if the uniform asymptotic stability condition is met and if there is a positive real γ such that:

$$\forall \left\|x(t_0)\right\| \le c, \ \exists Q(x(t_o)) \ \text{ such that } \ \forall t \ge t_0, \ \left\|x(t)\right\| \le Qt^{-\gamma} \tag{55}$$

$t^{-\gamma}$ stability will thus be used to refer to the asymptotic stability of fractional systems. As the components of the state $x(t)$ slowly decay towards 0 following $t^{-\gamma}$, fractional systems are sometimes called *long memory systems*.

3.3.1 A review on stability of fractional order time delay system

Despite intensive researches, the stability of fractional order including time-delay systems remains an open problem. As for linear time invariant integer order systems, it is now well-known that stability of a linear fractional order system depends on the location of the system poles in the complex plane. Applying Matignon's stability theorem [21] one can check the system stability through the location in the complex plane of the dynamic matrix eigenvalues of the state space like system representation. Also, in paper [33], the authors presented Linear Matrix Inequality (LMI) stability conditions for fractional order systems, where one can use the advantage of the LMI methods in control theory due to their connection with the Lyapunov method. But, in the case of fractional order time-delay system the characteristic function of a fractional-delay system involves fractional-order powers and exponential elements. As we know, due to the presence of the exponential function $e^{-\tau s}$, this equation has an infinite number of roots, which makes the analytical stability analysis of a time-delay system extremely difficult. In the field of infinite-dimensional fractional-delay systems most studies are

concerned with the stability of a class of distributed systems, whose transfer functions involve $\sqrt{s}$ and/or $e^{-\sqrt{s}}$,[53]. Many examples of fractional differential systems with delay can be found in the literature. Simple examples such as $G(s) = \exp(-a\sqrt{s})/s,\ a > 0$ arise in the theory of transmission lines [54], or one can find in [55] fractional delay systems with transfer function linked to the heat equation, which leads to transfer functions $G(s)$ such as

$$G(s) = \frac{\cosh(x\sqrt{s})}{\sqrt{s}\sinh(\sqrt{s})},\ \left(0 \le x \le 1\right) \text{ or } G(s) = \frac{2e^{-a\sqrt{s}}}{b(1-e^{-2a\sqrt{s}})} \tag{56}$$

In the literature few theorems are available for stability testing of fractional-delay systems.For example, Hotzel [54] presented the stability conditions for fractional-delay systems with the characteristic equation $\left(as^{\alpha}+b\right)+\left(cs^{\alpha}+d\right)e^{-\rho s}=0$. Chen and Moore [27] analyzed the stability of a class of fractional-delay systems, whose characteristic function can be represented as the product of factors of the form $\left(as^{\alpha}+b\right)e^{cs}+d=0$ where the parameters a,b,c,d, and *r* are all real numbers. They considered the following delayed fractional equation

$$\frac{d^{q}y(t)}{dt^{q}} = K_{p}y(t-\tau) \tag{57}$$

where *q* and *Kp* are real numbers and $0 < q < 1$, time-delay τ is a positive constant and all the initial values are zeros. The stability condition is that for all possible *q*, *r* and *Kp*

$$\frac{q}{\tau}W\left(\frac{\tau}{r}\left(K_{p}\right)^{1/q}\right) \le 0 \tag{58}$$

where in the inequality,*W(.)* denotes the Lambert function such that $W(x)e^{W(x)} = x$. However, such a bound remains analytic and is difficult to use in practice. In paper [56], the application of the Lambert *W* function to the stability analysis of time-delay systems is re-examined through actually constructing the root distributions of the derived transcendental characteristic equation's (TCE) of some chosen orders. It is found that the rightmost root of the original TCE is not necessarily a principal branch of the Lambert W function solution, and that a derived TCE obtained by taking the *n*-th power of the original TCE introduces superfluous roots to the system. Further, Matignon's theorem has been used in [57] to investigate fractional differential systems with multiple delays stability.They discovered that if all roots of the characteristic equation have negative parts, then the equilibrium of the above linear system with fractional order is Lyapunov globally asymptotical stable if the equilibrium exists that is almost the same as that of classical differential equations. Namely, the following *n*-dimensional linear fractional differential system with multiple time delays:

$$\begin{aligned}
\frac{d^{q_1}x_1(t)}{d^{q_1}t} &= a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \ldots + a_{1n}x_n(t-\tau_{1n}),\\
\frac{d^{q_2}x_2(t)}{d^{q_2}t} &= a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \ldots + a_{2n}x_n(t-\tau_{2n}),\\
&\ldots\ldots\ldots\ldots\\
\frac{d^{q_n}x_n(t)}{d^{q_n}t} &= a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \ldots + a_{nn}x_n(t-\tau_{nn}),
\end{aligned} \tag{59}$$

where q^{i} is real and lies in (0,1), the initial values $x_i\left(t\right) = \phi_i(t)$ are given for $-\max_{i,j}\ \tau_{ij} = -\tau_{\max} \le t \le 0$ and $i = 1,2,\ldots,n$,

$$\Delta(s)=\begin{pmatrix} s^{q_1}-a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \dots & -a_{1n}e^{-s\tau_{1n}} \\ -a_{21}e^{-s\tau_{21}} & s^{q_2}-a_{22}e^{-s\tau_{22}} & \dots & -a_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \dots & s^{q_n}-a_{nn}e^{-s\tau_{nn}} \end{pmatrix} \tag{60}$$

where $\Delta(s)$ denotes a *characteristic matrix* of the system (59) and $det(\Delta(s))$ a *characteristic polynomial* of (60). The distribution of $det(\Delta(s))$'s eigenvalues totally determines the stability of the system (59).

Theorem 7. *If all the roots of the characteristic equation* $det(\Delta(s))=0$ *have negative real parts, then the zero solution of the system (59) is Lyapunov globally asymptotically stable. If n = 1, then (60) is reduced to the system studied in* [57].

For forced fractional-delay systems, it is usually required that BIBO (bounded-input bounded-output) stability holds, or equivalently, the characteristic function has roots with negative real parts only, while for unforced autonomous fractional delay systems, the stability usually means asymptotical stability in the sense of Lyapunov, namely, the characteristic function has roots with negative real parts only. Bonnet and Partington [28,29] analyzed the BIBO stability of fractional exponential delay systems which are of retarded or neutral type. Stability conditions can be expressed in terms of the location of the poles of the system. Also, they have handled the robust stabilization of fractional exponential delay systems of retarded type. However, all these contributions do not provide universally acceptable practical effective algebraic criteria or algorithms for testing the stability of a given general fractional delay system. Although the stability of the given general characteristic equation can be checked with the Nyquist criterion or the Mikhailov criterion, it becomes sufficiently difficult when a computer is used since one should find an angle of turn of the frequency response plot for an infinite variation of the frequency ω. A visual conclusion on stability with respect to the constructed part of the plot is not practically reliable, since, along with an infinite spiral, the delay generates loops whose number is infinite. As evidenced from the literature, the lack of universally acceptable algebraic algorithms for testing the stability of the characteristic equation has hindered the advance of control system design for fractional delay systems. This is particularly true in the case of designing fixed-structure fractional order controller, e.g., $PI^{\alpha}D^{\beta}$.

On the other side, Hwang and Cheng [58] proposed a numerical algorithm that uses methods based on the Cauchy integral theorem and suggested the modified complex integral, where the stability of a given fractional-delay system can be achieved by evaluating the proposed integral and comparing its value with zero. In paper [25], an effective numerical algorithm for determining the location of poles and zeros on the first Riemann sheet is presented. The proposed method is based on *Rouche's theorem* and can be applied to all multi-valued transfer functions defined on a Riemann surface with the finite number of Riemann sheets, where the origin is a branch point. Recently, in paper [34] the authors have studied the stability of fractional order nonlinear time-delay systems for Caputo's derivative and they extended the Lyapunov- Krasovskii theorem for the fractional nonlinear systems. Also, the Razumikhin theorem for the fractional nonlinear time-delay systems for Riemann-Liouville and Caputo derivatives was extended in [35] because the Razumikhin stability theory is more widely used to prove the stability of time-delay systems, since the construction of Lyapunov-Krasovskii functional is more difficult than that of the Lyapunov-Razumikhin function. Further, in [36] the authors proposed and proved the Mittag-Leffler stability theorem in the presence of both the Riemann-Liouville or the Caputo fractional derivatives and delay. The obtained theorems contain particular cases of the fractional calculus versions as well as the time-delay ones.

3. 4 Finite-Time Stability of Fractional Order Time-Delay Systems

All classical stability concepts, e.g., Lyapunov stability, asymptotic stability, bounded-input-bounded-output (BIBO) stability, deal with systems operating over an *infinite* interval of time. Finite-time stability is a concept that was first introduced in the 1950s and it deals with systems whose operation is limited to a *fixed finite* interval of time and requires *prescribed bounds* on system variables. Here, it is important to point out that finite time stability and Lyapunov asymptotic stability are independent concepts, i.e a system can be finite-time stable but not Lyapunov asymptotic stable, and vice versa.

Moreover, the boundedness properties of the system responses are very important from the engineering point of view. That is to say, enable system trajectories to stay within *a priori* given sets for the fractional order time-delay systems in state-space form, i.e. system stability from the non-Lyapunov point of view is considered.

From this fact and our best knowledge, we firstly introduced and defined finite-time stability for fractional order time-delay systems, [37-42]. The paper extends some basic results from the area of finite time and practical stability to (non)linear, continuous, fractional order time-delay systems given in the state-space form. Sufficient conditions of this kind of stability, for particular classes of fractional time-delay systems are derived.We also need the following definitions to analyze the case of fractional order systems with time-delay from non-Lyapunov point of view. First, we introduce [37] the fractional order homogenous system with time-delay in state-space

$$ {}_{*}D_{to,t}^{\alpha}x(t)=\frac{d^{\alpha}\boldsymbol{x}(t)}{dt^{\alpha}}=A_0\boldsymbol{x}(t)+A_1\boldsymbol{x}(t-\tau),\quad 0<\alpha<1, \tag{61} $$

with the associated function of initial state:

$$ \boldsymbol{x}(t)=\psi_x(t)\in C\left[-\tau,0\right],\quad -\tau\le t\le 0. \tag{62} $$

Also, for the case of multiple time delays the state of fractional order systems can be presented as:

$$ {}_{*}D_{to,t}^{\alpha}\boldsymbol{x}(t)=A_o\boldsymbol{x}(t)+\sum_{i=1}^{n}A_i\boldsymbol{x}(t-\tau_i),\quad 0\le\tau_1<\tau_2<\tau_3<\ldots<\tau_i<\ldots<\tau_n=\Delta, \tag{63} $$

and with the associated function of initial state:

$$ \boldsymbol{x}(t)=\psi_x(t),\qquad -\Delta\le t\le 0\,. \tag{64} $$

Here, ${}_{*}D_{to,t}^{\alpha}(.)$ denotes either the Caputo fractional derivative ${}_{C}D_{to,t}^{\alpha}(.)$ or the Riemann-Liouville fractional derivative ${}_{RL}D_{to,t}^{\alpha}(.)$. Also, Lorenzo and Hartley [59] considered variable prehistories of $x(t)$ in $t<0$, and its effects were taken into account in the fractional derivative in terms of the initialization function. Moreover, using the short memory principle [47] and taking into account the initial function (62) one can obtain correct initial function, where it is assumed that there is no difficulty with questions of continuity of solutions with respect to initial data (function).

Definition 9.[37] *The system given by (61), satisfying the initial condition (62) is finite stable w.r.t* $\{t_o,J,\delta,\varepsilon,\tau\}$, $\delta<\varepsilon$ *if and only if:*

$$ \left\|\psi_x\right\|_C<\delta, \tag{65} $$

implies:

$$ \left\|\mathbf{x}(t)\right\|<\varepsilon\ ,\qquad \forall t\in J, \tag{66} $$

Definition 10.[37] *The system given by (63),satisfying the initial condition (64) is finite stable w.r.t* $\{t_o,J,\delta,\varepsilon,\Delta\}$, $\delta<\varepsilon$ *if and only if*:

$$ \left\|\boldsymbol{\psi}_x\right\|_C<\delta,\quad \forall t\in J_{\Delta},\quad J_{\Delta}=\left[-\Delta,0\right]\in R, \tag{67} $$

implies:

$$ \left\|\mathbf{x}(t)\right\|<\varepsilon\ ,\qquad \forall t\in J, \tag{68} $$

Theorem 8.*(A)*[37] *The autonomous system given by (61) satisfying the initial condition (62) is finite-time stable w.r.t.* $\{\delta,\varepsilon,\tau,t_o,J,\}$, $\delta<\varepsilon$, *if the following condition is satisfied*:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}\leq\varepsilon/\delta,\qquad\forall t\in J. \tag{69}$$

where $\sigma_{max}(.)$ *is the largest singular value of matrix (.), namely:*

$$\sigma_{\max}^{A}=\sigma_{\max}\left(A_0\right)+\sigma_{\max}\left(A_1\right), \tag{70}$$

and $\Gamma(.)$ *is the Euler's gamma function.*

B) *The autonomous system given by (63) satisfying the initial condition (64) is finite-time stable w.r.t.* $\{\delta,\varepsilon,\Delta,t_o,J,\}$, $\delta<\varepsilon$, *if the following condition is satisfied*:

$$\left[1+\frac{\sigma_{\Sigma\max}^{A}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right]\cdot e^{\frac{\sigma_{\Sigma\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}\leq\varepsilon/\delta,\qquad\forall t\in J. \tag{71}$$

where $\sigma_{\Sigma\max}^{\mathrm{A}}(.)=\sum_i\sigma_i\left(\mathrm{A}_i\right)$ *of matrices* A_i, $i=0,1,2,...,n$ *where* $\sigma_{max}(.)$*is the largest singular value of matrix* A_i, $i=0,1,2,...,n$.

The above stability results for linear time-delay fractional differential systems are derived by applying Bellman - Gronwall's inequality. In that way, one can check system stability over finite-time interval.

*Proof***:** In accordance with the property of the fractional order $0<\alpha<1$, one can obtain the solution in the form of the equivalent Volterra integral equation:

$$x\left(t\right)=x\left(t_0\right)+\frac{1}{\Gamma\left(\alpha\right)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(A_0x(s)+A_1x(s-\tau)\right)ds \tag{72}$$

Applying the norm $\|(.)\|$ on the equation (72) and using the appropriate property of the norm, it follows:

$$\left\|x\left(t\right)\right\|\leq\left\|x\left(t_0\right)\right\|+\frac{1}{\Gamma\left(\alpha\right)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\left\|\left(A_0x(s)+A_1x(s-\tau)\right)\right\|ds \tag{73}$$

Also, applying the norm $\|(.)\|$ on the equation (61), one can obtain:

$$\left\|\frac{d^{\alpha}x(t)}{dt^{\alpha}}\right\|=\left\|A_0x(t)+A_1x(t-\tau)\right\|\leq\left\|A_0\right\|\left\|x(t)\right\|+\left\|A_1\right\|\left\|x(t-\tau)\right\|, \tag{74}$$

where $\|A\|$ denotes the induced norm of a matrix A, as well as,

$$\left\|x\left(t-\tau\right)\right\|\leq\sup_{t-\tau\leq t^{\bullet}\leq t}\left\|x(t^{\bullet})\right\|, \tag{75}$$

Applying this inequality, the equation (74) can be presented in the following manner:

$$\left\|A_0x(t)+A_1x(t-\tau)\right\|\leq\sigma_{\max}\left(A_0\right)\left\|x(t)\right\|+\sigma_{\max}\left(A_1\right)\sup_{t-\tau\leq t^{\bullet}\leq t}\left\|x(t^{\bullet})\right\|\leq\sigma_{\max}\sup_{t-\tau\leq t^{\bullet}\leq t}\left\|x(t^{\bullet})\right\|, \tag{76}$$

$t>t_0+\tau$

$$\left\|A_0x(t)+A_1x(t-\tau)\right\|\leq\sigma_{\max}\left(\sup_{t-\tau\leq t^{\bullet}\leq t}\left\|x(t^{\bullet})\right\|+\left\|\psi_x\right\|_C\right),\quad t>t_{0^+} \tag{77}$$

After combining (77) with (73), one yields:

$$\left\|x(t)\right\| \le \left\|x(t_0)\right\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} |(t-s)|^{\alpha-1}\, \sigma_{\max}^{A}\left(\sup_{t-\tau\le t^{\bullet}\le t} \left\|x(t^{\bullet})\right\| + \left\|\psi_x\right\|_C \right) ds \tag{78}$$

or,

$$\left\|x(t)\right\| \le \left\|\psi_x\right\|_C \left[1 + \frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} \right] + \frac{\sigma_{\max}^{A}}{\Gamma(\alpha)}\int_{t_0}^{t} \left|(t-s)^{\alpha-1}\right| \sup_{t-\tau\le t^{\bullet}\le t} \left\|x(t^{\bullet})\right\| ds\,, \tag{79}$$

One can introduce nondecreasing function $\mu(t)$ such as:

$$\mu(t) = \left\|\psi_x\right\|_C \left(1 + \frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} \right), \tag{80}$$

and one can apply the known Bellman-Gronwall lemma, [60] where it is easy to show that:

$$\left\|x(t)\right\| \le \sup_{t-\tau\le t^{\bullet}\le t} \left\|x(t^{\bullet})\right\| \le \mu(t) e^{\frac{\sigma_{\max}^{A}}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right| ds} = \mu(t) e^{\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}\,, \tag{81}$$

According to the equation (80) and (65), one yields:

$$\left\|x(t)\right\| \le \delta\left(1 + \frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} \right) e^{\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}\,, \tag{82}$$

Lastly, using the basic condition of the main theorem (part A), the equation (81), it follows:

$$\left\|x(t)\right\| < \varepsilon, \quad \forall t \in J. \tag{83}$$

which had to be proved.

Proof B). The proof immediately follows from part A of the main theorem by applying the same procedure, taking into account (67) and (71).

Remark 1. If $\alpha = 1$, one can obtain the same conditions related to integer order time-delay systems

$$\frac{d\boldsymbol{x}(t)}{dt} = A_0\boldsymbol{x}(t) + A_1\boldsymbol{x}(t-\tau), \tag{84}$$

as follows ([61]):

$$\left[1 + \frac{\sigma_{\max}^{A}(t-t_0)^{1}}{1} \right] \cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^{1}}{1}} \le \varepsilon/\delta, \qquad \forall t \in J.\,, \Gamma(2) = 1 \tag{85}$$

Example 1. Using Time-Delay PD^{α} compensator on the linear system of equations with respect to the small perturbation $e(t) = y(t) - y_d(t)$ one can obtain:

$$\dot{e}(t) + \omega e(t) = K_P e(t-\tau) + K_D d e^{(\alpha)}(t-\tau)/dt^{\alpha}\,, \tag{86}$$

where: $\alpha = 1/2, \omega = 2,\ K_p = 3,\ K_D = 4.$ Also, all initial values are zeros and introducing:

$$x_1(t) = e(t), \quad x_2(t) = d^{1/2}e(t)/dt^{1/2}\,, \tag{87}$$

where $\boldsymbol{x}(t) = (x_1, x_2)^T$ one can write (86) in the state-space form:

$$D_t^{1/2}\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}, \tag{88}$$

with the associated function of initial state:

$$\boldsymbol{x}(t)=\psi_x(t)=0, \qquad -\tau\le t\le 0, \tag{89}$$

Now, one can check the finite-time stability w.r.t $\{t_0=0, J=\{0,2\}, \delta=0.1, \varepsilon=50, \tau=0.1\}$, where $\psi_x(t)=0,\ \forall t\in[-0.1,0]$. From initial data and the equation (88), and condition of Theorem 8 it follows

$$\|\psi_x(t)\|_C<0.1,\ \ \sigma_{\max}(A_0)=2, \sigma_{\max}(A_1)=5, \Rightarrow \sigma^A_{\max}=7 \tag{90}$$

$$\left[1+\frac{7T_e^{0.5}}{0.886}\right]\cdot e^{\frac{7T_e^{0.5}}{0.886}}\le 50/0.1=500 \quad \Rightarrow T_e=0.325s. \tag{91}$$

T_e being the "estimated time" of finite-time stability.

Further, it is shown in [38] that fractional order time-delay state-space model of PD^{α} control of Newcastle robot can be presented by (61) in a homogenous state-space form. It is suggested using the Caputo version of the fractional derivative, where introducing $\boldsymbol{x}(t)=(x_1,x_2,x_3,x_4)^T$ one can obtain:

$$D_t^{1/2}\boldsymbol{x}(t)=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-17.8&0&-12.8&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-0.04&-0.04&0&0\end{bmatrix}\begin{bmatrix}x_1(t-\tau)\\x_2(t-\tau)\\x_3(t-\tau)\\x_4(t-\tau)\end{bmatrix} \tag{92}$$

Also, one has to check the finite-time stability w.r.t $\{t_o=0, J=\{0,1\}, \delta=0.06, \varepsilon=100, \tau=0.1\}$, where $\psi_x(t)=(0.05,0,0,0)^T$, $\forall t\in[-0.1,\ 0]$. From initial data and the equation (92) , one can easily obtain:

$$\|\psi_x(t)\|_C<0.06,\quad \sigma_{\max}(A_0)=21.95,\ \ \sigma_{\max}(A_1)=0,\ \sigma_{\max}=21.95,$$
$$\Gamma(1+1/2)=\frac{\sqrt{\pi}\Gamma(2+1)}{2^2\Gamma(1+1)}=\frac{\sqrt{\pi}(2)!}{2^2 1!}=\frac{\sqrt{\pi}}{2}=0.886, \tag{93}$$

From the Theorem 8(A), it immediately follows

$$\left[1+\frac{21.95T_e^{1/2}}{0.866}\right]\cdot e^{\frac{21.95T_e^{1/2}}{0.866}}\le 100/0.06,\ \Rightarrow Te=0.05\ \ s, \tag{94}$$

T_e being the "estimated time" of finite-time stability.

Also, in paper [62], a stability test procedure is proposed for nonhomogeneous fractional order systems with pure time-delay

$${}_*D^{\alpha}_{to,t}x(t)=\frac{d^{\alpha}\boldsymbol{x}(t)}{dt^{\alpha}}=A_0\boldsymbol{x}(t)+A_1\boldsymbol{x}(t-\tau)+B_0\boldsymbol{u}(t), \quad 0<\alpha<1, \tag{95}$$

with the associated function of initial state:

$$\boldsymbol{x}(t)=\psi_x(t)\in C[-\tau,0], \quad -\tau\le t\le 0. \tag{96}$$

Definition 11. The system given by (95) satisfying the initial condition (96) is finite stable w.r.t $\{\delta,\varepsilon,\beta,\alpha_u,t_o,J,\}$, $\delta<\varepsilon$ if and only if:

$$\|\psi_x\|_C<\delta \tag{97}$$

$$\|\boldsymbol{u}(t)\|<\alpha_u\ , \qquad \forall t\in J,\ \ \alpha_u>0 \tag{98}$$

implying:

$$\|\boldsymbol{x}(t)\|<\varepsilon, \qquad \forall t\in J \tag{99}$$

Theorem 9. The nonautonomous system given by (95) satisfying the initial condition (96) is finite-time stable *w.r.t.* $\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\}$, $\delta<\varepsilon$, if the following condition is satisfied:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}+\gamma^{\bullet}\frac{(t-t_0)^{\alpha}}{\Gamma\left(\alpha+1\right)}\le\varepsilon/\delta,\quad \forall t\in J. \tag{100}$$

where $\gamma^{\bullet}=b_0\alpha_u/\delta$, $\|B_0\|=b_0$ and $\Gamma(.)$ Euler's gamma function.

Proof **:** The proof follows from the previous Theorem 8. Applying the same procedure and using (98), one can obtain

$$\|x(t)\|\le\|\psi_x\|_C\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right]+\frac{\sigma_{\max}^{A}}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\sup_{t-\tau\le t^{\bullet}\le t}\left\|x(t^{\bullet})\right\|ds+\frac{1}{\Gamma\left(\alpha+1\right)}(\alpha_u b_0)\left(t-t_0\right)^{\alpha} \tag{101}$$

Taking into account (97) and (99) and applying the Bellman-Gronwall lemma with the basic condition of Theorem 9 it follows

$$\|x(t)\|<\varepsilon,\ \forall t\in J. \tag{102}$$

Remark 2. If $\alpha=1$, see (95), one can obtain the same conditions related to integer order time-delay systems (3) as follows [19]:

$$\left[1+\frac{\sigma_{\max}^{A}\left(t-t_0\right)^{1}}{1}\right]\cdot e^{\frac{\sigma_{\max}^{A}(t-t_0)^{1}}{1}}+\gamma^{\bullet}\frac{(t-t_0)^{1}}{1}\le\varepsilon/\delta,\quad \forall t\in J,\ \Gamma(2)=1 \tag{103}$$

Moreover, in the same paper [62], finite-time stability criteria are proposed for a class of fractional non-linear nonautonomous system with time-delay in state and in control as follows:

$$\frac{d^{\alpha}\boldsymbol{x}(t)}{dt^{\alpha}}=A_0\boldsymbol{x}(t)+A_1\boldsymbol{x}(t-\tau)+B_0\boldsymbol{u}(t)+B_1\boldsymbol{u}(t-\tau)+f_1\left(\boldsymbol{x}(t)\right)+g_1\left(\boldsymbol{x}(t-\tau)\right), \tag{104}$$

with the associated function of initial state and control:

$$\boldsymbol{x}(t)=\psi_x(t),\quad \boldsymbol{u}(t)=\psi_u(t),\qquad -\tau\le t\le 0 \tag{105}$$

and vector functions f_1,g_1 satisfying (6).

Definition 12. *The system given by (104) satisfying the initial condition (105) is finite stable w.r.t* $\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\}$, $\delta<\varepsilon$ *if and only if*:

$$\|\psi_x\|_C<\delta,\quad \|\psi_u\|_C<\alpha_0, \tag{106}$$

$$\|\boldsymbol{u}(t)\|<\alpha_u\ ,\quad \forall t\in J \tag{107}$$

imply:

$$\|\boldsymbol{x}(t)\|<\varepsilon\ ,\quad \forall t\in J \tag{108}$$

Theorem 10. The nonlinear nonautonomous system given by (104) satisfying the initial condition (105) is finite-time stable w.r.t. $\{\delta,\varepsilon,\alpha_u,\alpha_0,t_o,J,\}$, $\delta<\varepsilon$ if the following condition is satisfied:

(109)

$$\left(1+\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)e^{\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}+\frac{\gamma^{\bullet}{}_{u0}\left(t-t_0\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\gamma^{\bullet}{}_{u1}\left(t-t_0-\tau\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\gamma^{\bullet}{}_{01}\left(\tau\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\le\varepsilon/\delta,\ \forall t\in J$$

where $\gamma^{\bullet}{}_{u0}=\alpha_u b_0/\delta,\ \gamma^{\bullet}{}_{u1}=\alpha_u b_1/\delta,\ \gamma^{\bullet}{}_{01}=\alpha_0 b_1/\delta$.

Proof. Taking into account the fractional order $0<\alpha<1$, one can present the solution in the form of the equivalent Volterra integral equation:

$$x(t)=x(t_0)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\begin{pmatrix}A_0\boldsymbol{x}(t)+A_1\boldsymbol{x}(t-\tau)+\\+B_0\boldsymbol{u}(t)+B_1\boldsymbol{u}(t-\tau)+f_0\left(\boldsymbol{x}(t)\right)+f_1\left(\boldsymbol{x}(t-\tau)\right)\end{pmatrix}ds, \tag{110}$$

Applying the norm $\|(.)\|$ on the equations (110) and (104), using the appropriate property of the norm, and (6), one yields:

$$\|x(t)\|\le\|x(t_0)\|+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}|(t-s)|^{\alpha-1}\left\{\sigma_{\max c01}\left(\sup_{t-\tau\le t^{\bullet}\le t}\|x(t^{\bullet})\|+\|\psi_x\|_C\right)+b_0\|u(t)\|+b_1\|u(t-\tau)\|\right\}ds, \tag{111}$$

or,

$$\begin{aligned}\|x(t)\|\le\|\psi_x\|_C\left[1+\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{\sigma_{\max c01}}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\sup_{t-\tau\le t^{\bullet}\le t}\|x(t^{\bullet})\|ds+\\+\frac{1}{\Gamma(\alpha+1)}(\alpha_u b_0)(t-t_0)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}(\alpha_u b_1)(t-t_0-\tau)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}(\alpha_0 b_1)(\tau)^{\alpha}\end{aligned}, \tag{112}$$

Further, one can conclude that $\mu(t)$ is nondecreasing function given as:

$$\mu(t)=\|\psi_x\|_C\left(1+\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right), \tag{113}$$

where using the Bellman-Gronwall lemma it is easy to show that

$$\|x(t)\|\le\sup_{t-\tau\le t^{\bullet}\le t}\|x(t^{\bullet})\|\le\mu(t)e^{\frac{\sigma_{\max c01}}{\Gamma(\alpha)}\int_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|ds}=\mu(t)e^{\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}, \tag{114}$$

and

$$\begin{aligned}\|x(t)\|\le\delta\left(1+\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right)e^{\frac{\sigma_{\max c01}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}}+\frac{1}{\Gamma(\alpha+1)}(\alpha_u b_0)(t-t_0)^{\alpha}+\\+\frac{1}{\Gamma(\alpha+1)}(\alpha_u b_1)(t-t_0-\tau)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}(\alpha_0 b_1)(\tau)^{\alpha}\end{aligned}, \tag{115}$$

Finally, if one uses the relation (109) it follows:

$$\|x(t)\|<\varepsilon,\quad\forall t\in J. \tag{116}$$

which had to be proved.

Recently, we have studied and reported in paper [39] a stability test procedure for linear nonhomogeneous fractional order systems with pure time-delay. New stability criteria for this class of fractional order systems were derived by applying Bellman-Gronwall`s approach using for the starting point a recently obtained *generalized Gronwall inequality* reported in [43].In that way, one can check system stability over finite-time, which is illustrated using a suitable illustrative example.

Theorem 11. ([43] *Generalized Gronwall inequality*) . Suppose $x(t),a(t)$ are nonnegative and local integrable on $0\le t<T,$*some* $T\le+\infty$,and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0\le t<T,\ g(t)\le M=const$, $\alpha>0$ with

$$x(t) \le a(t) + g(t)\int_0^t (t-s)^{\alpha-1} x(s)ds \tag{117}$$

on this interval.Then

$$x(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{\left(g(t)\Gamma(\alpha)\right)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right] ds, \quad 0 \le t < T \tag{118}$$

Corollary 2.1 of Theorem 9, [43] Under the hypothesis of Theorem 11, let $a(t)$ be a nondecreasing function on $[0,T)$. Then it holds:

$$x(t) \le a(t) E_\alpha \left(g(t)\Gamma(\alpha) t^\alpha \right) \tag{119}$$

Theorem 12. The linear nonautonomous system given by (95) satisfying the initial condition $\mathbf{x}(t) = \boldsymbol{\psi}_x(t),\ -\tau \le t \le 0$ is finite-time stable w.r.t. $\{\delta, \varepsilon, \alpha_u, J_0, \}$, $\delta < \varepsilon$ if the following condition is satisfied:

$$\left(1 + \frac{\sigma_{\max 01} t^\alpha}{\Gamma(\alpha+1)}\right) E_\alpha \left(\sigma_{\max 01} t^\alpha\right) + \frac{\gamma^\bullet_{u0} t^\alpha}{\Gamma(\alpha+1)} \le \varepsilon/\delta,\ \forall t \in J_0 = \{0,T\}, \tag{120}$$

where $\gamma^\bullet_{u0} = \alpha_u b_0 / \delta$, and $\sigma_{max}(.)$ being the largest singular value of the matrix (.), where: $\sigma_{\max 01} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1)$ and $E_\alpha(.)$ denotes Mittag-Leffler function (see Appendix C2).

Proof. The proof immediately follows from the proof of the Theorem 9 applying the same procedure.

Remark 3. If $\alpha = 1$, see (95), one can obtain the same conditions related to integer order time-delay systems (3) as follows [19]:

$$\left[1 + \frac{\sigma^A_{\max}(t-t_0)^1}{1}\right] \cdot e^{\frac{\sigma^A_{\max}(t-t_0)^1}{1}} + \gamma^\bullet \cdot \frac{(t-t_0)^1}{1} \le \varepsilon/\delta, \quad \forall t \in J,\ \Gamma(2) = 1,\ E_{\alpha=1}(z) = e^z \tag{121}$$

Theorem 13. The linear autonomous system given by (95) satisfying the initial condition $\mathbf{x}(t) = \boldsymbol{\psi}_x(t),\ -\tau \le t \le 0$ is finite-time stable w.r.t. $\{\delta, \varepsilon, J_0, \}$, $\delta < \varepsilon$ if the following condition is satisfied:

$$\left(1 + \frac{\sigma_{\max 01} t^\alpha}{\Gamma(\alpha+1)}\right) E_\alpha \left(\sigma_{\max 01} t^\alpha\right) \le \varepsilon/\delta,\ \forall t \in J_0, \tag{122}$$

Proof. The proof immediately follows from the proof of the Theorem 12.

Remark 4. If $\alpha = 1$, one can obtain the same conditions related to integer order time-delay systems

$$\frac{d\boldsymbol{x}(t)}{dt} = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-\tau), \tag{123}$$

as follows (see [19]):

$$\left[1 + \frac{\sigma_{\max 01} t^1}{1}\right] \cdot e^{\frac{\sigma_{\max 01} t^1}{1}} \le \varepsilon/\delta, \quad \forall t \in J_0 = [0,T], \tag{124}$$

Example 2. We consider the following nonhomogeneous fractional order systems with pure time-delay

$$D_t^{1/2} \boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{ff}(t), \tag{125}$$

or

$$D_t^{1/2}\boldsymbol{x}(t) = A_0\boldsymbol{x}(t) + A_1\boldsymbol{x}(t-\tau) + B_0\boldsymbol{u}(\mathrm{t}), \tag{126}$$

with the associated function of initial state:

$$\boldsymbol{x}(t) = \boldsymbol{\psi}_x(t) = 0, \qquad -\tau \le t \le 0, \tag{127}$$

Here, the task is to check the finite-time stability w.r.t $\{t_0 = 0,\ J = \{0,2\}, \delta = 0.1,\ \varepsilon = 100,\ \tau = 0.1, \alpha_u = 1\}$, where $\psi_x(t) = 0,\ \forall t \in [-0.1, 0]$. From initial data and properties of $\sigma_{\max}(..)$ one can calculate:

$$\|\psi_x(t)\|_C < 0.1,\ \ \sigma_{\max}(A_0) = 2, \sigma_{\max}(A_1) = \sqrt{3^2 + 4^2} = 5, \Rightarrow \sigma_{\max 0,1} = 7 \tag{128}$$

Applying the condition of the Theorem 12 (120) one can get:

$$\left[1 + \frac{7T_e^{0.5}}{0.886}\right] \cdot E_{0.5}\left(7T_e^{0.5}\right) + \frac{10 \cdot T_e^{0.5}}{0.886} \le 100/0.1 \quad \Rightarrow T_e \approx 0.1s. \tag{129}$$

T_e being the "estimated time" of finite-time stability.

Further, paper [41] presents natural extension of our paper [37] where new stability criteria are obtained. We considered a class of fractional non-linear perturbed autonomous system with time-delay described by the state-space equation:

$$\frac{d^\alpha \boldsymbol{x}(t)}{dt^\alpha} = \left(A_0 + \Delta A_0\right)\boldsymbol{x}(t) + \left(A_1 + \Delta A_1\right)\boldsymbol{x}(t-\tau) + f_0\left(\boldsymbol{x}(t)\right), \tag{130}$$

with the initial functions (62) of the system and vector functions f_0 satisfying (6).

Theorem 14. *The nonlinear perturbed autonomous system given by (130) satisfying the initial condition (62) and (6) is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_o, J,\}$, $\delta < \varepsilon$, *if the following condition is satisfied:*

$$\left(1 + \frac{\mu_p (t-t_0)^\alpha}{\Gamma(\alpha+1)}\right) e^{\frac{\mu_p (t-t_0)^\alpha}{\Gamma(\alpha+1)}} \le \varepsilon/\delta,\ \forall t \in J, \tag{131}$$

where $\Gamma(.)$ *is Euler's gamma function, and* $\mu_{Aoco} = \sigma_{Ao} + \gamma_{\Delta A_0} + c_0$, $\sigma_{A1\Delta} = \sigma_{A1} + \gamma_{\Delta A_1}$, $\mu_p = \mu_{Aoco} + \sigma_{A1\Delta}, \sigma_{\Delta Ao} \le \gamma_{\Delta Ao}$, $\sigma_{\Delta A1} \le \gamma_{\Delta A1}$.

Remark 5. If we have no perturbed system $\Delta\mathbf{A}_0 = 0, \Delta\mathbf{A}_1 = 0, \boldsymbol{f}_0(\boldsymbol{x}(\mathrm{t})) = 0$ one can obtain the same conditions which are related to Theorem 8.

Recently,[63] we discussed the case of the particular class of nonlinear perturbed nonautonomous fractional order , $0 < \alpha < 1$ *time-varying delay* system which is presented in the state space form (132) and we examined the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets.

$$\frac{d^\alpha x(t)}{dt^\alpha} = \left(A_0 + \Delta A_0\right)x(t) + \left(A_1 + \Delta A_1\right)x(t-\tau(t)) + B_0 u(t) + f_0\left(x(t), x(t-\tau(t))\right), \tag{132}$$

with the initial function (62) of the system and vector function f_0 satisfied (6). Here, $\tau(t)$ is an unknown time–varying parameter which satisfies

$$0 \le \tau(t) \le \tau_M, \quad \forall t \in J,\ J = [t_o, t_o + T], J \subset R \tag{133}$$

Theorem 15. The nonlinear nonautonomous system given by (132) satisfying initial condition $x(t) = \psi_x(t), \quad -\tau_M \le t \le 0$ is finite time stable *w.r.t.* $\{\delta, \varepsilon, \alpha_u, J_0,\}$, $\delta < \varepsilon$, if the following condition is satisfied

$$\left(1+\frac{\mu_\Sigma t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left(\mu_\Sigma t^\alpha\right)+\frac{\gamma^\bullet_{u0}}{\Gamma(\alpha+1)}t^\alpha \le \varepsilon/\delta,\ \ \forall t\in J_0=\{0,T\} \tag{134}$$

where μ_Σ is defined by (135), $\gamma^\bullet_{uo}=\alpha_u b_0/\delta$,(see (109)), $\sigma(.)$ being the largest singular value of matrix, and $E_\alpha(z)$ the Mittag-Leffler function.

$$\mu_{Aoco}=\sigma_{Ao}+\sigma_{\Delta Ao}+c_0,\quad \mu_{Aoc1}=\sigma_{A1}+\sigma_{\Delta A_1}+c_1,\quad \Rightarrow \quad \mu_\Sigma=\mu_{Aoco}+\mu_{Aoc1}, \tag{135}$$

Remark 6. If we have $\tau(t)=const=\tau_M$ and $f_0\left(x(t),x(t-\tau(t))\right)\equiv f_0\left(x(t)\right)$, we obtain same conditions reported in [42].

3. 5 Conclusion

While Lyapunov methods have been developed for stability analysis and control law synthesis of integer linear systems and have been extended to stability of fractional systems, only few studies deal with non-Lyapunov stability of fractional systems. First of all, some basic results are presented on the stability of fractional order systems including time-delays. Further, in this chapter, we have studied and presented the finite time stability of (non)perturbed (non)linear fractional order time-delay systems. We have employed the "classical" and the generalization of Belmann-Gronwall lemma to obtain finite-time stability criteria for the proposed class of time-delay systems. Finally, numerical examples are given to illustrate the validity of the proposed procedure.

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Stability of Discrete-Time Fractional Order Systems: An Approach Based on Stability of Discrete-Time Integer Order Time-Delay Systems

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Abstract: - In this paper the problem of stability of linear discrete-time fractional order systems is addressed. It was shown that some stability criteria for discrete time-delay systems could be applied with small changes to discrete fractional order state-space systems. Accordingly, simple conditions for the stability and robust stability of a particular class of linear discrete time-delay systems are derived. These results are modified and used for checking the stability of discrete-time fractional order systems. The systems under consideration involve time delays in the state and parameter uncertainties. The parameter uncertainties are assumed to be time-varying and norm bounded. New Lyapunov–Krasovskii functional is constructed to derive some delay-dependent stability criteria. The proposed methods give both sufficient and necessary and sufficient stability conditions. Numerical examples have been worked out to show the applicability of the derived results.

Key-Words: - Discrete-time fractional order systems, discrete time-delay systems, stability, robust stability, Lyapunov functional, delay-independent stability, delay-dependent stability.

4. 1 Introduction

In recent years, the studies of time-delay systems have received considerable attention since time delay is frequently a source of instability and commonly exists in various engineering, biological and economical systems due to the finite speed of information processing.

The various techniques of stability analysis of time-delay systems have been utilized by many researchers. However, less attention has been drawn to the corresponding results obtained for discrete time-delay systems [1-14]. This is mainly due to the fact that such systems can be transformed into augmented high dimensional systems (equivalent systems) without delay [15-18]. This augmentation of the systems is, however, inappropriate for the systems with unknown delays or systems with time-varying delays. Moreover, for the systems with known,large delays, this augmentation leads to large-dimensional systems. Therefore, in these cases the stability analysis of discrete time-delay systems cannot be reduced to stability of discrete systems without delay.

The existing stability criteria of time-delay systems can be classified into two categories: delay-independent [11-14, 19-28] and delay-dependent criteria, [29-39]. It has been shown that the delay-dependent stability criteria which take into account the size of delay, are generally less conservative compared to the delay-independent ones, which do not include any information on the size of delay. The delay-dependent stability criteria usually provide an upper bound of the delay, $h_{\max}$, such that system is stable for any delay less than this upper bound. For a system whose stability does not depend on the time-delay value, the analysis performed by delay-dependent conditions can be very conservative. Also, delay-independent conditions cannot be obtained as a limit case of delay-dependent ones just by imposing a maximum delay $h_{\max} \to \infty$, leading to a gap between these two types of delay-stability conditions.

The majority of stability conditions, for both continual and discrete time-delay systems, are sufficient conditions independent of time delay. Only a small number of works provide necessary and sufficient conditions [34-41]. These conditions do not possess conservatism but often require more complex numerical computations. In our study, we present some necessary and sufficient stability conditions.

The stability analysis and robust stabilization of systems with parameter uncertainties are the problems of recurring interest over the past years, [11, 23-24, 42-52]. Especially, in accordance with the advances of the robust control theory, a number of robust stability and stabilization methods have been proposed for the uncertain time-delay systems [53-59].

On the other hand, the theory of fractional calculus has a long and prominent history. In fact, one may trace it back to the very origins of differential calculus itself. However, its complexity prevented it from being put to practical application until only very recently. Over the last decade, the results of work on the theory of chaos revealed connections of this field with fractional derivatives and integrals, renewing interest in them, in their turn. The fundamental facts of the theory of fractional calculus, and its properties, are now widely known (see, e.g., [60,61]). Nowadays the concept of non-integer derivative or integral is used increasingly to model the behavior of real systems in various fields of science and engineering. The fractional calculus is a generalization of (integer order) differential calculus, allowing defining derivatives (and integrals) of a real or complex order [62]. Several definitions of fractional derivatives, including the Riemann-Liouville, Caputo, Riesz, Riesz-Caputo,Weyl, Grunwald-Letnikov, Hadamard, and Chen derivatives, are available (see Chapter 1) as well as in the literature [62-63]. Analogously, one can define a discrete fractional derivative in different ways. In 1989, Miller and Ross [64] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator [63]. Regarding other fractional discrete definitions, we refer the reader to [65-66] and the references therein.

Also, one of the fundamental problems of control is the stability analysis a the dynamic system.The stability problem for linear, continuous-time, fractional order state-space systems has been considered for some time and some properties and stability results for these systems are presented and discussed by, for example, [67-68].

For discrete-time fractional order systems, however, the discussion of this problem is much less common. Very few results have been published dealing with the stability of such systems, this is even more so for the state-space description of these systems. Some basic results of defining fractional order state-space systems have been presented by [69].Remarks on the poles and zeros of fractional order systems were given in the work of [70]. Linear discrete-time fractional-order systems modeled by a state space representation were introduced in [71-72]. These contributions are devoted to controllability and observability analysis, the design of a Kalman filter and observers, plus adaptive feedback control for discrete fractional order systems.In addition, the concept of practical stability of positive fractional discrete-time systems was introduced and the conditions for practical stability were given by [73].

The objective of this work is to present the results of a further investigation of the analysis of discrete-time fractional order systems. It was shown that some obtained results for the integer order discrete time delay systems can be applied to the discrete fractional order state-space system. Using Lyapunov techniques, new results concerning asymptotic stability of discrete-time fractional order systems are developed based on results of asymptotic stability of discrete time-delay systems.

4. 2 Problem Formulation

4.2.1 The discrete-time fractional order system as discrete-time linear integer order time delay system: stability issue

For the discrete-time fractional order systems the discussion of stability is much less common. There are very few results dealing with the stability of such systems. It is even more so for the state-space description of these systems. Some basic results of defining the fractional order state-space systems are presented in e.g.[69-73]. Also other system properties of fractional order systems like controllability and observability have been addressed only in recent years (see e.g. [72]).

As we know, Grunwald and Letnikov developed an approach to fractional differentiation i.e the left Grunwald-Letnilov (GL) derivative as a limit of a fractional order backward difference. Here, (GL) definition of the fractional order difference [60,61] is used and presented. If we consider $n = (t - a)/h$ where a is a real constant that expresses a limit value one can write

$$ {}_{GL}D^{\alpha}_{a,t}f(t) = \lim_{h\to 0}\frac{\left(\Delta^{\alpha}_{h}f(t)\right)}{h^{\alpha}} = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^{j}\binom{\alpha}{j}f(t-jh),\ h>0 \tag{1} $$

where $[x]$ means the integer part of x, a and t are the bounds of operation for ${}_{GL}D^{\alpha}_{a,t}f(t)$. For generalized binomial coefficients calculation for $\alpha \in R$ as previously defined is a fractional degree and $k \in \mathbb{N}_0$ we can use the relation between Euler's *Gamma* function and factorial, defined as

$$ \binom{\alpha}{j} = \begin{cases} 1 & for\ \ j=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} = \dfrac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, & for\ \ j>0 \end{cases} \tag{2} $$

Also, the following definition of the fractional discrete derivative will be used

$$ \Delta^{\alpha}x_k = \sum_{j=0}^{k}(-1)^{j}\binom{\alpha}{j}x_{k-j}, \qquad 0<\alpha<1 \tag{3} $$

where $\alpha \in R$ is the order of the fractional difference and $k \in N$, is the number of a samples for which the approximation of the n-derivative is calculated. Here, we consider that a linear continuous-time fractional order state-space system is given by the following set of equations:

$$ {}^{GL}_{a}D^{\alpha}_{t}x(t) = A_c x(t) \tag{4} $$

where $A_c \in R^{n\times n}$, $m-$ is a number of outputs, $p-$ is a number of inputs, $n-$ is a number of state equations. One can obtain a discrete-time fractional order state-space system by substituting the previous definition of GL derivative into (4):

$$ \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{[t/h]}(-1)^{j}\binom{\alpha}{j}x(t-jh) = A_c x(t) \tag{5} $$

Using the Eq. (3) with some relatively small value of h we can obtain the following structure of the discrete-time fractional order state-space model. In the general case, the values of the discrete system matrices are not the same as in continuous case and have to be found by the discretization process or by identification. Consider the autonomous discrete-time fractional order linear system, described by the state-space equations as follows:

$$ \Delta^{\alpha}x_{k+1} = Ax_k \tag{6} $$

and from (3) one obtains:

$$ \Delta^{\alpha}x_{k+1} = x_{k+1} + \sum_{j=1}^{k+1}(-1)^{j}\binom{\alpha}{j}x_{k-j+1} \tag{7} $$

For practical realization the number of samples taken into consideration has to be reduced to the predefined number N, where N is a number of samples taken into account, called *memory length.*

$$\Delta^{\alpha} x_{k+1} = x_{k+1} + \sum_{j=1}^{N} (-1)^j \binom{\alpha}{j} x_{k-j+1} \tag{8}$$

In this case the Eq. (6) is rewritten as

$$\begin{aligned} x_{k+1} &= (A + \alpha I_n) x_k + \sum_{j=1}^{N} (-1)^j \binom{\alpha}{j+1} x_{k-j} \\ &= A_0 x_k + \sum_{j=1}^{N} \beta_j(\alpha) x_{k-j} = A_0 x_k + \sum_{j=1}^{N} A_j x_{k-j}, \quad k \in Z_+ \end{aligned} \tag{9}$$

where are $A_0 = A + \alpha I_n$, $A_1 = \beta_1(\alpha) I_n$, $A_2 = \beta_2(\alpha) I_n$, …, $A_j = \beta_j(\alpha) I_n$, ..., $A_N = \beta_N(\alpha) I_n$ or in the form

$$x(k+1) = A_0 x(k) + \sum_{j=1}^{N} A_j x(k-j), \quad k \in Z_+ \tag{10}$$

Remark 1. The model described by (10) can be classified as a linear, autonomous, discrete-time system (linear autonomous discrete-time integer order system) with multiple time-delays: $h_1 = 1, h_2 = 2, \cdots, h_N = N$. Therefore, the discrete time-delay system (10) represents a practical implementation of the discrete-time fractional order linear system (6). Based on of this fact, we can apply various the existing stability criteria for a class of linear discrete systems with multiple time-delay to check stability of the discrete-time fractional order systems. In this sense, let us consider the general form of a linear, discrete multiple time-delay system described by:

$$x(k+1) = A_0(k) x(t) + \sum_{j=1}^{N} A_j(k) x(k-h_j), \quad 0 < h_1 < \cdots h_N \tag{11}$$

with an associated function of the initial state:

$$x(\theta) = \psi(\theta), \theta \in \{-h_N, -h_N + 1, \ldots, 0\} \triangleq \Delta \tag{12}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $A_j(k) \in \mathbb{R}^{n \times n}$, $j = 0,1,\cdots,N$, $B(k) \in \mathbb{R}^{n \times m}$ and $h \in \mathbb{Z}^+$ is unknown time delay in general case.The following two cases are considered:

a) the matrices $A_j(k)$, $j = 0,1,\cdots,N$ are stationary matrices:

$$A_j(k) = A_j, j = 0,1,\cdots,N \tag{13}$$

b) the matrices $A_j(k)$, $j = 0,1,\cdots,N$ are uncertain matrices

$$A_j(k) = A_j + \Delta A_j(k), \quad j = 0,1,\cdots,N \tag{14}$$

where A_j, $j = 0,1,\cdots,N$ are known real constant matrices, $\Delta A_j(k)$, $j = 0,1,\cdots,N$ are the time-varying parameter uncertainties, and are assumed to be of the form:

$$\begin{bmatrix} \Delta A_0(k) & \cdots & \Delta A_N(k) \end{bmatrix} = M F(k) \begin{bmatrix} N_{A_0} & \cdots & N_{A_N} \end{bmatrix} \tag{15}$$

where M, N_{A_j}, $j = 0,1,\cdots,N$ are constant matrices, and $F(k) \in \mathbb{R}^{i \times j}$ is the uncertain matrix satisfying:

$$F(k) F^T(k) \le I \tag{16}$$

Remark 2. The uncertain discrete-time fractional order system can be described by the following state-space equations:

$$\Delta^{\alpha} x_{k+1} = A(k) x_k \tag{17}$$

The Eq. (17) can be rewritten as

$$x(k+1) = A_0(k) x(k) + \sum_{j=1}^{N} A_j x(k-j), \quad k \in Z_+ \tag{18}$$

where:

$$A_0(k) = (A + \alpha I_n) + \Delta A(k),\ A_1 = \beta_1(\alpha) I_n, \cdots, A_N = \beta_N(\alpha) I_n \tag{19}$$

$$\Delta A(k) = M F(k) N_A \tag{20}$$

Clearly, the matrices A_j, $j = 1,2,\cdots,N$ of the uncertain discrete-time fractional order system (17) are independent from the discrete time k.

4. 3 The Problem Solution

Based on Remark 1, we first consider the problem of asymptotic stability for linear discrete time systems with multiple time-delay. Simple delay-independent and delay-dependent criteria of stability and robust stability of time-delay systems are proposed (Theorem 2-8). Then, using these results, we present the stability criteria for discrete-time fractional order systems (Corollary 1-7).

4.3.1 Stability of linear discrete time delay systems

Consider system (11)-(13) with stationary matrices $A_j(k)$, $j = 0,1,\cdots,N$ and $B(k)$.

Let $x_k \triangleq \left[x^T(k)\ \ x^T(k-1)\ \ x^T(k-h_N) \right]^T$, $k \in Z^+$ be state vector, $\mathbb{D}\left(\Delta, \mathbb{R}^n\right)$ – space of continuous functions mapping the discrete interval Δ into $\mathbb{R}^n$ and $\|\phi\|_D = \sup_{\theta \in \Delta} \|\phi(\theta)\|$, $\mathbb{D} \ni \phi(\theta) : \Delta \mapsto \mathbb{R}^n$ - the norm of an element ϕ in $\mathbb{D}$. Furthermore, $\mathbb{D}^\gamma = \left\{ \phi \in \mathbb{D} : \|\phi\|_D < \gamma, \gamma \in \mathbb{R} \right\} \subset \mathbb{D}$. For the initial state, the next condition is assumed:

$$\|\psi\|_D \in \mathbb{D}^\infty \tag{21}$$

Evidently, $x_k : \Delta \ni \theta \ \rightarrow \ x_k(\theta) \triangleq x(k+\theta) \in \mathbb{D}$ and $x(k) = x(k,\psi)$.

Definition 1. The equilibrium state $x = 0$ of system (11)-(13) is *globally asymptotically stable* if any initial $\psi(\theta)$ which satisfies:

$$\psi(\theta) \in \mathbb{D}^\infty \tag{22}$$

Then,it holds:

$$\lim_{k \to \infty} x(k,\psi) \to 0 \tag{23}$$

Theorem 1. [13] If there exist positive numbers α and β and a continuous functional $V : \mathbb{D} \to \mathbb{R}$ such that:

$$0 < V(x_k) \le \alpha \|x_k\|_D^2, \quad \forall x_k \ne 0, \quad V(0) = 0 \tag{24}$$

$$\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) \le -\beta \|x(k)\|^2 \tag{25}$$

$\forall x_k \in \mathbb{D}$ satisfying (11)-(13),then the solution $x = 0$ of equation (11)-(13) is globally asymptotically stable, [36].

Proof. From (25) it follows:

$$\sum_{j=0}^{k} \Delta V(x_j) = V(x_{k+1}) - V(x_0) \le -\beta \sum_{j=0}^{k} \|x(j)\|^2 \tag{26}$$

and from (21), (22) and (26) it holds:

$$V(x_0) \ge V(x_{k+1}) + \beta \sum_{j=0}^{k} \|x(j)\|^2 \ge \beta \sum_{j=0}^{k} \|x(j)\|^2 \ge \beta \|x(k)\|^2 \tag{27}$$

$$\beta \|x(k)\|^2 \le \sum_{j=0}^{k} \beta \|x(j)\|^2 \le V(x_0) \le \alpha \|x_0\|_D^2 = \alpha \|\psi(\theta)\|_D^2 < \infty$$
$$\forall k \in \mathbb{Z}^+, \quad \psi(\theta) \in \mathbb{D}^\infty \tag{28}$$

Moreover, from (28) it follows: $\lim_{k\to\infty}\sum_{j=0}^{k}\beta\|x(j)\|^2<\infty$ and, hence, $\|x(k)\|^2\to 0$ for $k\to\infty$ i.e. (23) holds for $\forall\psi(\theta)\in\mathbb{D}^{\infty}$. **Q.E.D.**

Definition 2. Discrete system with time delay (11)-(13) is globally asymptotically stable if and only if its solution $x=0$ is globally asymptotically stable.

Lemma 1. For any two matrices F and G of dimensions $(n\times m)$ and for any square matrix $P=P^T>0$ of dimension n, the following statement is true:

$$(F+G)^T P(F+G)\le(1+\varepsilon)F^T PF+\left(1+\varepsilon^{-1}\right)G^T PG \tag{29}$$

where ε is some positive constant.

Lemma 2. Tchebyshev's inequality holds for any real vector v_i

$$\left(\sum_{i=1}^{m}v_i\right)^T\left(\sum_{i=1}^{m}v_i\right)\le m\sum_{i=1}^{m}v_i{}^Tv_i \tag{30}$$

Lemma 3. Given matrices $Q=Q^T$, H, E and $R=R^T>0$ of appropriate dimensions,

$$Q+HFE+E^TF^TH^T<0 \tag{31}$$

for all F satisfying $F^TF\le R$, if and only if there exists some $\lambda>0$ such that

$$Q+\lambda HH^T+\lambda^{-1}E^TRE<0 \tag{32}$$

4.3.1.1 Delay-independent stability

Theorem 2. [74] System (11)-(13) is asymptotically stable if:

$$\sum_{j=1}^{M}\|A_j\|<1 \tag{33}$$

Proof. The proof is based on contradiction. Let $|x|(\cdot)$ be any vector norm (e.g., $\cdot$ = 1, 2, ∞) and $\|(\cdot)\|$ the matrix

norm induced by this vector. Here, we use $|x|_2=\left(x^Tx\right)^{1/2}$ and $\|(.)\|_2=\lambda_{\max}^{1/2}\left(A^{\bullet}A\right)$ where upper indices $(\ )^{\bullet}$ and T denote transpose conjugate and transpose, respectively. Suppose that system to be considered is not stable which is equivalent to:

$$\det(zI_n-A_0-\sum_{j=1}^{M}A_jz^{-j})=0,\quad |z|\ge 1, \tag{34}$$

or:

$$\exists v\in R^n\qquad (zI_n-A_0-\sum_{j=1}^{M}A_jz^{-j})v=0,\quad |z|\ge 1, \tag{35}$$

$$A_0v=\ (zI_n-\sum_{j=1}^{M}A_jz^{-j})v=0,\quad |z|\ge 1, \tag{36}$$

$$\|A_0\|\|v\|\ge\|A_0v\|=\left\|zI_n-\sum_{j=1}^{M}A_jz^{-j})v\right\|=|z|\|v\|-\left\|\sum_{j=1}^{M}z^{-j}A_jv\right\| \tag{37}$$

$$\begin{aligned}\|A_0\|\|v\| \geq \|A_0 v\| &= \left\| zI_n - \sum_{j=1}^{M} A_j z^{-j})v \right\| = |z|\|v\| - \left\| \sum_{j=1}^{M} z^{-j} A_j v \right\| \\ &\geq |z|\|v\| - \sum_{j=1}^{M} \left| z^{-j} \right| \|A_j\| \|v\| \geq \|v\| \left(1 - \sum_{j=1}^{M} \|A_j\| \right), \qquad |z| \geq 1\end{aligned} \tag{38}$$

So if the system is not stable, then:

$$\sum_{j=1}^{M} \|A_j\| \geq 1 \tag{39}$$

This is in contradiction, so the system under consideration is stable. **Q.E.D.**

Corollary 1. Discrete-time fractional order system (6) is asymptotically stable if:

$$\|A + \alpha I_n\| + \sum_{j=1}^{N} |\beta_j(\alpha)| < 1 \tag{40}$$

Theorem 3. [75] The linear discrete time-delay system (11)-(13) with $\|A_0\|_2 \neq 0$ is asymptotically stable if there exists real symmetric matrix $P > 0$ such that

$$\begin{aligned}&(1+\varepsilon_m) A_0{}^T P A_0 + \frac{N(1+\varepsilon_m)}{\varepsilon_m} \sum_{j=1}^{N} A_j{}^T P A_j - P < 0 \\ &\varepsilon_m = \left(N \sum_{j=1}^{N} \|A_j\|_2^2 \right)^{\frac{1}{2}} \|A_0\|_2^{-1}\end{aligned} \tag{41}$$

Proof. Let the Lyapunov functional be:

$$\begin{aligned}V(x_k) &= x^T(k) P x(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_j} x^T(k-l) S_j x(k-l) \\ P &= P^T > 0, \;\; S_j = S_j{}^T \geq 0\end{aligned} \tag{42}$$

where:

$$x_k = x(k+\theta), \; \theta \in \{-h_N, -h_N + 1, \ldots, 0\} \tag{43}$$

The forward difference along the solutions of system (11)-(13) is:

$$\begin{aligned}\Delta V(x_k) = &\left[A_0 x(k) + \sum_{j=1}^{N} A_j x(k-h_j) \right]^T P \cdot \left[A_0 x(k) + \sum_{j=1}^{N} A_j x(k-h_j) \right] - \\ &- x^T(k) P x(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_j} x^T(k+1-l) S_j x(k+1-l) - \sum_{j=1}^{N} \sum_{l=1}^{h_j} x^T(k-l) S_j x(k-l)\end{aligned} \tag{44}$$

Applying *Lemma* 1 to (44), one can obtain:

$$\begin{aligned}\Delta V(x_k) \leq &(1+\varepsilon) x^T(k) A_0{}^T P A_0 x(k) + (1+\varepsilon^{-1}) \sum_{j=1}^{N} x^T(k-h_j) A_j{}^T P \sum_{j=1}^{N} A_j x(k-h_j) \\ &- x^T(k) P x(k) + x^T(k) \sum_{j=1}^{N} S_j x(k) - \sum_{j=1}^{N} x^T(k-h_j) S_j x(k-h_j)\end{aligned} \tag{45}$$

Based on *Lemma* 2 it follows:

$$\begin{aligned}\Delta V(x_k) \leq &x^T(k) \left[(1+\varepsilon) A_0{}^T P A_0 + \sum_{j=1}^{N} S_j - P \right] x(k) \\ &+ (1+\varepsilon^{-1}) N \sum_{j=1}^{N} x^T(k-h_j) A_j{}^T P A_j x(k-h_j) \\ &- \sum_{j=1}^{N} x^T(k-h_j) S_j x(k-h_j)\end{aligned} \tag{46}$$

$$\begin{aligned}\Delta V(x_k) \le x^T(k)\left[(1+\varepsilon)A_0{}^T PA_0 + \sum_{j=1}^{N} S_j - P\right]x(k) \\ + \sum_{j=1}^{N} x^T(k-h_j)\left[N(1+\varepsilon^{-1})A_j{}^T PA_j - S_j\right]x(k-h)\end{aligned} \tag{47}$$

If one adopts:

$$S_j = N(1+\varepsilon^{-1})A_j{}^T PA_j \tag{48}$$

then:

$$\Delta V(x_k) \le x^T(k)\left[(1+\varepsilon)A_0{}^T PA_0 + N(1+\varepsilon^{-1})\sum_{j=1}^{N} A_j{}^T PA_j - P\right]x(k) \tag{49}$$

Let us define the following function:

$$f(\varepsilon, x(k)) = x^T(k)\left[(1+\varepsilon)A_0{}^T PA_0 + N(1+\varepsilon^{-1})\sum_{j=1}^{N} A_j{}^T PA_j\right]x(k) \tag{50}$$

Since all matrices $A_j{}^T PA_j$, $j = 0,1,\cdots,N$ are symmetric and positive semidefinite then, based on *Rayleigh* and *Amir-Moez* inequalities [76-77]:

$$\begin{aligned}f(\varepsilon, x(k)) &\le x^T(k)\left[(1+\varepsilon)\lambda_{\max}(A_0{}^T PA_0) + N(1+\varepsilon^{-1})\sum_{j=1}^{N}\lambda_{\max}(A_j{}^T PA_j)\right]x(k) \\ &= x^T(k)\lambda_{\max}(P)g(\varepsilon)x(k)\end{aligned} \tag{51}$$

Scalar function:

$$g(\varepsilon) = (1+\varepsilon)\sigma^2_{\max}(A_0) + N(1+\varepsilon^{-1})\sum_{j=1}^{N}\sigma^2_{\max}(A_j) \tag{52}$$

possesses its minimum at:

$$\varepsilon = \varepsilon_{\mathrm{m}} = \left(N\sum_{j=1}^{N}\sigma_{\max}(A_j)\right)^{\frac{1}{2}}\sigma^{-1}_{\max}(A_0) = \left(N\sum_{j=1}^{N}\|A_j\|_2^2\right)^{\frac{1}{2}}\|A_0\|_2^{-1} \tag{53}$$

from where:

$$f(\varepsilon_{\mathrm{m}}, x(k)) \le f(\varepsilon, x(k)) \tag{54}$$

Having in mind that:

$$\Delta V(x_k) \le x^T(k)\left(\lambda_{\max}(P)g(\varepsilon_{\mathrm{m}})I_n - P\right)x(k) \le x^T(k)\left(\lambda_{\max}(P)g(\varepsilon)I_n - P\right)x(k) \tag{55}$$

one can put $\varepsilon_{\min}$ into (49) instead of ε, which t yields:

$$\Delta V(x_k) \le x^T(k)\left[(1+\varepsilon_m)A_0{}^T PA_0 - P + N(1+\varepsilon_m{}^{-1})\sum_{j=1}^{N} A_j{}^T PA_j\right]x(k) \tag{56}$$

If the condition (41) is satisfied then:

$$\Delta V(x_k) \le -\lambda_{\min}\{Q\}\|x(k)\|_2^2 = -\beta\|x(k)\|_2^2 < 0, \quad \beta \triangleq \lambda_{\min}\{Q\} > 0 \tag{57}$$

Likewise, for $x_k \neq 0$ it holds:

$$\begin{aligned}0 < V(x_k) &\le \max\left\{x^T(k)Px(k) + N(1+\varepsilon_m{}^{-1})\sum_{j=1}^{N}\sum_{l=1}^{h_j} x^T(k-l)A_j{}^T PA_j x(k-l)\right\} \\ &\le \lambda_{\max}\{P\}\|x(k)\|_2^2 + N(1+\varepsilon_m{}^{-1})\sum_{j=1}^{N} h_j\lambda_{\max}\{A_j{}^T PA_j\}\|x(k)\|_D^2 \\ &\le \left[\lambda_{\max}\{P\} + N(1+\varepsilon_m{}^{-1})\sum_{j=1}^{N} h_j\lambda_{\max}\{A_j{}^T PA_j\}\right]\|x(k)\|_D^2 \\ &= \alpha\|x(k)\|_D^2 \\ &\alpha \triangleq \lambda_{\max}\{P\} + N(1+\varepsilon_m{}^{-1})\sum_{j=1}^{N} h_j\lambda_{\max}\{A_j{}^T PA_j\} > 0\end{aligned} \tag{58}$$

thus, based on *Theorem* 1, system (11)-(13) is asymptotically stable. **Q.E.D.**

Corollary 2. Discrete-time fractional order systems (6) is asymptotically stable if there exists real symmetric matrix $P>0$ such that

$$\begin{aligned}&(1+\varepsilon_m)\left(A+\alpha I_n\right)^T P\left(A+\alpha I_n\right)+\frac{N(1+\varepsilon_m)}{\varepsilon_m}\sum_{j=1}^{N}\beta_j{}^2(\alpha)P-P<0\\&\varepsilon_m=\left(N\sum_{j=1}^{N}\beta_j{}^2(\alpha)\right)^{\frac{1}{2}}\left\|A+\alpha I_n\right\|_2^{-1}\end{aligned}\tag{59}$$

Theorem 4. [75] The linear discrete time-delay system (11)-(13) is asymptotically stable if there exist real symmetric matrix $P>0$ and scalar $\varepsilon>0$ such that

$$\begin{bmatrix}N(1+\varepsilon)P & N(1+\varepsilon)A_0^TP & N(1+\varepsilon)A_1^TP & \cdots & N(1+\varepsilon)A_N^TP\\ * & NP & 0 & \cdots & 0\\ * & * & \varepsilon P & \cdots & (*)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \cdots & \varepsilon P\end{bmatrix}>0\tag{60}$$

Proof. From(41), for

$$\hat{A}_0=A_0\sqrt{1+\varepsilon},\quad \hat{A}_j=A_j\sqrt{N\left(1+\varepsilon\right)/\varepsilon},\ j=1,2,\cdots,N\tag{61}$$

it follows

$$P-\sum_{j=1}^{N}\hat{A}_j{}^TP\hat{A}_j-\hat{A}_0{}^TP\left(P^{-1}\right)P\hat{A}_0>0,\quad P>0\tag{62}$$

By using Schur complements it is easy to see that the condition (62) is equivalent to

$$\begin{bmatrix}P-\sum_{j=1}^{N}\hat{A}_j{}^TP\hat{A}_j & \hat{A}_0^TP\\ * & P\end{bmatrix}>0,\quad P>0\tag{63}$$

Similarly, condition (63) is equivalent to

$$\begin{bmatrix}P-\sum_{j=2}^{N}\hat{A}_j{}^TP\hat{A}_j & \hat{A}_0^TP & \hat{A}_1^TP\\ * & P & 0\\ * & * & P\end{bmatrix}>0\tag{64}$$

Finally, the condition (62) is equivalent to

$$\begin{bmatrix}P & \hat{A}_0^TP & \hat{A}_1^TP & \cdots & \hat{A}_N^TP\\ * & P & 0 & \cdots & 0\\ * & * & P & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \cdots & P\end{bmatrix}>0\tag{65}$$

By using (61) and pre and post multiply (65) with *blockdiag*$\{I,\ I\sqrt{1/(1+\varepsilon)}\ ,\ I\sqrt{\varepsilon/N(1+\varepsilon)}\ ,\ \ldots\ ,\ I\sqrt{\varepsilon/N(1+\varepsilon)}\ \}$ we obtain

$$\begin{bmatrix}P & A_0^TP & A_1^TP & \cdots & A_N^TP\\ * & \frac{1}{1+\varepsilon}P & 0 & \cdots & 0\\ * & * & \frac{\varepsilon}{N(1+\varepsilon)}P & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \cdots & \frac{\varepsilon}{N(1+\varepsilon)}P\end{bmatrix}>0\tag{66}$$

With $P/(N(1+\varepsilon))$ replaced by P we obtain (60).

Example 1. Let us consider a discrete delay system described by

$$x(k+1) = A_0 x(k) + A_1 x(k-h_1) + A_2 x(k-h_2) \tag{67}$$

$$A_0 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & \alpha \end{bmatrix},\ A_1 = \gamma \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.01 & 0.05 \\ 0.03 & 0.02 \end{bmatrix} \tag{68}$$

where γ is an adjustable parameter and the system scalar parameter α takes the following values: -0.15 and 0.5. To determine the largest parameter γ for various values of ε by Theorem 4, the feasibility of equation (66) with γ as a variable can be cast into a generalized eigenvalue problem

$$\min_{P>0}\ \alpha,\ \gamma = 1/\alpha \tag{69}$$

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ -N(1+\varepsilon)PA_1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} < \alpha \begin{bmatrix} N(1+\varepsilon)P & * & * & * \\ N(1+\varepsilon)PA_0 & NP & * & * \\ 0 & 0 & \varepsilon P & * \\ N(1+\varepsilon)PA_2 & 0 & 0 & \varepsilon P \end{bmatrix} \tag{70}$$

The delay-independent asymptotic stability conditions are characterized by means of the range of parameter γ and are summarized in *Table* 1.

Table 1. Stability Conditions

	Parameter α			
Conditions	- 0.15		+ 0.50	
Theorem 3	$\lvert\gamma\rvert < 1.370$	$\varepsilon = 1$	$\lvert\gamma\rvert < 1.022$	$\varepsilon = 1$
Theorem 3	$\lvert\gamma\rvert < 1.468$	$\varepsilon = \varepsilon_m = 2.066$	$\lvert\gamma\rvert < 1.023$	$\varepsilon = \varepsilon_m = 0.886$
Theorem 4	$\lvert\gamma\rvert < 1.469$	$\varepsilon = \varepsilon_{op} = 2.144$	$\lvert\gamma\rvert < 1.050$	$\varepsilon = \varepsilon_{op} = 0.767$

Corollary 3. Discrete-time fractional order systems (6) is asymptotically stable if there exist a real symmetric matrix $P > 0$ and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} N(1+\varepsilon)P & N(1+\varepsilon)(A+\alpha I_n)^T P & N(1+\varepsilon)\beta_1(\alpha)P & \cdots & N(1+\varepsilon)\beta_N(\alpha)P \\ * & NP & 0 & \cdots & 0 \\ * & * & \varepsilon P & \cdots & (*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \varepsilon P \end{bmatrix} > 0 \tag{71}$$

Theorem 5. [23] The discrete time-delay system (11)-(13) is asymptotically stable if there exist matrices $P > 0$, and $Q_i > 0$, $i = 1, 2, \cdots, N$ such that the following LMI holds:

$$\begin{bmatrix} -P + \sum_{i=1}^{N} Q_i & 0 & \cdots & 0 & A_0^T P \\ * & -Q_1 & \cdots & 0 & A_1^T P \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & -Q_N & A_N^T P \\ * & * & \cdots & * & -P \end{bmatrix} < 0 \tag{72}$$

Proof. Let the Lyapunov functional be:

$$V(k) = x^T(k) P x(k) + \sum_{i=1}^{N} \sum_{j=k-h_i}^{k-1} x^T(j) Q_i x(j) \tag{73}$$

The forward difference along the solutions of system (11) –(12), (14)-(15) is:

$$\Delta V(k)=\left(A_0x(t)+\sum_{j=1}^{N}A_jx(k-h_j)\right)^T P\left(A_0x(t)+\sum_{j=1}^{N}A_jx(k-h_j)\right)$$
$$-x^T(k)Px(k)+\sum_{i=1}^{N}\sum_{j=k+1-h_i}^{k}x^T(j)Q_ix(j)-\sum_{i=1}^{N}\sum_{j=k-h_i}^{k-1}x^T(j)Q_ix(j) \tag{74}$$
$$=\begin{bmatrix}x(k)\\x(k-h_1)\\\vdots\\x(k-h_N)\end{bmatrix}^T\begin{bmatrix}A_0^TPA_0-P+\sum_{i=1}^{N}Q_i & A_0^TPA_1 & \cdots & A_0^TPA_N\\ * & A_1^TPA_1-Q_1 & \cdots & A_1^TPA_N\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & A_N^TPA_N-Q_N\end{bmatrix}\begin{bmatrix}x(k)\\x(k-h_1)\\\vdots\\x(k-h_N)\end{bmatrix}$$

Condition (25) and relation (74) are equivalent to the existence of the variables $P>0$ and $Q_i>0$, $i=1,2,\cdots,N$ satisfying:

$$\begin{bmatrix}A_0^TPA_0-P+\sum_{i=1}^{N}Q_i & A_0^TPA_1 & \cdots & A_0^TPA_N\\ * & A_1^TPA_1-Q_1 & \cdots & A_1^TPA_N\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & A_N^TPA_N-Q_N\end{bmatrix}<0 \tag{75}$$

Further:

$$\begin{bmatrix}A_0^TPA_0-P+\sum_{i=1}^{N}Q_i & A_0^TPA_1 & \cdots & A_0^TPA_N\\ * & A_1^TPA_1-Q_1 & \cdots & A_1^TPA_N\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & A_N^TPA_N-Q_N\end{bmatrix}$$
$$=\begin{bmatrix}-P+\sum_{i=1}^{N}Q_i & 0 & \cdots & 0\\ * & -Q_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & -Q_N\end{bmatrix}-\begin{bmatrix}A_0^TP\\A_1^TP\\\vdots\\A_N^TP\end{bmatrix}\left(-P^{-1}\right)\begin{bmatrix}PA_0 & PA_1 & \cdots & PA_N\end{bmatrix}<0 \tag{76}$$

By using Schur complements it is easy to see that the condition (76) is equivalent to (72). **Q.E.D.**

Corollary 4. Discrete-time fractional order systems (6) is asymptotically stable if there exist matrices $P>0$, and $Q_i>0$, $i=1,2,\cdots,N$ such that the following LMI holds

$$\begin{bmatrix}-P+\sum_{i=1}^{N}Q_i & 0 & \cdots & 0 & (A+\alpha I_n)^TP\\ * & -Q_1 & \cdots & 0 & \beta_1(\alpha)P\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ * & * & \cdots & -Q_N & \beta_N(\alpha)P\\ * & * & \cdots & * & -P\end{bmatrix}<0 \tag{77}$$

Next, starting from (72) we shall develop sufficient conditions for robust stability of the uncertain discrete time-delay system (11)-(12), (14)-(15).

Theorem 6. [23] The uncertain discrete time-delay system (11)-(12),(14)-(15) is asymptotically stable if there exist matrices $P>0$, $Q_i>0$, $i=1,2,\cdots,N$ and a scalar $\lambda>0$ such that the following LMI holds:

$$\begin{bmatrix} -P+\sum_{i=1}^{N} Q_i + \lambda N_{A_0}^T N_{A_0} & \lambda N_{A_0}^T N_{A_1}^T & \cdots & \lambda N_{A_0}^T N_{A_N} & A_0^T P & 0 \\ * & -Q_1 + \lambda N_{A_1}^T N_{A_1} & \cdots & \lambda N_{A_1}^T N_{A_N} & A_1^T P & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -Q_N + \lambda N_{A_N}^T N_{A_N} & A_N^T P & 0 \\ * & * & \cdots & * & -P & PM \\ * & * & \cdots & * & * & -\lambda I \end{bmatrix} < 0 \quad (78)$$

Proof. Replacing A_j, $j = 0,1,\cdots,N$ in (72) with $A_j + \Delta A_j(k) = A_j + MF(k)N_{A_j}$, respectively, for uncertain system (11)-(12),(14)-(15) we find that (72) is equivalent to the following condition:

$$\begin{bmatrix} -P+\sum_{i=1}^{N} Q_i & 0 & \cdots & 0 & A_0^T P \\ * & -Q_1 & \cdots & 0 & A_1^T P \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & -Q_N & A_N^T P \\ * & * & \cdots & * & -P \end{bmatrix} + \begin{bmatrix} N_{A_0}^T \\ N_{A_1}^T \\ \vdots \\ N_{A_N}^T \\ 0 \end{bmatrix} F^T(k) \begin{bmatrix} 0 & 0 & \cdots & 0 & M^T P \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ PM \end{bmatrix} F(k) \begin{bmatrix} N_{A_0} & N_{A_1} & \cdots & N_{A_N} & 0 \end{bmatrix} < 0 \quad (79)$$

By using Lemma 3, we have

$$\begin{bmatrix} -P+\sum_{i=1}^{N} Q_i & 0 & \cdots & 0 & A_0^T P \\ * & -Q_1 & \cdots & 0 & A_1^T P \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & -Q_N & A_N^T P \\ * & * & \cdots & * & -P \end{bmatrix} + \begin{bmatrix} N_{A_0}^T \\ N_{A_1}^T \\ \vdots \\ N_{A_N}^T \\ 0 \end{bmatrix} \lambda \begin{bmatrix} N_{A_0} & N_{A_1} & \cdots & N_{A_N} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ PM \end{bmatrix} \lambda^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & M^T P \end{bmatrix} < 0 \quad (80)$$

By using the Schur complement, from (80) there follows (78). **Q.E.D.**

Corollary 5. The uncertain discrete-time fractional order systems (17),(19)-(20) is asymptotically stable if there exist matrices $P > 0$, $Q_i > 0$, $i = 1,2,\cdots,N$ and a scalar $\lambda > 0$ such that the following LMI holds

$$\begin{bmatrix} -P+\sum_{i=1}^{N} Q_i + \lambda N_A^T N_A & 0 & \cdots & 0 & (A+\alpha I_n)^T P & 0 \\ * & -Q_1 & \cdots & 0 & \beta_1(\alpha)P & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -Q_N & \beta_N(\alpha)P & 0 \\ * & * & \cdots & * & -P & PM \\ * & * & \cdots & * & * & -\lambda I \end{bmatrix} < 0 \tag{81}$$

The next theorem gives sufficient condition for asymptotic stability which is equivalent with condition (78) presented in *Theorem* 6. This condition is suitable one for direct application of LMI in order to derive stabilization conditions of the uncertain system, given by (11)-(12),(14)-(15).

Theorem 7. [23] The uncertain discrete time-delay system (11)-(12),(14)-(15) is *asymptotically stable* if there exist matrices $L>0$, $W_i>0$, $i=1,2,\cdots,N$ and a scalar $\eta>0$ such that the following LMI holds:

$$\begin{bmatrix} -L+\sum_{i=1}^{N} W_i & 0 & \cdots & 0 & LA_0^T & N_{A_0}^T \\ * & -W_1 & \cdots & 0 & LA_1^T & N_{A_1}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -W_N & LA_N^T & N_{A_N}^T \\ * & * & \cdots & * & -L+\eta MM^T & 0 \\ * & * & \cdots & * & * & -\eta I \end{bmatrix} < 0 \tag{82}$$

Proof. By using the Schur complement, from (80) for $\eta=\lambda^{-1}$ it follows

$$\begin{bmatrix} -P+\sum_{i=1}^{N} Q_i & 0 & \cdots & 0 & A_0^T P & N_{A_0}^T \\ * & -Q_1 & \cdots & 0 & A_1^T P & N_{A_1}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -Q_N & A_N^T P & N_{A_N}^T \\ * & * & \cdots & * & -P+\eta PMM^T P & 0 \\ * & * & \cdots & * & * & -\eta I \end{bmatrix} < 0 \tag{83}$$

By pre and post multiplying (83) by $diag\left\{P^{-1},\cdots,P^{-1},I\right\}$ and applying the change of variables such that $L \triangleq P^{-1}>0$ and $W_i \triangleq P^{-1}Q_i P^{-1}>0$, $i=1,2,\cdots,N$ we obtain the LMI condition (82). **Q.E.D.**

Corollary 6. The uncertain discrete-time fractional order system (17),(19)-(20) is asymptotically stable if there exist matrices $L>0$, $W_i>0$, $i=1,2,\cdots,N$ and a scalar $\eta>0$ such that the following LMI holds:

$$\begin{bmatrix} -L+\sum_{i=1}^{N} W_i & 0 & \cdots & 0 & L(A+\alpha I_n)^T & N_A^T \\ * & -W_1 & \cdots & 0 & L\beta_1(\alpha) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ * & * & \cdots & -W_N & L\beta_N(\alpha) & 0 \\ * & * & \cdots & * & -L+\eta MM^T & 0 \\ * & * & * & * & * & -\eta I \end{bmatrix} < 0 \tag{84}$$

4.3.1.2 Delay-dependent stability

Consider the system (11)-)(13). Characteristic polynomial of system (11)-(13) is given by:

$$f(\lambda) \hat{=} \det M(\lambda) = \sum_{j=0}^{N} a_j \lambda^j, a_j \in R, \quad M(\lambda) = I_n \lambda^{h_N+1} - \sum_{j=0}^{N} A_j \lambda^{h_N - h_j} \tag{85}$$

Denote by

$$\Omega \hat{=} \{ \lambda \mid f(\lambda) = 0 \} \tag{86}$$

the set of all characteristic roots of system (11)-(13). A root λ_m of Ω having maximal module is:

$$\lambda_m \in \Omega : |\lambda_m| = \max|\lambda| \tag{87}$$

let us call maximal root (eigenvalue). If scalar variable λ in the characteristic polynomial is replaced by matrix $X \in \mathbb{C}^{n\times n}$ the two following monic matrix polynomials are obtained

$$M(X) = X^{h_N+1} - \sum_{j=0}^{N} A_j X^{h_N - h_j} \tag{88}$$

$$F(X) = X^{h_N+1} - \sum_{j=0}^{N} X^{h_N - h_j} A_j \tag{89}$$

A matrix $S \in \mathbb{C}^{n\times n}$ is a *right solvent* of $M(X)$ if

$$M(S) = 0 \tag{90}$$

If

$$F(R) = 0 \tag{91}$$

then $R \in \mathbb{C}^{n\times n}$ is a *left solvent* of $M(X)$ [78]. We will further use *S* to denote the right solvent and *R* to denote left solvent of $M(X)$.Let F^* denotes conjugate transponse of matrix $F \in \mathbb{C}^{n\times n}$.

Remark 3. For discrete-time fractional order linear system (6) we have: $h_1 = 1, h_2 = 2, \cdots, h_N = N$ and by using (88) and (89) we can define the following matrix polynomials:

$$M(X) = X^{N+1} - \sum_{j=0}^{N} A_j X^{N-j} = X^{N+1} - (A + \alpha I_n) X^N - \beta_1(\alpha) X^{N-1} - \ldots - \beta_{N-1}(\alpha) X - \beta_N(\alpha) I_n \tag{92}$$

$$F(X) = X^{N+1} - \sum_{j=0}^{N} X^{N-j} A_j = X^{N+1} - X^N (A + \alpha I_n) - \beta_1(\alpha) X^{N-1} - \ldots - \beta_{N-1}(\alpha) X - \beta_N(\alpha) I_n \tag{93}$$

In the present paper, majority of the presented results starts from the left solvents of $M(X)$. In contrast, in the existing literature the right solvents of $M(X)$were mainly studied. The mentioned discrepancy can be overcome by the following lemma.

Lemma 4. The conjugate transpose value of the left solvent of $M(X)$ is also, at the same time, the right solvent of the following matrix polynomial

$$M_T(X) = X^{h_N+1} - \sum_{j=0}^{N} A_j^T X^{h_N - h_j} \tag{94}$$

Proof. Let *R* be the left solvent of $M(X)$. Then it holds

$$M_T(R^*) = (R^*)^{h_N+1} - \sum_{j=0}^{N} A_j^T R^{h_N - h_j} = \left(R^{h_N+1} - \sum_{j=0}^{N} R^{h_N - h_j} A_j \right)^* = F^*(R) = 0 \tag{95}$$

so R^* is the right solvent of $M_T(X)$. **Q.E.D**

Remark 4. Based on *Lemma* 4, all characteristics of the left solvents of $M(X)$ can be obtained by the analysis of the conjugate transpose value of the right solvents of $M_T(X)$.

The following proposed factorization of the matrix $M(\lambda)$ will help us to better understand the relationship between eigenvalues of the left and right solvents and roots of the system.

Lemma 5. The matrix $M(\lambda)$ can be factorized in the following way

$$M(\lambda)=(\lambda I_n - S)\cdot\left(\sum_{i=0}^{h_N}\lambda^{h_N-i}S^i - \sum_{j=0}^{N-1}\sum_{i=0}^{h_N-h_j-1}\lambda^{h_N-h_j-1-i}S^i A_j\right) \tag{96}$$

Proof.

$$\begin{aligned} M(\lambda)-M(X) &= \lambda^{h_N+1}I_n - X^{h_N+1} - \sum_{j=0}^{N-1}\left(\lambda^{h_N-h_j}I_n - X^{h_N-h_j}\right)A_j \\ &= (\lambda I_n - X)\left(\sum_{i=0}^{h_N}\lambda^{h_N-i}X^i - \sum_{i=0}^{h_N-h_j-1}\lambda^{h_N-h_j-1-i}X^i\sum_{j=0}^{N-1}A_j\right) \end{aligned} \tag{97}$$

If S is a right solvent of $M(X)$, from (97), there follows (96). **Q.E.D**

Remark 5. From (85) and (96),there follows $f(S)=f(R)=0$, e.g. the characteristic polynomial $f(\lambda)$ is *annihilating polynomial* for the right and left solvents of $M(X)$. Therefore, $\lambda(S)\subset\Omega$ and $\lambda(R)\subset\Omega$ holds.

Eigenvalues and eigenvectors of the matrix have a crucial influence on the existence, enumeration and characterization of the solvents of the matrix equation (90) [78].

Definition 3. Let $M(\lambda)$ be a matrix polynomial in terms of λ. If $\lambda_i\in\mathbb{C}$ is such that $\det(M(\lambda_i))=0$, then we say that λ_i is a *latent root* or an *eigenvalue* of $M(\lambda)$. If a nonzero $v_i\in\mathbb{C}^n$ is such that

$$M(\lambda_i)v_i = 0 \tag{98}$$

then we say that v_i is a (right) *latent vector* or a (right) *eigenvector* of $M(\lambda)$, corresponding to the eigenvalue λ_i [78].

Eigenvalues of matrix $M(\lambda)$ correspond to the characteristic roots of the system, i.e. eigenvalues of its block companion matrix A_a [78]. Their number is $n(h_N+1)$. Since $F^*(\lambda)=M_T(\lambda^*)$ holds, it is not difficult to show that matrices $M(\lambda)$ and $M_T(\lambda)$ have the same spectrum.

In papers [78] some sufficient conditions for the existence, enumeration and characterization of the right solvents of $M(X)$ were derived. They show that the number of solvents can be *zero*, *finite* or *infinite*. For the needs of system stability (11) only the so called maximal solvents are usable, whose spectrums contain maximal eigenvalue λ_m. A special case of the maximal solvent is the so called dominant solvent which, unlike maximal solvents, can be computed in a simple way [78].

Definition 4. Every solvent S_m of $M(X)$, whose spectrum $\sigma(S_m)$ contains maximal eigenvalue λ_m of Ω is a *maximal solvent*.

Definition 5. Matrix A dominates matrix B if all the eigenvalues of A are greater, in modulus, than those of B. In particular, if the solvent S_1 of $M(X)$ dominates the solvents $S_2,\ldots,S_l$ we say it is a *dominant solvent*. [78].

The dominant solvent S_1 of $M(X)$, under certain conditions, can be determined by the *Traub* and *Bernoulli iteration*. The necessary and sufficient conditions for asymptotic stability of linear discrete time-delay systems (11) are to follow.

Theorem 8. [35] Suppose that there exists at least one maximal left solvent of $M(X)$ and let R_m denotes one of them. Then, linear discrete time-delay system (11)-(13) is *asymptotically stable* if and only if there exists symmetric matrix $P^*=P>0$ such that

$$R_m^* P R_m - P < 0 \tag{99}$$

Proof. *Sufficient condition*.

Define the following vector discrete functions

$$x_k = x(k+\theta),\ \theta\in\{-h_N, -h_N+1,\ldots,0\} \tag{100}$$

$$z(x_k) = x(k) + \sum_{j=1}^{N}\sum_{l=1}^{h_j} T_j(l)\,x(k-l) \tag{101}$$

where, $T_j(k) \in \mathbb{C}^{n\times n}$, $j = 1,2,\cdots,N$ are, in general, some time-varying discrete matrix functions. The conclusion of the theorem follows immediately after defining Lyapunov functional for the system (11)-(13) as

$$V(x_k) = z^*(x_k)\,P\,z(x_k),\ P = P^* > 0 \tag{102}$$

It is obvious that $z(x_k) = 0$ if and only if $x_k = 0$, so it follows that $V(x_k) > 0$ for $\forall x_k \neq 0$. The forward difference of (102), along the solutions of system (11)-(13) is

$$\Delta V(x_k) = \Delta z^*(x_k)\,Pz(k) + z^*(x_k)\,P\Delta z(x_k) + \Delta z^*(x_k)\,P\Delta z(x_k) \tag{103}$$

A difference of $\Delta z(x_k)$ can be determined in the following manner

$$\Delta z(x_k) = \Delta x(k) + \sum_{j=1}^{N}\sum_{l=1}^{h_j} T_j(l)\,\Delta x(k-l) \tag{104}$$

with

$$\Delta x(k) = (A_0 - I_n)\,x(k) + \sum_{j=1}^{N} A_j\,x(k-h_j) \tag{105}$$

and

$$\sum_{l=1}^{h_j} T_j(l)\,\Delta x(k-l) = T_j(1)\,x(k) - T_j(h_j)\,x(k-h_j) + \sum_{l=1}^{h_j-1} \Delta T_j(l)\,\Delta x(k-l) \tag{106}$$

Therefore:

$$\begin{aligned}\Delta z(x_k) = &\left(A_0 - I_n + \sum_{j=1}^{N} T_j(1)\right)x(k) + \sum_{j=1}^{N}\left(A_j - T_j(h_j)\right)x(k-h_j)\\ &+ \sum_{j=1}^{N}\sum_{l=1}^{h_j-1}\Delta T_j(l)\,x(k-l)\end{aligned} \tag{107}$$

Define a new matrix *R* by

$$R \triangleq A_0 + \sum_{j=1}^{N} T_j(1) \tag{108}$$

If

$$\Delta T_j(h_j) = A_j - T_j(h_j) \tag{109}$$

then $\Delta z(x_k)$ has a form

$$\Delta z(x_k) = (R - I_n)\,x(k) + \sum_{j=1}^{N}\sum_{l=1}^{h_j}\Delta T_j(l)\,x(k-l) \tag{110}$$

If one adopts

$$\Delta T_j(l) = (R - I_n)\,T_j(l),\quad l = 1,2,\ldots,h_j \tag{111}$$

then $\Delta z(x_k)$ becomes

$$\Delta z(x_k) = (R - I_n)\left(x(k) + \sum_{j=1}^{N}\sum_{j=1}^{h_j} T_j(l)\,x(k-l)\right) = (R - I_n)\,z(x_k) \tag{112}$$

Therefore, (103) becomes

$$\Delta V(x_k) = z^*(x_k)\left(R^*PR - P\right)z(x_k) \tag{113}$$

It is obvious that if the following inequality is satisfied $R^*PR - P < 0$ then $\Delta V(x_k) < 0, x_k \neq 0$. In the Lyapunov matrix inequality (99), of all possible solvents *R* of $M(X)$, only one of the maximal solvents is of importance, since it is the only one that contains maximal eigenvalue $\lambda_m \in \Omega$ (*Remark* 5), which has dominant

influence on the stability of the system. Therefore, (99) represent stability *sufficient condition* for system given by (11)-(13). The maximal left solvent R can be determined as follows. From (109) and (111) follows:

$$T_j(l+1) = RT_j(l), \quad T_j(h_j) = R^{h_j-1}T_j(1), \ 1 \le j \le N \tag{114}$$

$$A_j - T_j(h_j) = \Delta T_j(h_j) = (R - I_n)T_j(h_j)R, \quad RT_j(h_j) = A_j \tag{115}$$

By combining the two last equations, one can obtain the following system of matrix equations

$$R^{h_j}T_j(1) = A_j, \ 1 \le j \le N \tag{116}$$

By using (108) and (116), we have:

$$R^{h_N} \cdot / \ 0 = R - \left(A_0 + \sum_{j=1}^{N} T_j(1)\right) \Rightarrow R^{h_N+1} - R^{h_N}A_0 - \sum_{j=1}^{N} R^{h_N-h_j}R^{h_j}T_j(1) = 0 \tag{117}$$

Therefore:

$$R_m{}^{h_N+1} - \sum_{j=0}^{N} R_m{}^{h_N-h_j}A_j = 0 \tag{118}$$

Necessary condition. If the system (11)-(13) is asymptotically stable then all roots $\lambda_i \in \Omega$ are located within unit circle. Since $\sigma(R_m) \subset \Omega$, there follows $\rho(R_m) < 1$, so the positive definite solution of Lyapunov matrix equation (99) exists. **Q.E.D.**

Example 2. Let us consider linear discrete system with delayed state (11)-(13) with

$$A_0 = \begin{bmatrix} 0.01 & 0 \\ 0.2 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad h_1 = 2, \quad h_2 = 10 \tag{119}$$

From (118) follows

$$R^{11} - R^{10}A_0 - R^8A_1 - A_2 = 0 \tag{120}$$

In this case the maximal solvent coincides with the dominant solvent

$$R_m = R_1 = \begin{bmatrix} 0.9621 & 0.0347 \\ 0 & 0.9768 \end{bmatrix} \tag{121}$$

The same solution can be obtained by using Traub or Bernoulli algorithm. So $\rho(R_1) = 0.9768 < 1$ and based on *Theorem* 8 the system is asymptotically stable.

Corollary 7. Suppose that there exists at least one maximal solvent of (93) and let R_m denote one of them. Then, discrete-time fractional order systems (6) is *asymptotically stable* if and only if there exists symmetric matrix $P^* = P > 0$ such that

$$R_m^* P R_m - P < 0 \tag{122}$$

4. 4 Conclusion

This paper proposes a simple stability and robust stability analysis for the linear discrete-time fractional order systems. It was shown that some results obtained for the stability of the discrete time-delay systems can be applied to the discrete fractional order state-space system. Two classes of stability criteria are derived: delay-independent and delay-dependent. The delay-dependent criteria are given in the form of the necessary and sufficient conditions. The approach is based on the idea of constructing novel Lyapunov–Krasovskii functionals combined with free-weighting matrices or algebraic methods.

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Part III

“Modeling”

Modeling of Human Skin Using Distributed Order Fractional Derivative Model-Frequency Domain

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Abstract: - Bioelectro-mechanical properties of the skin can be very valid data for analyzing fractal characteristics of skin structure. In this paper, bioelectrical impedance method and fractional calculus have been utilized for characterization of the human skin. Our fractional model presents the generalized continuous Cole model which can predict structural – functional parameters as a lot of Cole complex relaxation times. It is shown using the proposed model and experimental results that these parameters depend on the fractional indexes as a degree of the fractional derivatives in the interval (0,1). These relaxation time constants correspond to structural – functional characteristics of the skin. The integral approximation of continuous fractional Cole model was done from ten points corresponding to orderly connected known reduced Cole elements. It was observed that five reduced Cole elements had significant values of corresponding relaxation times. Lastly, the advantages of the proposed model are discussed.

Key-Words: - skin layer, fractional calculus, frequency analysis, viscoelastic, electric impedance.

5.1 Introduction

Bioelectro-physical properties of human skin tissue, like most other soft tissues, exhibit electroviscoelastic behavior. To obtain complete information about the electroviscoelastic behavior of the human skin, it is also necessary to have experimental data over a wide range of time scales.

If electricity is applied from an external source outside the living organism, we can measure bio-impedance. To analyze skin impedance effectively, it is very desirable to introduce the skin impedance model. Also, the complex modulus concept is a powerful and widely used tool for characterizing the electroviscoelastic behavior of materials in the frequency domain. In this case, according to the proposed concept, bioimpendance moduli can be regarded as complex quantities.

In the *BIS* technique impedance (or, without the geometric factors, such as surface of electrode and distance between them, complex specific resistance of material) or admittance (as a capacitor, or , without the geometric factors, complex permittivity), measurements are done at each frequency, and then are plotted forming a circular arc [1], [2]. Using electrical engineering modeling mathematics, the points on a circular arc can be transformed into an equivalent electrical model, where the values correspond to specific compositional elements. Also, mathematically, the fractional integro-differential operators (*fractional calculus*) represent a generalization of integration and derivation to non-integer order (fractional) operators.

On the other hand, a memory function equation, scaling relationships and structural–fractal behavior of biomaterials and, here, the mathematical model based on fractional calculus, were used for the physical interpretation of the Cole-Cole and Cole-Davidson exponents [3] with permittivity. As it is well-known, three expressions for the complex permittivity allow one to describe a wide range of experimental data: Cole–Cole function, Cole–Davidson function and Havriliak–Negami function ([1], [3], [4],[5]). Impedance material properties (complex electrical resistance) also describe, in many cases, the relations of the same type. Both of these behaviors (i.e. admittance, impedance), in the approximation of linear isotropic and homogeneous Maxwell's medium, are functionally connected.

According to literature data, at low frequencies [1], the skin is usually observed as a simple structure, and equivalent electrical model of the skin doesn't include tissue lamination. In this paper, we propose the skin structure as a more complex system consisting of several layers (see Appendix D). We obtain the mathematical model of skin structure applying fractional calculus, which describes series of structures via new generalizing the Cole equation. According to this model and experimental data of the skin bioimpedance measurements, one may predict more complex equivalent electrical circuit. In approximation, the new continuous single-pole Cole model, shown here, better describes electrical behavior of human skin as continuum skin layers. The new parameter is introduced at the same time, characterizing the width of the interval in which there are a lot of continuous electrical Cole elements.

There is a wide range of models in literature [6],[7], due to numerous factors involved in tissue behavior under various conditions. Generally, three expressions for the impedance allow one to describe a wide range of experimental data: Cole function, Cole–Davidson function and Havriliak –Negami function [3],[4],[5],[6]. To obtain complete information about the electrical behaviour of human skin, it is also necessary to have experimental data over a wide range of time scales. In one of the effective experimental methods, the bio–impedance spectroscopy (*BIS*), which we used, measurements are done at each frequency. Each is plotted forming a circular arc and then is transformed into an equivalent electrical model where the values correspond to specific compositional elements, [8]. Usefulness of BIS is in that it determines the characteristic time constants of the bio-electrical processes in the tissues. Recently, fractional differential equations (*FDE*s) have been the focus of many studies due to their appearance in various fields, such as physics, chemistry, biophysics, engineering control theory, bioengineering, etc. in [6],[7],[9][10],[11]. Mathematically, the fractional integro-differential operators (*fractional calculus*) represent a generalization of integration and derivation to non-integer order (fractional) operators. Here, we suggest the mathematical model of skin structure applying fractional calculus, which describes series of structures *via continuous* generalizing (distributed order type) the Cole equation,[12]. The basic hypothesis is that bioelectrical skin properties have been described as the serial connection impedance and continually many reduced Cole elements that correspond to infinitely many functional-structural skin characteristics. According to this model and experimental data of the skin bioimpedance measurements, one may predict more complex equivalent electrical circuit and define new time parameters which correspond to each reduced Cole element.

5.2 Distributed Order Type Fractional Derivative Model of Impedance

5.2.1 Some basic results related to dielectric properties of materials

In [13], it is shown that the capacitive component of the polarization admittance-conductivity–dielectric information is a proper electrical component to monitor the material as an insulator or semiconductor. The electrical impedance method was used as a quantitative technique for evaluating changes in the skin. Dielectric information, in general, can be presented in a number of equivalent ways and it is important to use the most appropriate form of presentation to suit particular requirements. The following principal dielectric functions can be defined:

(a) the complex permittivity ε*(ω) and susceptibility χ*(ω),

$$\chi^*(\omega) = \left[\varepsilon^*(\omega) - \varepsilon_\infty\right]/\varepsilon_0 = \chi'(\omega) - j\chi''(\omega), \qquad j^2 = -1 \tag{1}$$

where ε_0 is the permittivity of free space, and ε_∞ is a suitable high-frequency permittivity contributing to the real and imaginary components of the polarization. So, Debye (D), Cole–Cole (C-C), Cole-Davidson (C-D) and Havriliak–Negami (G-H) dispersion functions are presented as follows:

$$\chi^*(\omega)\big|_D = \frac{\chi_0}{1 + j\cdot\omega/\omega_p}, \qquad \chi^*(\omega)\big|_{C-C} = \frac{\chi_0}{1 + \left(j\cdot\omega/\omega_p\right)^\alpha},$$
$$\chi^*(\omega)\big|_{C-D} = \frac{\chi_0}{\left(1 + j\cdot\omega/\omega_p\right)^\nu}, \qquad \chi^*(\omega)\big|_{G-N} = \frac{\chi_0}{\left(1 + \left(j\cdot\omega/\omega_p\right)^\alpha\right)^\nu} \tag{2}$$

Here, χ_0 is constant, $\omega_p = 1/\tau$ is the loss peak frequency, τ denotes characteristic damped time, $0 < \alpha, \nu \le 1$. The experimental data show that α and ν are strictly dependent on temperature, structure, composition and other controlled physical parameters [13]. However, until recently the reasons underlying such dependencies have not been clear ,[14]. The α and ν were discussed as the parameters of the distribution of the relaxation times, or mentioned as broadening parameters without further discussion. For $\alpha = 1$ in the Cole-Cole function one can obtain the Debye function (Eq.(2)). The Cole-Cole equation described by means of permittivity [15] is

$$\varepsilon^* = \varepsilon_\infty + \frac{\varepsilon_S - \varepsilon_\infty}{1 + \left(j\cdot\omega\cdot\tau\right)^{1-\alpha}} \tag{3}$$

where ε_s is the static permittivity of material. One of them, the Cole impedance-complex resitivity model, was introduced in its final form by [16], introducing constant phase element (*CPE*). In [6], as shown by *CPE,* in the equivalent fractional circuit diagrams, (Fig.1):

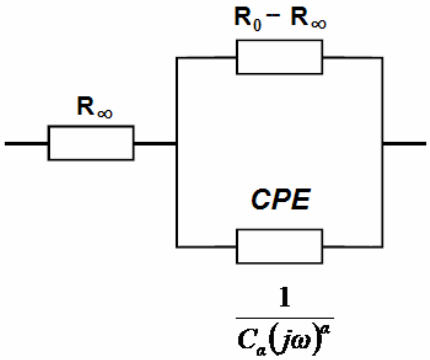

Fig. 1. Equivalent circuit single-pole Cole model for calculating the electrical impedance of the skin

The papers [1],[4] present the circuit which was used to model the biomaterials, and after some adaptation it will be applied for this study

$$\underline{Z}_{\alpha}(\omega) = R_{\infty} + \frac{R_0 - R_{\infty}}{1 + (j \cdot \omega \cdot \tau_{\alpha})^{\alpha}} \tag{4}$$

where R_0 denotes a low-frequency resistor and R_∞ is a high-frequency resistor, α is fractional *CPE* exponent-index, τ_α is characteristic relaxation time. In the paper [6] the expression for τ_α is given as follows, (C_α is a fractional order capacitance).

$$\tau_{\alpha} = \sqrt[\alpha]{(R_0 - R_{\infty}) \cdot C_{\alpha}} \tag{5}$$

All mentioned models are used to describe the behavior of biological tissue. Material constant C_α in the case of living tissue is determined indirectly, through the time τ_α.

5.2.2 Basic facts related to bio-impedance of human skin

Epithelia are cells organized as layers, the skin is an example. Cells in epithelia form gap junctions. Particularly, in tight membranes these junctions are special tight junctions. The transmembrane admittance is dependent on both the type of cell junctions and extent to which the epithelium is shunted by channels or specialized organs (e.g. sweat ducts in the skin).

The impedance of the skin is dominated by the stratum corneum at low frequencies. It has generally been stated that skin impedance is determined mainly by the stratum corneum at frequencies below 10 kHz and by the viable skin at higher frequencies [17]. This will of course be dependent on factors like skin hydration, electrode size and geometry, etc. but may nevertheless serve as a rough guideline. A finite element simulation on a concentric two-electrode system used by [1] showed that the stratum corneum accounted for about *50%* of the measured skin impedance at *10 kHz*, but only about *10%* at *100 kHz,* [18].

Stratum corneum can have the thickness from about 0.01 mm to 1 mm or more under the foot. The stratum corneum is a solid state substance, not necessarily containing liquid water, but with moisture content dependent on the surrounding air humidity. It is not soluble in water, but the surface will be charged and a double layer will be formed in the water side of the interphase. Stratum corneum can absorb large amounts of water (e.g. doubling its weight). Stratum corneum may be considered as a solid state electrolyte, perhaps with few ions free to move and contribute to DC conductance. The stratum corneum contains such organic substances as proteins and lipids, which may be highly charged but bound, and therefore contributing only to AC admittance. An open question is whether the conductance in stratum corneum, in addition to the ionic component, also has an electronic component (e.g. as a semiconductor).While the impedance of the stratum corneum is much higher at low frequencies than the impedance of the living skin, the differences in dispersion mechanisms make the electrical properties converge as the frequency is increased. This is the main reason why increased frequency, in general, leads to measurements at deeper layers in the skin.Living tissue is considered as a dispersive medium because both permittivity and conductivity are functions of frequency. In paper [19] three major dispersions, α-, β-, and γ-dispersion, were identified and named. Later, in work [20] a forth dispersion was identified and named δ- dispersion. The α-dispersion takes place at low frequencies, *10 Hz – 10 kHz*. Although all elements contributing to this frequency dependence have not been clearly identified yet, in [21] three main causes were established. First, the effect of the endoplasmic reticulum contributes to this frequency dependence. Second, the channel proteins present in the plasma membrane also cause the frequency-dependent conductance. Finally, the relaxation of counter-ions on the charged cellular surface is another mechanism that produces this frequency dependence. Authors, [22] established the range of β-dispersion from approximately *10 kHz* to *100 kHz*. It is caused by the low conductivity and capacitivity properties of the cell membrane and other internal membrane structures and their interactions with the extra and intra-cellular electrolytes. This paper will consider the human skin impedance dispersion in the area of frequency mostly covered by α-, β-dispersion.

5. 3 Distributed Order Type Fractional Derivative Model Of Impedance

5.3.1 Fractional calculus preliminaries

5.3.1.1 Basic definitions

Let $f \in L\left(\left[t_0, t_1\right]\right)$ and $\alpha > 0$, ([11],[23]), so that, in the left Riemann-Liouville integral of *f(t)* of fractional order α which is:

$$ {}^{RL}_{t_0}I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-t')^{\alpha-1} f(t')\, dt' \tag{6} $$

where $\Gamma(.)$ is the well-known Euler's gamma function. For the initial moments $t_0 = -\infty$ usually refers to integral as *a left Weyl fractional integral* of the order $\alpha \in (0,1]$. Also, *left Riemann-Liouville* and *Caputo derivative* of $f(t)$ of the order α, can be presented as follows, $\alpha \in [0,1)$:

$$ {}^{RL}_{t_0}D_t^{\alpha} f(t) \equiv \frac{d}{dt}\left({}^{RL}_{t_0}I_t^{1-\alpha} f(t)\right) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} (t-t')^{-\alpha} f(t')\, dt' \tag{7} $$

$$ {}^{C}_{t_0}D_t^{\alpha}\left(f(t)\right) := {}_{t_0}I_t^{1-\alpha} f'(t) \tag{8} $$

In the case $t_0 = -\infty$, expression (7) i.e (8) represents *a left Weyl fractional derivative* (in turn, *Riemann-Liouville-Weyl* and *Caputo-Weyl derivative*). Apart from linearity and derivative of the constant being zero, a left Caputo-Weyl fractional derivative has the following characteristics [7], [11],[23], that are used in this paper $f(t) = C \cdot \exp(p \cdot t),\ \Re(p) = 0,\ p = j \cdot \omega$

$$ {}^{CW}_{-\infty}D_t^{\alpha} f(t) = (p)^{\alpha} \cdot f(t) \tag{9} $$

In addition, one may obtain ([11] ,[23])

$$ {}^{CW}_{-\infty}D_t^{0} f(t) := f(t) \tag{10} $$

The initial conditions problems of fractional differential equations, which were compared to the given fractional derivatives, were also considered and presented in [24]. In line with the recent work, if the the input or output system are known as the paper case, it is possible to calculate physically acceptable initialization function. The Caputo derivative was used as the initial moment $t_0 = 0$, but was not usable for distant initial moments t_0=-∞, as Caputo-Weyl's, which was used to describe harmonic processes in this work.

5.3.1.2 Distributed Caputo derivatives and integrals

The left-sided fractional Caputo derivative of absolute continuous function $\varphi(t)$ of the order $\alpha \in (0,1)$, $T > 0$, is defined by the following relation:

$$ \left({}^{C}_{t_0}D_t^{\alpha}\varphi\right)(t) := \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \cdot \varphi'(\tau) \cdot d\tau,\ t_0 < t \le T,\ t_0 \le 0 \tag{11} $$

Let $C(t),\ t \in (0,1]$ be the continuous non-negative function different from zero on a set of positive measure. The distributed Caputo fractional derivative [25] is then defined by the relation:

$$ \left({}^{C}_{t_0}D_t^{C}\varphi\right)(t) := \int_{t_0}^{t} k(t-\tau) \cdot \varphi'(\tau) \cdot d\tau \tag{12} $$

where

$$ k(s) := \int_{0}^{1} \frac{s^{-\alpha}}{\Gamma(1-\alpha)} \cdot C(\alpha) \cdot d\alpha,\ s > t_0 \tag{13} $$

Since $\varphi(t)$ is absolute continuous function, it holds:

$$\frac{d}{dt}\int_{t_0}^{t} k(t-\tau)\cdot\varphi(\tau)\cdot d\tau = \frac{d}{dt}\int_{t_0}^{t} k(s)\cdot\varphi(t-s)\cdot ds = k(t)\cdot\varphi(0) + \int_{t_0}^{t} k(s)\cdot\varphi'(t-s)\cdot ds \tag{14}$$

respectively,

$$\left({}_{t_0}^{C}D_t^{C}\varphi\right)(t) = \frac{d}{dt}\int_{t_0}^{t} k(t-\tau)\cdot\varphi(\tau)\cdot d\tau - k(t)\cdot\varphi(0) \tag{15}$$

Consider only the case when the derivative of integral of the preceding equation right-side makes sense and it is the basis for further considerations. Also, it is necessary to define the Laplace transform $t_0 = 0$ that is used in this work:

$$k_L(p) = \int_0^{\infty} k(s)\cdot e^{-p\cdot s}\cdot ds,\ \mathrm{Re}(p) > 0 \tag{16}$$

It is logical to consider the inverse operation of distributed Caputo derivative. Let $\left({}_0^{C}D_t^{C}u\right)(t) = f$, $u(0) = 0$. Applying formally the Laplace transform, $u \to u_L$ will be $u_L = f_L(p)/\left(p\cdot k_L(p)\right)$. Asymptotic behavior of $k_L(p)$ was considered in [26] and, based on this paper, via the Laplace transform $1/\left(p^2\cdot k_L(p)\right)$, the kernel of the function $\kappa(t)$ is obtained:

$$\kappa(t) = \frac{1}{2\cdot\pi\cdot j}\int_{\gamma-j\cdot\infty}^{\gamma+j\cdot\infty}\frac{e^{p\cdot t}}{p}\cdot\frac{1}{p\cdot k_L(p)}\cdot dp,\ \gamma > 0 \tag{17}$$

The preceding formula is used to define the distributed integral via the convolution operation as follows:

$$\left({}_0^{C}I_t^{C}f\right)(t) := \int_0^{t}\kappa(t-s)\cdot f(s)\cdot ds \tag{18}$$

Previously described derivatives and integrals are well-known in literature as well as those based on Riemann-Liouville fractional operators. Furthermore, there follow new formulations of the corresponding operators and statements that are useful in considerations below.

5.3.1.3 Distributed Caputo-Weyl derivatives and integrals

We will consider here the distributed and, similar to them, fractional derivatives on a set of functions $\varphi(t)\in L^2\left((-\infty,\infty)\right)$ or on their suitable subset if $\omega > 0$ that the Caputo-Weyl fractional derivative ($t_0 = -\infty,\ \alpha\in(0,1)$) will be applied to.

$$\left({}_{-\infty}^{CW}D_t^{\alpha}\varphi\right)(t) := \int_{-\infty}^{t} f(t-\tau,\alpha)\cdot\varphi'(\tau)\cdot d\tau,\ -\infty < t \le T \tag{19}$$

where

$$f(t-\tau,\alpha) := \begin{cases}\dfrac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} & t\ge\tau\\ 0 & t<\tau\end{cases} \tag{20}$$

Certainly, there could have remained the condition that the function belongs to those absolute continuous functions that the Caputo Weyl fractional derivative makes sense for (where it can be defined). Additionally, the Caputo-Weyl fractional derivative can be represented as follows:

$$\left({}_{-\infty}^{CW}D_t^{\alpha}\varphi\right)(t) := \int_{-\infty}^{\infty} f(s,\alpha)\cdot\varphi'(t-s)\cdot ds,\ -\infty < t \le T \tag{21}$$

with the condition

$$f(s,\alpha)=\begin{cases}\dfrac{s^{-\alpha}}{\Gamma(1-\alpha)} & s\geq 0\\ 0 & s<0\end{cases} \tag{22}$$

i.e. if $m(s)$ is defined here as:

$$m(s):=\int_0^1 f(s,\alpha)\cdot C(\alpha)\cdot d\alpha \tag{23}$$

It follows that the corresponding Caputo-Weyl distributed derivative can be represented in the following form:

$$\left({}_{-\infty}^{CW}D_t^C\varphi\right)(t):=\int_{-\infty}^{\infty} m(\tau)\cdot\varphi'(t-\tau)\cdot d\tau \tag{24}$$

Also, the Fourier transform of *m(s)* is

$$m_F(\omega)=\int_{-\infty}^{\infty} m(s)\cdot e^{-2\cdot\pi\cdot j\cdot s\cdot\omega}\cdot ds \tag{25}$$

Let $\left({}_{-\infty}^{CW}D_t^C w\right)(t)=g$. Applying the Fourier transform $w\to w_F$ it follows that $w_F=g_F(\omega)/\left(j\cdot\omega\cdot m_F(\omega)\right)$. Then the kernel of the function $\mu(t)$ has the form:

$$\mu(t)=\int_{-\infty}^{\infty}\frac{e^{2\cdot\pi\cdot j\cdot t\cdot\omega}}{j\cdot\omega\cdot m_F(\omega)}\cdot d\omega \tag{26}$$

such that the distributed Caputo-Weyl integral can be represented in the form as follows:

$$w(t)=\int_{-\infty}^{\infty}\mu(t-s)\cdot g(s)\cdot ds \tag{27}$$

Analogous to distributed fractional derivatives, it is possible to define the corresponding integrals too. The Riemann-Liuoville distributed integral:

$${}_{t_0}^{RL}I_t^{\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-t')^{\alpha-1}f(t')dt' \tag{28}$$

Let $C(t),\ t\in(0,1]$ be the continuous non-negative function, different from zero, on a set of positive measure. The infinitely distributed Caputo fractional derivative of negative degree is then defined by the relation:

$$\left(\left({}_{t_0}^{C}D_t^C\right)^{-1}\varphi\right)(t):=\int_{t_0}^{t}k(t-\tau)\cdot\varphi'(\tau)\cdot d\tau \tag{29}$$

where

$$k(s):=\int_0^1\frac{s^{\alpha}}{\Gamma(1-\alpha)}\cdot C(\alpha)\cdot d\alpha,\ s>t_0, \tag{30}$$

Let $C(t),\ t\in(0,1]$ be the continuous non-negative function, different from zero, on a set of positive measure. Then

$$\left({}_{t_0}^{RL}I_t^C\varphi\right)(t):=\int_{t_0}^{t}k(t-\tau)\cdot\varphi(\tau)\cdot d\tau \tag{31}$$

where

$$k(s):=\int_0^1\frac{s^{\alpha-1}}{\Gamma(\alpha)}\cdot C(\alpha)\cdot d\alpha,\ s>t_0 \tag{32}$$

In the case that $t_0\to-\infty$ holds, for $f(t)=\exp(j\cdot\omega\cdot t)$

$$ {}^{RL}_{-\infty}I_t^{\alpha} f(t) = {}^{CW}_{-\infty}D_t^{-\alpha} f(t) \tag{33} $$

Also, in that case Eqs (29) and (31) describe the same operator.

5.3.2 Cole and distrubeted order Cole element

In case of relaxation in the electric circuit consisting of parallel connected resistor *R* and *CPE*, the suitable fractional differential equation is

$$ C_{\alpha} \cdot {}^{C}_{0}D_t^{\alpha} V(t) + V(t)/R = 0, V(0) = V_0, \quad \sqrt[\alpha]{R \cdot C_{\alpha}} = \tau_{R\alpha} \tag{34} $$

where voltage on *CPE* element was marked with *V(t)*, and *V(0)* presents given initial condition. The solution is given as

$$ V(t) = V_0 \cdot \sum_{k=0}^{\infty} \frac{\left(-(t/\tau_{R\alpha})^{\alpha}\right)^k}{\Gamma(k \cdot \alpha + 1)} = V_0 \cdot E_{\alpha}\left(-(t/\tau_{R\alpha})^{\alpha}\right), \tag{35} $$

where $E_{\alpha}(t)$ denotes Mittag-Leffler's function. If we connected the complex alternating - oscillating voltage to the same electric circuit in the shape of $V_0 \exp(j(\omega t+\varphi))$, the Weyl derivative can be used, (V_0 is the voltage amplitude, ω is the source frequency, and φ is the phase angle between the voltage and the current). Then, if the dependence of the electric current of amplitude i_0 on time is introduced as $i(t) = i_0 exp(j\omega t)$, it yields

$$ \begin{aligned} & i(t) = C_{\alpha} \cdot {}_{-\infty}D_t^{\alpha} V(t) + V(t)/R, \\ & i_0 = V_0\left(C_{\alpha} \cdot (j \cdot \omega)^{\alpha} + 1/R\right)e^{j\theta} = \frac{V_0 e^{j\theta}}{\underline{Z}}, \end{aligned} \tag{36} $$

where $\underline{Z}$ is a complex impedance of the system. Introducing the sign "| |" for the parallel connection of complex resistance, we can write

$$ \begin{aligned} & \underline{Z} = R \,||\, C_{\alpha} \cdot (j \cdot \omega)^{\alpha} = R/\left((j \cdot \tau_{\alpha} \cdot \omega)^{\alpha} + 1\right), \quad \underline{Z} = |\underline{Z}| e^{j \cdot \theta}, \\ & \cos\theta = R/|\underline{Z}|. \end{aligned} \tag{37} $$

Then, the (4) which describes the electric Cole circuit influenced by the aforementioned alternating voltage actually models the system consisting of orderly connection of resistance R_{∞} and reduced Cole element $(R_0 - R_{\infty})||C_{\alpha}(j\omega)^{\alpha}$

$$ \underline{Z}_{\alpha}(\omega) = R_{\infty} + (R_0 - R_{\infty}) || \frac{1}{(j \cdot \omega)^{\alpha} \cdot C_{\alpha}} \tag{38} $$

The basic assumptions for which this generalization is done are that there are neither inductive resistances, nor active elements, connected serially or in parallel. In that case, the skin is, in the electric sense, taken as serially continually many connected linear, reduced Cole elements $R_{\alpha}||C_{\alpha}(j\omega)^{\alpha}$ and one R_{∞} (Fig.2). Resistance R_{α} can be presented as $R_{\alpha}=p(\alpha)(R_0 - R_{\infty})$ and characterized each individual reduced Cole element, where $p(\alpha)$ is a function of α. The equivalent total impedance $\underline{Z}_{C;c}$ of this new electric circuit is given by the equation

$$ \underline{Z}_{C;c}(\omega) = R_{\infty} + (R_0 - R_{\infty}) \cdot \int_{0+}^{1} \frac{p(\alpha) \cdot d\alpha}{1 + (j\omega\tau_{\alpha})^{\alpha}}, \; \tau_{\alpha} = \sqrt[\alpha]{p(\alpha) \cdot (R_0 - R_{\infty}) \cdot C_{\alpha}} \tag{39} $$

or, this expression (39) is *the continuous Cole generalization equation*s (distributed order model) , where $\tau_{\alpha} = \tau'_{\alpha} + j\tau''_{\alpha}$, $0 < \alpha \le 1$ are the corresponding complex time constants. In the characteristic case $\alpha = 1$, the relaxation constant $\tau_1 = \tau'_1 + j \cdot \tau''_1$ is in correlation with complex dielectric constant. For more details

about the concept of a complex characteristic time, see [27]. In our work, the cases $\alpha \neq 1$ correspond to the analogous fractional processes in the skin.

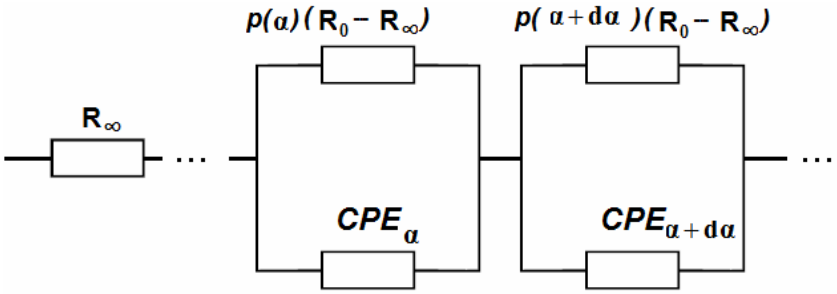

Fig. 2. Electrical continuum, distributed order model of the skin, based on the Cole equation, *p(α)* is a fraction of $(R_0 - R_\infty)$.

The above expression (39) can have its geometrical representation, i.e. the following integral is considered:

$$\underline{Z}_{C;c,N} = \frac{\underline{Z}_{C;c} - R_\infty}{(R_0 - R_\infty)} = \int_{0+}^{1} \frac{p(\alpha)\cdot d\alpha}{1+(j\cdot\omega\cdot\tau_\alpha)^\alpha} \tag{40}$$

where $p(\alpha)$ and τ_α are complex functions. Geometrically, in the $\left(y_1 = Re(\underline{y})\right) - O - \left(y_2 = Im(\underline{y})\right)$ plane the previous expression represents the continuous development-integral in coordinate system $y' = y'(\alpha)$ i $y'' = y''(\alpha)$

$$y' = \frac{\mathrm{Re}\left(\underline{Z}_{C;c,N}\right)_\alpha}{p(\alpha)},\ y'' = \frac{\mathrm{Im}\left(\underline{Z}_{C;c,N}\right)_\alpha}{p(\alpha)},\ p(\alpha) \neq 0 \tag{41}$$

respectively, for $\tau_\alpha = |\tau_\alpha| \cdot \exp(j \cdot \arg\tau_\alpha)$ and the central angle $\varphi = \alpha \cdot (\pi/2 + \arg\tau_\alpha)$ one obtains circular arcs with the centers at points T*(1/2, tgφ/2)* and radii

$$r_\alpha = \frac{\sqrt{1+tg^2\varphi}}{2} \tag{42}$$

The sides of the mentioned central angle that determine the circular arc make semi-straight lines from the vertex T through the coordinate origin and point (1,0). An example of a circular arc for $\alpha = 0.7$ is presented in Fig.3.

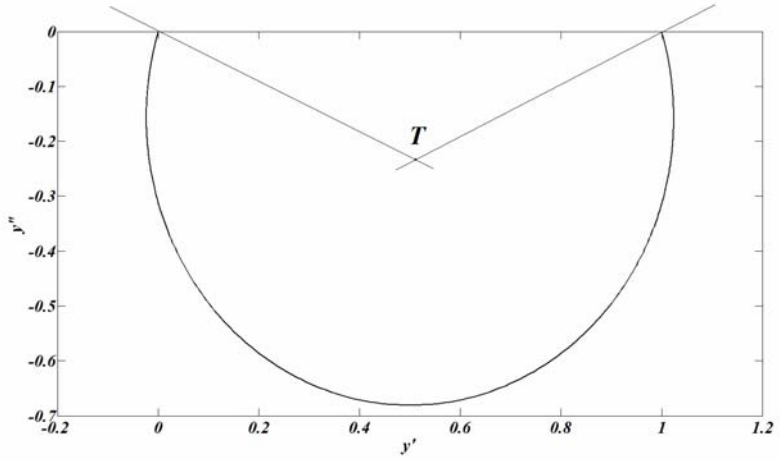

Fig.3. The graph of the function $\underline{y(\omega)} = \left(1+(j\cdot\omega\cdot\tau_\alpha)^\alpha\right)^{-1}$ for $\alpha = 0.7$ i $\tau_\alpha = 10{+}20\cdot\mathrm{j}$.

To prove this statement, it is necessary to consider the locus of points described by the expression for the fixed value of index α:

$$\underline{y(\omega)} = \frac{1}{1+(j\cdot\omega\cdot\tau_\alpha)^\alpha} \tag{43}$$

Here, it is also introduced a new complex structure constants

$$K(\alpha) = (R_0 - R_\infty)\cdot C_\alpha \cdot p(\alpha) \tag{44}$$

$$\underline{Z}_{C;c} = R_\infty + (R_0 - R_\infty)\cdot \int_{0+}^{1} \frac{p(\alpha)d\alpha}{1+K(\alpha)\cdot(j\cdot\omega)^\alpha} \tag{45}$$

Mathematically, Eqs. (39) and (45) correspond to the application of continually many derivatives, which have not been distributed (the application of concept of the distributed derivatives on oscillating movement can be found in [28]. For $(R_0 - R_\infty) \to \infty$, there will be

$$\underline{Z}_{C;c} = R_\infty + \int_{0+}^{1} \frac{d\alpha}{C_\alpha (j\omega)^\alpha}, \tag{46}$$

which corresponds to Caputo-Weyl distributed derivatives (Eqs. (29) and (31))

$$i(t) = V(t)/R_\infty + \left({}^{CW}_{-\infty}D_t^c\right)^{-1}(V(t)) = V(t)/R_\infty + \int_{0+}^{1}\frac{1}{C_\alpha}\cdot\left({}^{CW}_{-\infty}D_t^{-\alpha}(V(t))\right)d\alpha. \tag{47}$$

Our case, due to $R_0 - R_\infty \neq \infty$, describes one generalization of Weyl distributed derivatives (integrals). On the other hand, if

$$\frac{p(\alpha)}{1+K(\alpha)\cdot(j\cdot\omega)^\alpha} = \sum_{i=1}^{n}\frac{p(\alpha_i)}{1+K(\alpha_i)\cdot(j\cdot\omega)^{\alpha_i}}\cdot\delta(\alpha-\alpha_i),\ 0<\alpha_i\le 1 \tag{48}$$

then, (45) changes to

$$\underline{Z} = R_\infty + (R_0 - R_\infty)\cdot\sum_{i=1}^{n}\frac{p(\alpha_i)}{1+K(\alpha_i)\cdot(j\cdot\omega)^{\alpha_i}} \tag{49}$$

and represents discrete series of Cole elements. For numerical calculations, we can obtain the Cole equation (45) in a discrete form (the integral approximation by the integral sum of five members) as follows

$$\underline{Z}_{C;c}(\omega_l) \approx R_\infty + (R_0 - R_\infty)\cdot\sum_{i=1}^{5}\frac{p(0.2\cdot i)\cdot 0.2}{\left(1+K(0.2\cdot i)\cdot(j\cdot\omega_l)^{0.1i}\right)},\ i,l = 1,\ldots,5 \tag{50}$$

The structure parameters $K(i) \equiv K(0.2\,i)$ and $p(i)$ are calculated from the corresponding system of the ten nonlinear equations, where we used $R_\infty \approx R_{\omega=100KHz}, R_0 \approx R_{\omega=0.1Hz}$. Numerical calculations are done using suitable the least squares method in MATLAB environment.

5.3.3 Materials and methods

The proposed experimental method uses a two-electrode technique with a constant amplitude sinusoidal voltage. The skin of the upper arm impedance mesaurement was carried out in healthy young men. The electrodes were made of stainless steel, 2.0 *cm* and the distance betwen the electrodes was 5.0 *cm*. The electrode paste used was a cream (*EC 33* skin conductance). The measuring system Solartron *1255* Frequency Response Analyser in combination with Solatron *1286* Pstat/Gstat was used for measuring the components of impedance and characteristic frequency of the skin in the frequency range of *0.1 Hz* to *100.0 KHz.*

Fig. 4. Solartron *1255* Frequency Response Analyser in combination with Solartron *1286* Pstat/Gstat

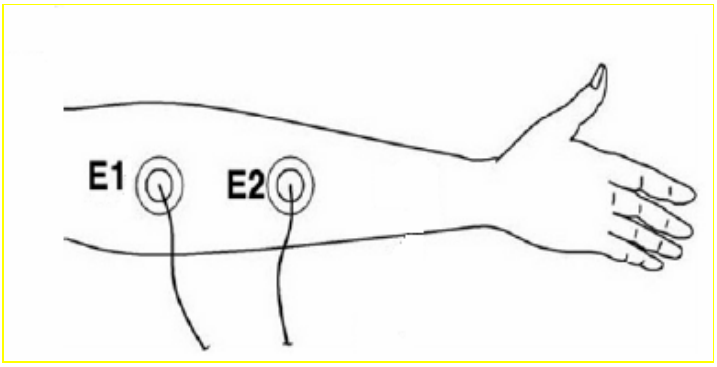

Fig. 5. Two - electrode system in impedance measurement by constant amplitude sinusoidal voltage method.

Measurements were taken at *61* different frequencies for twenty young men, frequencies between *0.1 Hz* and *100.0 KHz* and the applied voltage was of the amplitude *0.1 V*. Experimental data were processed using the *Zview software*. Calculated impedance error in the population for given frequencies is *2%*. For this analysis, we had ten random impendances and the total required time for the frequncy sweep measurement was about 10 minutes.

5.3.4 Results and discussion

Consider an assembly of Cole systems like the one described in [1]. This electrical equivalent corresponds to the object under investigation, when doing impedance measurements on human skin. Measured data may represent contributions from electrode polarization, stratum corneum, sweat ducts and deeper tissue, and furthermore several dispersions of some of these components. Only one Cole system is shown for the electrode polarization, although two dispersions have been found in some studies [29]. The stratum corneum is dominated by a single broad dispersion [8] and the sweat ducts may exhibit dispersion due to countering relaxation [4]. In work [8], two orderly connected *CPEs* with indices of about 0.71 and relaxation times of 0.001s were observed. It was assumed that the relaxation times were positive real numbers. Also, in [30] they are suggested and proposed layer models (series layer model, parallel layer model, brick layer model,…) as well as effective medium models. Here, in relation to our experimental *in vivo* conditions, the structure and complexity of the considered system - human skin, we decided to have its electrical behavior described by the (continuum) series layer model. In addition, we will consider the *Zview software*, the chi-square goodness-of-fit test normally distributed random changes in various standard deviations for each measured impedance frequency and determine the acceptability of this model, which can be one of the techniques for establishing its relevance [31], [32]. Against, in the above stated approximation of integrals it was carried out where the interpolation coefficients are determined *K(i)* and *p(i)*. In the beginning of the discussion we shall present the basic experimental results on the population of twenty volunteers at 22 ^{0}C and 50% relative humidity. The

module of impedance decreases with increasing frequencies. The interval of values of modules of impedance is from *1 KΩ* to *1 M Ω*. The minimum of angle theta is at frequency of around *500 Hz* (see Fig. 6.)

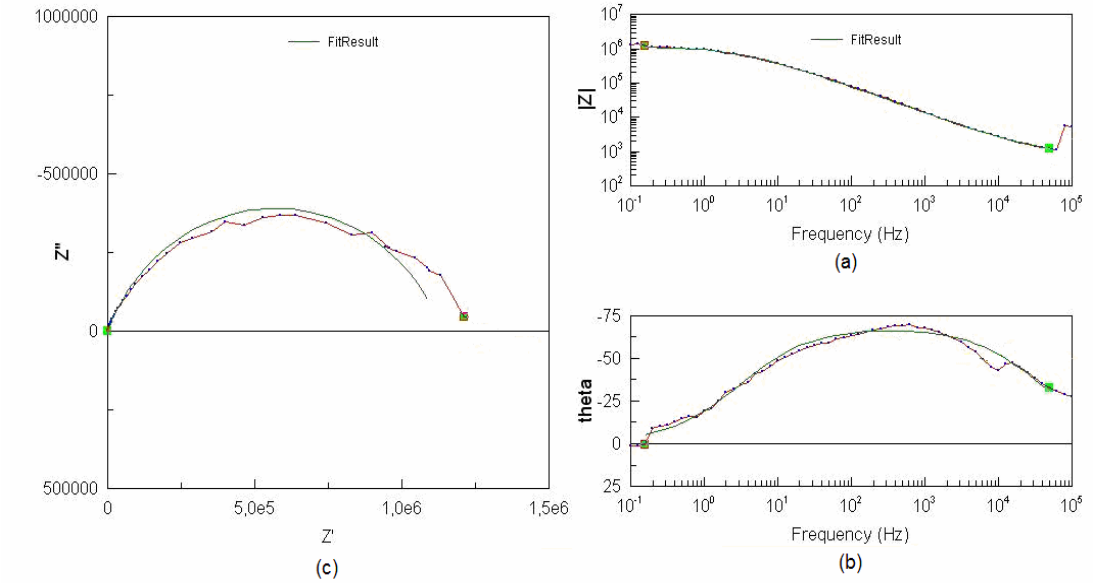

Fig. 6. Bioimpedance of human skin $Z'-Z''$, *Zview software*,(c); Amplitude characteristic (a) and phase characteristic (b) of impedance of human skin, voltage $V_0=0.1$V, diameter of eletrodes is *2 cm*

In our work, with *Zview software* (*61* different frequencies) we predicted one Cole element with parameters $\alpha=0.766(0.003)$ *(0.5%)*, $\tau=9.2(0.3)10^{-8}s(3\%)$, $(R_0-R_\infty)=1.24(0.02)$ *MΩ(2%)* and $R_\infty=0.82(0.04)k\Omega(4\%)$. Approximation error with impedance for each frequency is greater than *6%*.

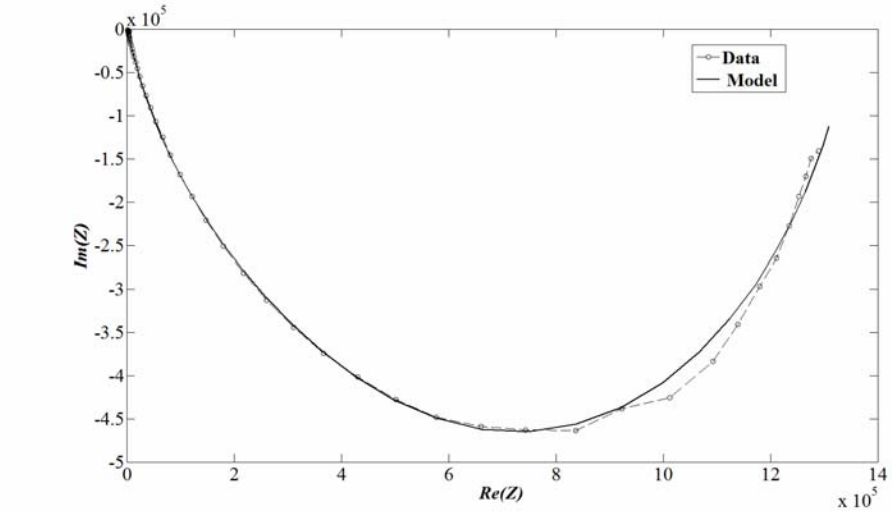

Fig. 7. $\mathrm{Re}(Z)-\mathrm{Im}(Z)$ plot for model being presented in this work,voltage amplitude $V_0=0.1$V, diameter of eletrodes is *2 cm,* distance between electrodes is *5 cm.*

Results of numerical solving of the (18) are: $R_\infty=0.72k\Omega$, $(R_0 - R_\infty)=1.27M\Omega$ and they are represented in Fig. 7. and Table 1.

Table 1 Calculated values $\tau(\alpha)$, $p(\alpha)$

α	$\tau(\alpha)(s)$	$p(\alpha)(s)$
0.2	-0.0052 - 0.0077 · j	0.0674 + 0.0592 · j
0.4	0.4660 + 3.6670 · j	-3.5138 - 2.3752 · j
0.6	-0.0047 - 0.0002 · j	-0.0012 + 0.0001 · j
0.8	-0.0165 - 0.0185 · j	-1.0832 - 0.4607 · j
1.0	1.2572 + 2.3425 · j	2.0076 + 6.0802 · j

In the MATLAB environment, the proposed program based on the least squares method quickly calculated the required 486 iterations to a solution. Approximation of integral of sum is obtained with five members and it is *4%* better (the sum of squares is $7.72 \cdot 10^9$) than in *Zview software (*the sum of squares is $8.05 \cdot 10^9$*)*.

5.4 Conclusion

In our paper, it is suggested the mathematical model of skin structure applying fractional calculus, which describes series of structures *via continuous* generalizing (distributed order type) the Cole equation. In that way, our fractional model can predict structural – functional parameters as a lot of Cole complex relaxation times. The main conclusion of our work is that continuous fractional single-pole Cole model, even the integral approximation of continuous fractional Cole model describes much better the electrical behavior of the human skin than using single-pole Cole model. This creates the basis for more precise analysis of the bioimpedance behavior of the skin based on continuous approximation. Also, it is shown using the proposed model and experimental results that these parameters depend on the fractional indexes as a degree of the fractional derivatives in the interval (0,1). Last, it is shown the integral approximation of the distributed order Cole model, in the sense of the proposed least squares method, is better than the *Zview software* by about *4 %.*

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A Thermodynamically Consistent Rheological Model for Engineering Applications

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Abstract: - In this chapter, it is our intention to convince the reader that a modified Zener model of viscoelastic body, i.e. standard fractional linear viscoelastic body, comprising both fractional derivatives of stress and strain and the restrictions on the coefficients that follow from Clausius Duhem inequality, seems to be a very tractable object in engineering applications. The principal advantages of the model are twofold. First, it takes an energy dissipation *ab initio*, and secondly, it can be used for rheological description of both new high performance materials, such as elastomers/polymers, as well as different biological tissues. In the following, the differences in approach to the existing models are highlighted. Then, after introducing the model, several rheological experiments that connect the model to real materials are listed. Four constants of the model will provide descriptions for a sufficiently wide range of engineering/bioengineering applications. We start with a viscoelastic compliant contact impact model that is followed by forced oscillations problem with fractional type of dissipation pattern. A simplified earthquake dynamics of a column-like structure is also examined. The model is also tractable within problems where geometric nonlinearity exists, as shown when the airplane landing problem is considered with dynamics represented by a single integral equation involving Mittag-Leffler-type function, whose solution is ensured by the fixed point theorem. Finally, the model can be coupled with nonsmooth mechanical problems. The impact problem is re-examined with dry friction as a multifunction, causing physical nonlinearity, and leading to a multivalued fractional differential equation that is solved by the slack variable algorithm in dealing with discontinuous motion phases. This time, the existence result ensuring the contractible solution set of the corresponding integro-differential inclusion will be used.

Key-Words: - Fractional Calculus, Standard Fractional Linear Viscoelastic Body, Clausius-Duhem Inequality, Forced Oscillations, Seismic Load, Impact, Geometric Nonlinearity, Dry Friction, Energy Dissipation, Discontinuous System Behaviors, Slack Variable, Multivalued Fractional Differential Equations, Integro-Differential Inclusions

6.1 Introduction

The new tendency in engineering favors the design of systems incorporating new high performance materials. Because of today's concern for liability, the use of such systems requires thorough knowledge of their physical properties. Thus, it is widely accepted that engineering innovations must be exhaustively tested and analytically proven to a degree unknown in the past. In doing so, a special attention is paid to energy loss within a particular system representing a major part in its modeling. Namely, the low cost of numerical versus expensive experimental simulations can be used to study a wide variety of loadings, materials and geometries before proceeding to its construction. Engineering models are usually generated by fundamental physical and geometrical principles. Initially, the number of unknown variables is not the same as the number of equations expressing the principles, so the model should be complemented with an appropriate set of constitutive equations. Among the variety of all possible choices the question that is posed at that stage is which rheological description of the system component contain enough information on the physical properties of the system to allow accurate predictions of its behavior. We here discuss the constitutive model known as the modified Zener model of viscoelastic body, [2] that comprises fractional derivatives of stress and strain, emphasizing the restrictions which the second law of thermodynamics places on the coefficients of the model. It seems that the

model that takes energy dissipation *ab initio* is very tractable in applications dealing with impact or oscillatory motions within mechanical systems.

The simplest models of mechanical systems are formed of rigid bodies in the ideal environment. Rigidity causes simplicity which is a desirable property but at the same time significantly narrows the class of problems that can be dealt with. The same is valid for an assumption of ideal constraints. For example, a physical pendulum will oscillate forever unless the dry friction is included. Note that if one uses viscous friction in the model, the motion ceases in infinite time. A problem of a rigid sphere in translatory motion impinging two rigid spheres at rest and in contact, even can not be well posed within the rigid body mechanics, [59]. So, we need models to understand nature. Thus, in order to improve the rigid body model, we introduce a deformable layer around a rigid core [58]. For simplicity, we deal only with isothermal and uniaxial deformation. A simple shear deformation pattern will appear later.

It is known that the Hooke law, describing strain (relative elongation) ε and stress σ, force per unit area of the body in the undeformed state, also called Piola-Kirchhoff stress, reads

$$\sigma = E\varepsilon \tag{1}$$

where the constant $E > 0$ is called the modulus of elasticity. Despite the fact that it is used in most of theories, it has never been experimentally proved. Namely, it is worth noting that the experiments of three hundred years have demonstrated amply for every solid substance examined with sufficient care that the strain resulting from small applied stress is not a linear function thereof, see [10], p.155. This leads us to viscoelastic layers.

A usual form of the rheological model corresponding to Kelvin-Voigt viscoelastic body reads

$$\sigma = E_1\varepsilon + E_1\tau_{\varepsilon 1}\varepsilon^{(1)}, \tag{2}$$

where $(\cdot)^{(1)} = d(\cdot)/dt$ denotes the first derivative with respect to time t, and E_1 and $\tau_{\varepsilon 1}$ are constants called the modulus of elasticity and strain relaxation time, respectively. This model has a chance to be suitable for specific applications. For example, two constants of the model are examined for several different types of vehicles, so it can be used in crash analysis and accident reconstructions, see [34]. However, if we analyze a real deformation stimulus pattern shown in Fig.1

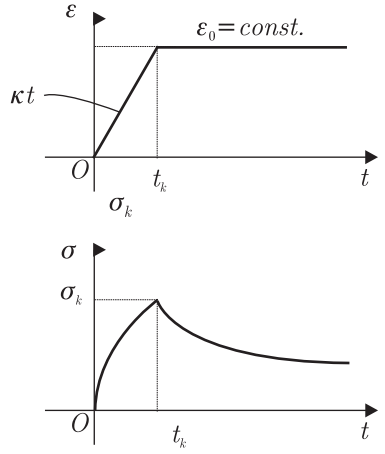

Fig. 1 The real stimulus pattern.

i.e. a ramp-and-hold strain, then stress relaxation as an inherent property of real materials, see [66] or [64]. Namely, after the loading phase with initial strain rate $\kappa = const.$ ends, and strain reaches a constant level ε_0 at instant t_k, leading to stress level σ_k, we note that there is no jump in stress-time diagram. It can be easily seen that the Kelvin-Voit model does not fit for the real deformation stimulus shown in Fig. 1.

The one that fits to Fig. 1 is so called standard linear viscoelastic body, i.e. the Zener model that has constitutive relation (stress-strain relation) in the form

$$\sigma + \tau_{\sigma 1}\sigma^{(1)} = E_1\varepsilon + E_1\tau_{\varepsilon 1}\varepsilon^{(1)}, \tag{3}$$

where the stress rate $\sigma^{(1)}$ and a constant called stress relaxation time $\tau_{\sigma 1}$ are introduced. Note, however, that there exists a fundamental restriction on the coefficients in (3) that follows from the second law of thermodynamics,

$$E_1 > 0, \;\; \tau_{\sigma 1} > 0, \;\; \tau_{\varepsilon 1} > \tau_{\sigma 1}, \tag{4}$$

as proposed in [9] and [2]. Although the Kelvin-Voigt model $\tau_{\sigma 1} = 0$ does not satisfy the restrictions (4) it could be used for certain materials and certain types of deformations, see [54]. Also, note that there will be no damping if $\tau_{\sigma 1} = \tau_{\varepsilon 1}$ in (3), see [24].

Starting from the virginal state $\sigma(0) = 0, \;\; \varepsilon(0) = 0,$ and applying the Laplace transform with

$$\bar{\sigma} = \bar{\sigma}(s) = \mathcal{L}\{\sigma(t)\} = \int_0^\infty e^{-st}\sigma(t)\;dt, \; \bar{\varepsilon} = \bar{\varepsilon}(s) = \mathcal{L}\{\varepsilon(t)\} = \int_0^\infty e^{-st}\varepsilon(t)\;dt, \tag{5}$$

from (3) one gets

$$\bar{\sigma} = \frac{1+\tau_{\varepsilon 1}s}{1+\tau_{\sigma 1}s}\bar{\varepsilon}. \tag{6}$$

The inverse transform of (6) yields the following stress-strain relation

$$\sigma(t) = \frac{\tau_{\varepsilon 1}}{\tau_{\sigma 1}}\varepsilon(t) + \frac{1}{\tau_{\sigma 1}}\left(1 - \frac{\tau_{\varepsilon 1}}{\tau_{\sigma 1}}\right)\int_0^t \varepsilon(\eta)\; e^{-\frac{t-\eta}{\tau_{\sigma 1}}}d\eta, \tag{7}$$

which for

$$\varepsilon(t) = \begin{cases} \kappa t \;\; 0 \le t \le t_k \\ \\ \varepsilon_0 = const. \; t > t_k \end{cases} \tag{8}$$

can be used to recover the behavior shown in Fig.1. Note that in (7) the history of deformation is taken into account.

To describe specific class of viscoelastic materials equations of the type (3) have been generalized by replacing the first derivative that appear in (3) with the fractional derivatives. In the 80s of the former century, in doing so, Bagley and Torvik proved experimentally that this generalization of (3) correctly describes the behavior of over 150 viscoelastic materials. Recently, it has been shown that it fits for several biological tissues too, see [49] and [19] for example, so it represents a strong candidate for a reliable rheological model, we are going to consider next. It should be noted that the price we pay for better rheological descriptions is the use of a slightly different mathematical tool.

6.2 The modified Zener model

Let us introduce $\alpha-$ the derivative, $0<\alpha<1,$ of a function $u(t)$ in the standard Riemann-Liouville form

$$\frac{d^{\alpha}}{dt^{\alpha}}u(t)=u^{(\alpha)}\equiv\frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{u(\xi)\ d\xi}{(t-\xi)^{\alpha}},$$

where Γ is the Euler gamma function, then the fractional derivative type generalization of (3), usually called a modified Zener body, was taken in the form

$$\sigma+\tau_{\sigma\alpha}\sigma^{(\alpha)}=E_{\alpha}\varepsilon+E_{\alpha}\tau_{\varepsilon\alpha}\varepsilon^{(\alpha)}, \tag{9}$$

where $0<\alpha<1,$ and $\tau_{\sigma\alpha}$, $\tau_{\varepsilon\alpha}$ and E_{α} are constants. The dimension of the relaxation constants $\tau_{\sigma\alpha}$, and $\tau_{\varepsilon\alpha}$ is time to the power of $\alpha.$ The constant E_{α} is usually given in MPa. The constitutive equation (9) describes uniaxial, isothermal deformation of the viscoelastic body of negligible mass, together with fundamental restrictions on the coefficients of the model that follow from the Clausius-Duhem inequality, see [9] and [2],

$$E_{\alpha}>0,\ \tau_{\sigma\alpha}>0,\ \tau_{\varepsilon\alpha}>\tau_{\sigma\alpha}. \tag{10}$$

Note that in (9) the nonlocal derivatives are used. Also, note that in the special case $\alpha=1,$ relations (9) and (10) reduce to (3) and (4) respectively. In that case, a conventional tool of mathematical analysis can be used.

Applying the Laplace transform once again with (5), from (9), also starting from virginal state, one gets

$$\bar{\sigma}=E_{\alpha}\frac{1+\tau_{\varepsilon\alpha}s^{\alpha}}{1+\tau_{\sigma\alpha}s^{\alpha}}\bar{\varepsilon}, \tag{11}$$

where the standard expression for the Laplace transform of $u^{(\alpha)}$ was used, that is,

$$\mathcal{L}\{u^{(\alpha)}\}=s^{\alpha}\bar{u}-\left[\frac{1}{\Gamma(1-\alpha)}\left(\int_0^t\frac{u(\eta)\ d\eta}{(t-\eta)^{\alpha-1}}\right)\right]\Bigg|_{t=0}, \tag{12}$$

with $\mathcal{L}\{u(t)\}=\bar{u}=\bar{u}(s),$ and where the integral in brackets vanish since $\lim_{t\to0^+}u(t)$ is bounded, see [9]. Following the standard preparatory procedure, as shown by Mainardi and Gorenflo in [41], the inversion of (11) yields

$$\sigma=\sigma(t)=E_{\alpha}\varepsilon(t)+\nu\mathcal{L}^{-1}\left\{\frac{s^{\alpha}}{\lambda+s^{\alpha}}\bar{\varepsilon}(s)\right\}, \tag{13}$$

where λ and ν denote $1/\tau_{\sigma\alpha}$ and $E_{\alpha}\left(\tau_{\varepsilon\alpha}/\tau_{\sigma\alpha}-1\right),$ respectively.

With this preparation done, we are ready to examine at least five types of rheological experiments. The classical stress-relaxation and creep experiments, with $\varepsilon=\varepsilon_0=const.$ and $\sigma=\sigma_0=const.$ respectively, for the generalized Zener model, were analyzed in [2] and applied for the O-Ring Sealing (Fluor-Elastomer), see [7], yielding the following values of the four constants in the model (11).

$$\alpha=0.490,\ E_{\alpha}=7.150,\ \tau_{\sigma\alpha}=0.945,\ \tau_{\varepsilon\alpha}=135.01.$$

For comparison, four constants describing human root dentin, obtained also from the experiments of the same type reported in [37], reads

$$\alpha = 0.136,\ E_\alpha = 350.7,\ \tau_{\sigma\alpha} = 0.525,\ \tau_{\varepsilon\alpha} = 0.762,$$

see [49] for details. Note that inertial limitations of physical testing devices prevent instantaneous strain applications and very fast ramp times are intractable due to the issues such as overshoot, vibration, and poorly approximated strain histories, see [64]. Thus, the classical stress-relaxation experiments with the stress behavior of a tissue measured in response to an instantaneous (Heaviside or step) strain application can not be considered as a real deformation stimulus, see [66]. Examples of the experiments with a constant strain rate type, i.e. $\varepsilon = \kappa t$, and $\kappa = const.$, and the stretching in steps type, with $\varepsilon = [t/\chi]$, where $\chi = const.$, and $[\cdot]$ denoting the maximal integer less then t/χ, can be found in various reports, and will not be considered here. The fifth type, the ramp-and-hold strain, then the stress relaxation experiment, with an input

$$\varepsilon(t) = \begin{cases} \varepsilon = \kappa t, \ \text{for} \ 0 \le t \le t_k, \\ \\ \varepsilon = \varepsilon_0 = \kappa t_k = const., \ \text{for} \ t \ge t_k. \end{cases} \tag{14}$$

with $\kappa = const.$ as the initial strain rate, is the most recognized type and will be used here, see Fig. 1.

Rewriting (14) in the following form

$$\varepsilon(t) = \kappa\left[t \cdot h(t) - (t - t_k)\ h(t - t_k)\right], \tag{15}$$

where $h(t)$ stands for the Heaviside step function, and applying the Laplace transform to (15), one obtains

$$\bar{\varepsilon}(s) = \frac{\kappa}{s^2}\left[1 - e^{-st_k}\right]. \tag{16}$$

Introducing (16) into (13), and referring again to [41] one gets

$$\sigma(t) = \begin{cases} E_\alpha \kappa t + \nu\kappa \int_0^t e_\alpha(\xi, \lambda)\ d\xi, \ \text{for} \ 0 < t \le t_k, \\ \\ E_\alpha \kappa t_k + \nu\kappa\left[\int_0^t e_\alpha(\xi, \lambda)\ d\xi - \int_0^{t-t_k} e_\alpha(\xi, \lambda)\ d\xi\right], \ \text{for} \ t > t_k, \end{cases} \tag{17}$$

where $e_\alpha(t; \lambda)$ denotes the Mittag-Leffler type function, defined as

$$e_\alpha(t; \lambda) = MLE_\alpha\left(-\lambda t^\alpha\right), \tag{18}$$

with $MLE_\alpha(t)$ being the Mittag-Leffler function, $(\alpha > 0)$,

$$MLE_\alpha(t) = \begin{cases} \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \ \text{for small values of} \ t, \\ \\ -\sum_{j=1}^{\infty} \frac{t^{-j}}{\Gamma(1 - \alpha j)}, \ \text{for large values of} \ t. \end{cases} \tag{19}$$

For the values of parameters satisfying (10) the expression (19) exhibits behavior shown in Fig. 1. It could be shown for several sets of recent experimental data presented in literature. Namely, several points were chosen

from the stress relaxation curves and then (17) was forced to pass through those points. In doing so, we used particle swarm optimization algorithm, see [38] and [55], with the least squares method as an optimality criterion. The suggested procedure yields four parameters of the constitutive model $\alpha, E_\alpha, \tau_{\sigma\alpha}$, and $\tau_{\varepsilon\alpha}$. For rough comparison the stress relaxation curves obtained only by four constants agree well with the classical Prony approximation involving significantly more constants.

We make two remarks here. First, note that the strain rate κ is an input parameter of the model. Secondly, following the lines of [22] and [23] we can apply the same analysis for the case of simple shear deformation pattern. In such a case σ and ε in (9) stand for the shear stress and strain, respectively, while the modulus of elasticity should be replaced by the shear modulus. For the latter, we give an example here. Namely, for the results presented in [57], for polymer used in seismic base isolation systems, for different values of $\kappa = 2, 1, 0.4, 0.2$ s $^{-1}$, in time domain, the suggested procedure yields the following average values

$$\alpha = 0.619, G_\alpha = 0.163, \tau_{\sigma\alpha} = 0.110, \tau_{\varepsilon\alpha} = 8.003, \tag{20}$$

as shown in [67]. The agreement between the experimental results and the model are shown in Fig. 2. In the following figures σ is given in MPa, while time t is measured in seconds.

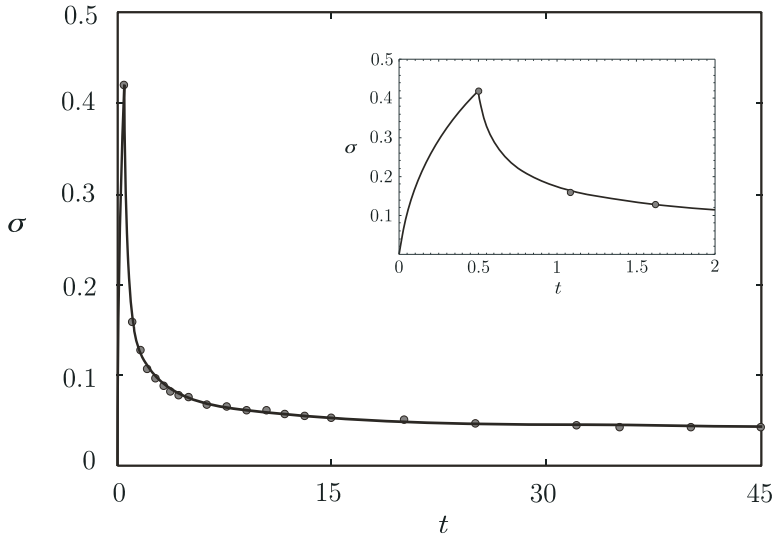

Fig. 2: The agreement between the experimental results of [57] and the modified Zener model with (9).

Note that in the used stress relaxation experiments of [57] only values for $t > t_k$, corresponding to relaxation phase are reported. This is due to relatively high strain rates. However, the suggested procedure is suitable in describing both loading $0 < t < t_k$ and relaxation phase $t > t_k$. We illustrate this by invoking experimental results of [12] who examined uniaxial stress relaxation of calf ulnar growth plate by ramp-and-hold strain, then stress relaxation tests with a displacement rate of 0.115×10^{-6} s $^{-1}$. The results presented in Fig. 8 of the mentioned paper are used to determine the four parameters of the modified Zener model. In doing so, one obtains

$$\alpha = 0.815, E_\alpha = 0.622, \tau_{\sigma\alpha} = 34.114, \tau_{\varepsilon\alpha} = 481.569. \tag{21}$$

In Fig. 3 we present the agreement between the experimental results presented in Fig. 8 of [12], and the fractional model (9) with (21).

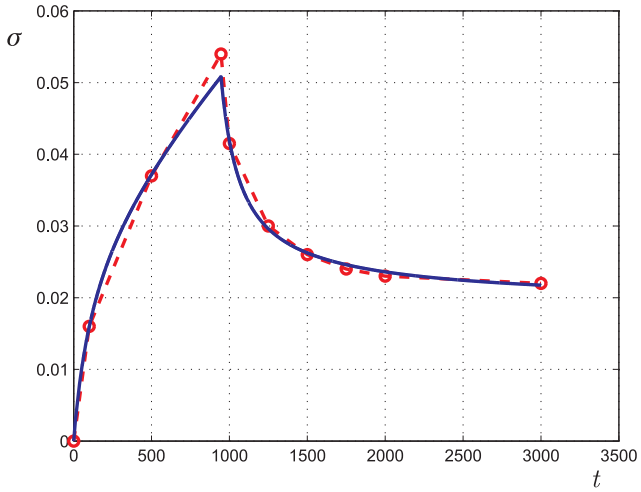

Fig. 3: Predicted (solid line) and measured (dashed line with marks) stress relaxation of Fig. 8 of [12] and (9) with (21).

It should be noted that on the basis of the several recently published results of the ramp-and-hold strain, then stress relaxation experiments for the middle ear structures: tympanic membrane, tensor tympany tendon, stapedial tendon and anterior malleolar ligament, see [15] to [18], the four parameters of modified Zener model (9) and the corresponding relations (10) are reported in [19] where good agreements between the measured values and the predicted values, as in Fig. 3, are shown. The application of the model in the analysis of the hamstring muscle group are given in [31].

The values of four constants in the modified Zener model, presented in this section as well as in [19], can be used within different engineering/bioengineering applications including impact and cyclic loading. The strategy for it will be clear soon. Before it, we give some results dealing with constitutive equation (9) that will be useful in the forthcoming analysis. First, rewriting (11) as

$$\bar{\sigma}=\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\bar{\varepsilon}+\frac{1}{\tau_{\sigma\alpha}}\left(1-\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\right)\frac{1}{\frac{1}{\tau_{\sigma\alpha}}+s^{\alpha}}\bar{\varepsilon}, \tag{22}$$

and following the lines of Gorenflo and Mainardi, as in [29] p. 267, after inversion one gets the following relation between stress and strain

$$\sigma(t)=\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\varepsilon(t)+\frac{1}{\tau_{\sigma\alpha}}\left(1-\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\right)\int_0^t\varepsilon(\xi)\cdot e_{\alpha,\alpha}\left(t-\xi,\frac{1}{\tau_{\sigma\alpha}}\right)d\xi, \tag{23}$$

where $e_{\alpha,\beta}(t;\lambda)$ stands for the generalized Mittag-Leffler function, that is $e_{\alpha,\beta}(t;\lambda)\equiv E_{\alpha,\beta}(-\lambda t^{\alpha})/t^{1-\beta}$ with $E_{\alpha,\beta}(t)=\sum_{n=0}^{\infty}t^n/\Gamma(\alpha n+\beta)$. Note that (23) generalizes (7) in a natural way, as expected. Secondly, for numerical analysis a slightly different form of fractional derivative will be useful. Introducing the time step h, $(t_m=m\,h, m=1, 2,\ldots)$, the fractional derivative $z_m^{(\alpha)}$ can be taken in the form

$$z_m^{(\alpha)}=h^{-\alpha}\sum_{j=0}^{m}\omega_{j,\alpha}z_{m-j},\ m=1, 2,\ldots, \tag{24}$$

with $\omega_{j,\alpha}$ calculated by the recurrence relationships

$$\omega_{0,\alpha}=1,\ \omega_{j,\alpha}=\left(1-\frac{\alpha+j}{j}\right)\omega_{j-1,\alpha},\ j=1, 2, 3,\ldots, \tag{25}$$

see [51]. The expression (24) can be used for discretization of (9)

$$\sigma_m + \tau_{\sigma\alpha} h^{-\alpha} \sum_{j=0}^{m} \omega_{j,\alpha} \sigma_{m-j} = E_\alpha \varepsilon_m + E_\alpha \tau_{\varepsilon\alpha} h^{-\alpha} \sum_{j=0}^{m} \omega_{j,\alpha} \varepsilon_{m-j}, \; m = 1, 2, \ldots, \tag{26}$$

which will, together with standard difference approximations for the first and second derivatives, lead to an algorithm for numerical integration of the systems containing fractional differential equations. Finally, we comment on the term fractional. It is a traditional word that could be replaced by arbitrary real order since the order of the derivative can be $\alpha = \sqrt{2} - 1$. Thus, the words differential equations of arbitrary real order can be used as well.

6.3 The Applications

As a first example, we analyze the viscoelastic compliant contact impact model with fractional derivative type of dissipation (9). We intend to show that the dynamics of the problem is governed by a single differential equation of real order. The obtained equation will be numerically solved. The comparison will be made to the solution obtained by the Laplace transform method and Post's inversion formula. The predictions of the model concerning the duration of the impact, maximal values of the impacting force and deformation as well as the restitution coefficient are determined for several values of the system parameters. The influence of the thermodynamical restrictions on constitutive equations (10) will be taken into account as well.

6.3.1 The impact against a rigid wall - an ideal case

The impact of solid bodies is a complicated phenomena and could be studied by several different approaches, see [11], [25] or [26], for example. If impacting bodies are taken to be deformable, then one has an advantage of being able to determine the impacting time, maximal deflections and impacting forces. Special feature of viscoelastic impacting body is that there exists hysteresis-like behavior in force displacement diagram. Such a behavior was explained either by nonlinear models [35], or by the use of the standard linear viscoelastic model, as done in [13]. We believe that generalized model of viscoelastic body used here is capable of describing impact in a more accurate way, while still remaining in the linear theory, [8].

Consider a block of mass m sliding on an ideal straight line with constant velocity v_0 and impacting against the rigid wall (infinite mass), through a deformable body, we model as a straight rod of negligible mass. We assume the rod is of constant cross-sectional area A that we assume remains the same during the deformation, and of length l in the undeformed state. The assumed value of the friction coefficient between the block and the line is zero, say $\mu = 0$. We use x to measure uniaxial deformation of that deformable body, see Fig.4. This deformation is assumed to be isothermal.

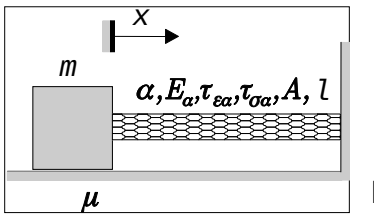

Fig. 4: The system under consideration.

Let f be the force between the body and the wall. This force also acts on the block, so its equation of motion following from fundamental axiom of dynamics [43], reads

$$m\, x^{(2)} = -f, \; x(0) = 0, \; x^{(1)}(0) = v_0, \; f(0) = 0, \tag{27}$$

where we used $(\cdot)^{(k)} = d^k(\cdot)/dt^k$ to denote the $k-$th derivative with respect to time t. The relation between $f = f(t)$ and $x = x(t)$ (constitutive equation of the deformable body) can be taken in different forms. Noting that the stress and strain used here are given as $\sigma = f/A$ and $\varepsilon = x/l$, respectively, and applying it to (9) we get the complementary constitutive equation in the following form

$$f + \tau_{\sigma\alpha} \cdot f^{(\alpha)} = \frac{E_\alpha A}{l}\left(x + \tau_{\varepsilon\alpha} \cdot x^{(\alpha)}\right), \tag{28}$$

that should be followed by (10). As above $(\cdot)^{(\alpha)}$ denotes the $\alpha-th$ derivative of a function $(\cdot)$ taken in the standard Riemann-Liouville form.

Introducing the dimensionless coordinate, force, and time

$$\bar{x} = \frac{xE_\alpha A}{mgl}, \ \bar{f} = \frac{f}{mg}, \ \bar{t} = t\sqrt{\frac{E_\alpha A}{ml}}, \tag{29}$$

as well as the dimensionless quantities

$$\bar{\tau}_{\sigma\alpha} = \tau_{\sigma\alpha}\left(\frac{E_\alpha A}{ml}\right)^{\alpha/2}, \ \bar{\tau}_{\varepsilon\alpha} = \tau_{\varepsilon\alpha}\left(\frac{E_\alpha A}{ml}\right)^{\alpha/2}, \ \xi = \frac{v_0}{g}\left(\frac{E_\alpha A}{ml}\right)^{1/2}, \tag{30}$$

into (27) and (28), after omitting the bar, we get the following equations describing the impact of the system presented in Fig. 4,

$$x^{(2)} = -f, \ x(0) = 0, \ x^{(1)}(0) = \xi, \ f(0) = 0, \tag{31}$$

$$f + \tau_{\sigma\alpha} \cdot f^{(\alpha)} = x + \tau_{\varepsilon\alpha} \cdot x^{(\alpha)}, \tag{32}$$

where derivatives are taken with respect to dimensionless time. Note that this model belongs to the class of continuous-dynamics models of collision, i.e., collision dynamics is treated as a continuous time dynamics phenomena restricted to local deformations (vibration effects of the solid body are not taken into account). In other words, it allows rapidly changing velocities without discontinuities. Also, note that thermodynamical restrictions $(10)_{2,3}$ in dimensionless form remain the same. The impact ends at instant $t = T$, when

$$f(T) = 0. \tag{33}$$

Actually, the study of considered fractional standard linear solid models, which is another name for the modified Zener model, possess an essential mathematical interest too. In this section, we start with simple numerical procedures for the solution of (31) and (32). First, we remove the non-homogeneous initial condition $(31)_3$ and eliminate f. Namely, by introducing the variable

$$z(t) = x(t) - \xi t, \tag{34}$$

and using basic properties of the Riemann-Liouville fractional differentiation, instead of (31), (32), we obtain the following differential equation of real order

$$\tau_{\sigma\alpha} z^{(2+\alpha)} + z^{(2)} + z + \tau_{\varepsilon\alpha} z^{(\alpha)} = -\xi t - \frac{\tau_{\varepsilon\alpha}\xi\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}, \tag{35}$$

with homogeneous boundary conditions

$$z^{(k)}(0)=0,\ k=0,1,2. \tag{36}$$

Using the first order approximation of the problem (24) for derivatives of the order α and $2+\alpha$ with corresponding coefficients (25) $\omega_{j,\upsilon}$, $\upsilon=\alpha, 2+\alpha,$ we derive the following algorithm to obtain the numerical solution, see [51],

$$z_0=0,\ z_1=0,\ z_2=0, \tag{37}$$

$$\begin{aligned} z_m = \frac{1}{1+h^{-2}+\tau_{\varepsilon\alpha}h^{-\alpha}+\tau_{\sigma\alpha}h^{-2-\alpha}}\times\Bigg\{&\frac{2z_{m-1}-z_{m-2}}{h^2} \\ &-\frac{\tau_{\varepsilon\alpha}\sum_{j=1}^{m}\omega_{j,\alpha}z_{m-j}}{h^{\alpha}}-\frac{\tau_{\sigma\alpha}\sum_{j=1}^{m}\omega_{j,2+\alpha}z_{m-j}}{h^{2+\alpha}} \\ &-m\,h\xi-\frac{\tau_{\sigma\alpha}\xi\Gamma(2)\,(m\,h)^{1-\alpha}}{\Gamma(2-\alpha)}\Bigg\},\ m=3,4,\ldots \end{aligned} \tag{38}$$

where h is the introduced time step. Noting that $z_m=z(t_m)=z(m\,h)$ from (34) we obtain

$$x_m=\xi m\,h+z_m\,. \tag{39}$$

Finally, using the second-order backward differences from $(31)_1$ we find

$$f_m=-\frac{(x_m-2x_{m-1}+x_{m-2})}{h^2}. \tag{40}$$

As an alternative algorithm of finding approximative solutions, we are going to apply the Laplace transform and Post's inversion formula. Namely, applying the Laplace transform to both (31), (32), after some calculations one gets the following images of the deformation and the force in viscoelastic rod

$$X(s)=\frac{\xi\left(1+\tau_{\sigma\alpha}s^{\alpha}\right)}{\tau_{\sigma\alpha}s^{2+\alpha}+s^2+\tau_{\varepsilon\alpha}s^{\alpha}+1},\ F(s)=\frac{\xi\left(1+\tau_{\varepsilon\alpha}s^{\alpha}\right)}{\tau_{\sigma\alpha}s^{2+\alpha}+s^2+\tau_{\varepsilon\alpha}s^{\alpha}+1}, \tag{41}$$

respectively. The originals $x(t)$ and $f(t)$ follow by the use of Post's inversion formula, see [45] p. 380, i.e.

$$x(t)=\lim_{n\to\infty}\frac{(-1)^n\left(\frac{n}{t}\right)^{n+1}X^{(n)}\left(\frac{n}{t}\right)}{n!},\ f(t)=\lim_{n\to\infty}\frac{(-1)^n\left(\frac{n}{t}\right)^{n+1}F^{(n)}\left(\frac{n}{t}\right)}{n!}. \tag{42}$$

Although Post's formula, discovered in 1930 [53], may be regarded as an analytical result, very useful for applications, difficulties, essentially technical in nature, prevented its usage in practical problems. However, nowadays the n -th derivatives of (41) needed for the right-hand-side of Post's formula (42) could be easily calculated by the use of standard software packages. In such a way, we can obtain the results useful for error estimations of numerical solutions. At the same time, the Post result could serve as an analytical approximation for $x(t)$ (and $f(t)$) provided the computer has enough memory and is fast enough to perform a large amount of symbolic differentiation.

The described numerical method was experimentally verified on a number of test problems for $\xi=1.$

In Table 1 we present the duration of impact T, determined by (33), the maximal values of x and f for several values of dimensionless relaxation times and for several values of constants α, $\tau_{\sigma\alpha}$ and $\tau_{\varepsilon\alpha}$. The values of dimensionless time corresponding to these maximums are given in parenthesis. The values x and $x^{(1)}$ at T, ($x^{(1)}(T)$ determining the restitution coefficient) are also presented. The numerical values of constants $0<\alpha<1$, $\tau_{\sigma\alpha}$ and $\tau_{\varepsilon\alpha}$ were taken from the paper of Fenander, where the railpad models were investigated, [24].

The values shown in Table 1 are taken from [5]. In all the calculations the time step was $h=10^{-3}$.

It should be noted that the agreement between the numerical results obtained by (39), (40) and (42) was satisfactory even for relatively small values of n ($n=40$ in case $\alpha<1$).

In Fig. 5 we present some solutions $x(t)$. The values obtained by applying Post's inversion formula for $\alpha=1$, $\tau_{\sigma 1}=0.04$ and $\tau_{\varepsilon 1}=0.2$ are also presented (squares in Fig. 5). The difference between the numerical solution (39), (40) and the solution obtained by Post's formula (42) for $n=70$, is less than 5×10^{-2}. Also, the values calculated by Post's formula for $\alpha=0.23$, $\tau_{\sigma\alpha}=0.004$, $\tau_{\varepsilon\alpha}=1.183$ for $n=40$ are also marked by circles in Fig. 5. It is worth noting that modern computers allow larger n and thus more accuracy.

Table 1

Material description	Impact description	
$\alpha=1$ $\tau_{\sigma 1}=0.01$ $\tau_{\varepsilon 1}=0.2$	$f_{max}(1.289)=0.887$ $x_{max}(1.480)=0.869$	$T=2.962$ $x=0.144$ $x^{(1)}=-0.756$
$\alpha=1$ $\tau_{\sigma 1}=0.04$ $\tau_{\varepsilon 1}=0.2$	$f_{max}(1.330)=0.907$ $x_{max}(1.490)=0.890$	$T=2.981$ $x=0.126$ $x^{(1)}=-0.792$
$\alpha=0.95$ $\tau_{\sigma\alpha}=0.01$ $\tau_{\varepsilon\alpha}=0.2$	$f_{max}(1.289)=0.899$ $x_{max}(1.475)=0.870$	$T=2.945$ $x=0.144$ $x^{(1)}=-0.762$
$\alpha=0.95$ $\tau_{\sigma\alpha}=0.01$ $\tau_{\varepsilon\alpha}=0.011$	$f_{max}(1.570)=0.999$ $x_{max}(1.571)=0.998$	$T=3.141$ $x=0$ $x^{(1)}=-0.997$
$\alpha=0.49$ $\tau_{\sigma\alpha}=5\times10^{-8}$ $\tau_{\varepsilon\alpha}=0.886$	$f_{max}(0.832)=1.140$ $x_{max}(1.110)=0.642$	$T=2.151$ $x=0.232$ $x^{(1)}=-0.596$
$\alpha=0.23$ $\tau_{\sigma\alpha}=0.004$ $\tau_{\varepsilon\alpha}=1.183$	$f_{max}(0.910)=1.356$ $x_{max}(1.034)=0.632$	$T=2.025$ $x=0.135$ $x^{(1)}=-0.771$

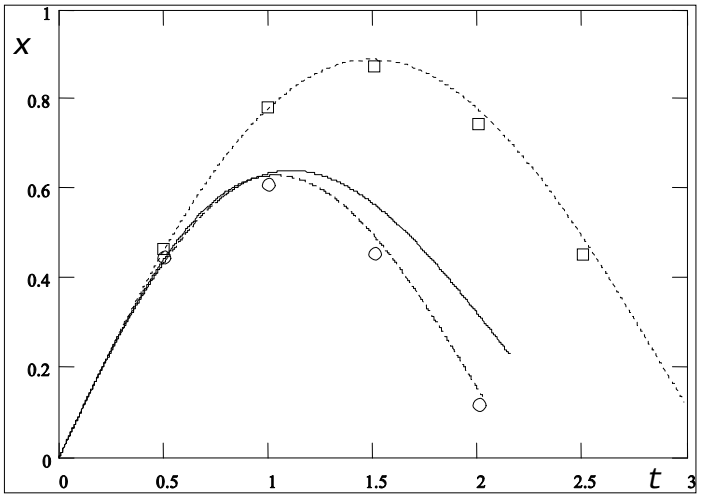

Fig. 5: Curves $x(t)$ for standard linear solid $\alpha = 1,\ \tau_{\sigma 1} = 0.04$ and $\tau_{\varepsilon 1} = 0.2$ (dotted), fractional standard solid with $\alpha = 0.49,\ \tau_{\sigma\alpha} = 5\times 10^{-8},\ \tau_{\varepsilon\alpha} = 0.886,$ (solid line) and for $\alpha = 0.23,\ \tau_{\sigma\alpha} = 0.004,\ \tau_{\varepsilon\alpha} = 1.183$ (dashed).

Finally, the hysteresis diagrams corresponding to the solutions presented in Fig. 5 are shown in Fig. 6.

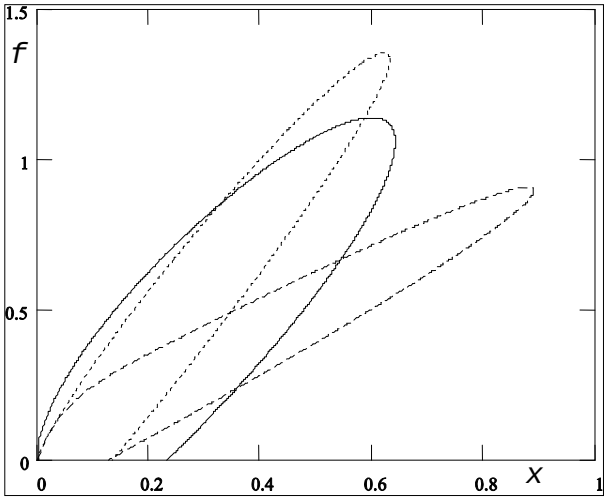

Fig. 6: Hysteresis diagrams for standard linear solid $\alpha = 1,\ \tau_{\sigma 1} = 0.04$ and $\tau_{\varepsilon 1} = 0.2$ (dotted), fractional standard solid with $\alpha = 0.49,\ \tau_{\sigma\alpha} = 5\times 10^{-8},\ \tau_{\varepsilon\alpha} = 0.886,$ (solid line) and for $\alpha = 0.23,\ \tau_{\sigma\alpha} = 0.004,$ $\tau_{\varepsilon\alpha} = 1.183$ (dashed).

From the values presented in Table 1 we conclude that for the case when $\alpha \to 1$ the solutions for the fractional standard linear solid are very close to the ones that describe the standard linear solid (for $\alpha = 1$ and the same values of the constants describing the material). In other words, the solution is continuous with respect to the order of the derivative. Note that when compared to standard linear viscoelastic solid, for the same values of relaxation constants, the solid described by fractional derivatives exhibits shorter duration of impact, smaller maximal deformation and larger maximal force and smaller amount of dissipated energy. Also, the calculations show that when $\tau_{\varepsilon 1} \to \tau_{\sigma 1}$ there is no damping as expected, i.e. the velocity after the rebound is almost of the same intensity as before the impact. In such a case, the presented values of T and x_{max} could be compared with the case of nonlinear spring as presented in [39]. Namely, in [39] the impact is modeled by the equation

$$x^{(2)} = -\rho x^{3/2},\ x(0) = 0,\ x^{(1)}(0) = 1, \tag{43}$$

For example, if we take $\rho=1$ we obtain $T=3.218,\ x^{(1)}(T)=-1$ and $x_{max}=1.093.$ Our final remark concerns the thermodynamical restrictions. Violating them could pose severe problems. Roughly speaking, putting $\tau_{\sigma\alpha}>\tau_{\varepsilon\alpha}$ will lead to the rebound velocity which is higher than the approaching velocity, i.e. $\left|x^{(1)}(T)\right|>x^{(1)}(0)=1$.

We close this section with two remarks. First, note that more details on the model of viscoelastic rod in unilateral contact with a rigid wall, with respect to mathematical analysis, can be found in [6]. Finally, we may cast the impact problem we are dealing with here in another form. Namely, by applying the similar steps as when (11) was transformed to (23), from (32) one obtains

$$f(t)=\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}x(t)+\frac{1}{\tau_{\sigma\alpha}}\left(1-\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\right)\int_0^t x(\eta)\cdot e_{\alpha,\alpha}\left(t-\eta,\frac{1}{\tau_{\sigma\alpha}}\right)d\eta, \tag{44}$$

and then using (31) the description of impact is given in the form of the Cauchy problem for integro-differential equation

$$\begin{gathered}x^{(2)}+\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}x(t)+\frac{1}{\tau_{\sigma\alpha}}\left(1-\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\right)\int_0^t x(\eta)\cdot e_{\alpha,\alpha}\left(t-\eta,\frac{1}{\tau_{\sigma\alpha}}\right)d\eta=0,\\ x(0)=0,\ x^{(1)}(0)=\xi.\end{gathered} \tag{45}$$

This form can be suitable for both different numerical procedures, as in [14], and the more general case when the problem is treated in the presence of dry friction, as in [32]. This case, $\mu>0,$ will be considered later.

The next engineering application is more demanding with respect to duration of integration procedure and will be treated by slightly different procedure. It considers a mass moving on a straight line under the action of a harmonic disturbing force. The mass is fixed to a viscoelastic rod whose other end is anchored, and whose description is given by (9), (10). It will be shown that the dynamics of the problem is governed by a single differential equation of real order. The obtained equation will be solved using the Laplace transform method.

6.3.2 The forced vibrations with fractional type of dissipation pattern

The study of forced vibrations is a classical problem. The interest in it increases if the materials included are taken to exhibit nonlinear behavior with or without damping. As a part of it, the problem of eliminating undesirable oscillations and vibrations has emerged. Namely, the new tendency in civil engineering favors the design of slender structures made of new viscoelastic materials which provide necessary extra damping. For example, in [22], an example of a forced oscillator including fractional damping elements was given and the solution was obtained by the Grünwald algorithm and the finite element method. The influence of the order of fractional derivative on the solution for one value of disturbing force frequency was considered. In this work, we intend to reexamine the problem. Namely, we are going to solve the problem by different methods. We plan to use the method of Laplace transformation with inversion performed by complex integration, see [29] or [56]. As alternative approaches in this subsection, we are going to show the numerical method described by Podlubny, [51] and Post's inversion formulae, see [45] or [53]. The alternative approaches are more convenient for engineers but are usually followed by problems concerning the convergence in the large time domain and the short memory principle. Our analysis closes with the influence of four parameters included in the viscoelastic material description on the solution for different values of disturbing force frequencies. Namely, taking into account the restrictions on the parameters, following from the Clausius Duhem inequality, the amplitude ratio (or magnification factor) will be analyzed. In doing so, the resonance recognition problem will be tackled.

Consider the mass m moving on a smooth straight line under the action of a harmonic disturbing force, say $F_0\sin\Omega t,$ where F_0 and Ω are positive constants and t is time. The mass is fixed to a viscoelastic body, which is assumed to be a rod of constant cross-sectional area A and of length $l.$ We assume that the other end of the rod is anchored and we use x to measure uniaxial, isothermal deformation of that rod. Let f be the

force between the rod and the mass. As before, we start with the fundamental axiom of dynamics [43], and the Newton-Laplace principle, i.e. we describe the considered motion by

$$m\,x^{(2)} = -f + F_0 \sin \Omega t,\; x(0) = 0,\; x^{(1)}(0) = 0,\; f(0) = 0, \tag{46}$$

where we assumed that the mass was at rest at initial instant of time $t = 0$ that correspond to the undeformed and unloaded state of the rod.

The relation between $f = f(t)$ and $x = x(t)$, i. e. the constitutive relation of the deformable body, will be taken in the form of modified Zener model of a viscoelastic body (28), with restrictions given by (10).

Introducing the dimensionless coordinate, force, and frequency of the excitation force, say

$$\bar{x} = \frac{x E_\alpha A}{F_0 l},\; \bar{f} = \frac{f}{F_0},\; \bar{\Omega} = \Omega \sqrt{\frac{m\,l}{E_a A}}, \tag{47}$$

respectively, as well as the dimensionless time as in (29), and relaxation constants as in $(30)_{1,2}$ from (46), (28) and (47) we get the system of equations describing the forced vibrations with fractional type of dissipation consisting of the equation

$$x^{(2)} = -f + \sin \Omega t,\; x(0) = 0,\; x^{(1)}(0) = 0,\; f(0) = 0, \tag{48}$$

that is complemented with (32) and the restrictions (10). As before, all the derivatives are taken with respect to dimensionless time and the bars are suppressed over the dimensionless variables. Note that as a consequence of the second law of thermodynamics from $(10)_3$ we have

$$\Delta \tau_\alpha = \tau_{\varepsilon\alpha} - \tau_{\sigma\alpha} > 0, \tag{49}$$

and there will be no damping if $\Delta \tau_\alpha = 0$, see [24]. Also, note that following the lines of the classical vibration theory, when $\tau_{\sigma\alpha} = \tau_{\varepsilon\alpha}$, we expect the resonance and the vibroisolation to be exhibited for $\Omega = 1$ and $\Omega \gg 1$ respectively.

In order to compute the solution of (48) and (32), as before we eliminate f, and then using basic properties of the Riemann-Liouville fractional differentiation, we obtain the following (single) differential equation of real order

$$\begin{gathered} \tau_{\varepsilon\alpha} x^{(2+\alpha)} + x^{(2)} + x + \tau_{\sigma\alpha} x^{(\alpha)} = \sin \Omega t + \tau_{\sigma\alpha} S_t(-\alpha, \Omega), \\ x(0) = 0,\; x^{(1)}(0) = 0,\; x^{(2)}(0) = 0, \end{gathered} \tag{50}$$

where $S_t(-\alpha, \Omega) = \sum_{j=0}^{\infty} (-1)^j \Omega^{2j+1} t^{2j+1-\alpha} \Gamma^{-1}(\alpha + 2j + 2)$ stands for the $\alpha - th$ Riemann-Liouville derivative of $\sin \Omega t$, see [46], p. 355. As in the previous case, introducing the discrete time $t_m = m\,h$, where h is time step, and using the first order approximations, as in [60], we derive the following algorithm to obtain the numerical solution

$$\begin{aligned} x_m = x(t_m) = \frac{1}{1 + h^{-2} + \tau_{x\alpha} h^{-\alpha} + \tau_{f\alpha} h^{-2-\alpha}} \times \Bigg\{ & \frac{2x_{m-1} - x_{m-2}}{h^2} - \frac{\tau_{x\alpha} \sum_{j=1}^{m} \omega_{j,\alpha} x_{m-j}}{h^\alpha} - \frac{\tau_{f\alpha} \sum_{j=1}^{m} \omega_{j,2+\alpha} x_{m-j}}{h^{2+\alpha}} \\ & + \sin(\Omega m h) + \tau_{\sigma\alpha} S_{mh}(-\alpha, \Omega) \Bigg\},\; m = 3, 4, \ldots, \end{aligned} \tag{51}$$

where homogeneous initial conditions $(50)_{2,3,4}$ correspond to $x_0 = x_1 = x_2 = 0.$

The described numerical method was experimentally verified on a number of test problems. In the case of equation (50), with $S_{mh}(-\alpha, \Omega)$ given as above, it seems that it will work provided the time mh does not leave the convergence domain of that series. Since we know that $S_t(-1, \Omega)$ coincide with $\Omega\cos(\Omega t)$, see [46], p.318, we may speculate that if $S_{mh}(-1, \Omega)$ does not coincide with $\Omega\cos(\Omega mh)$, for say $m > m_c$, then we are not to expect the series $S_{mh}(-\alpha, \Omega)$ to be convergent for $m > m_c$ and $\alpha < 1$, and thus the algorithm (51) may fail for $m > m_c$, or $t > m_c h$. This really does happen in practice. For example, the numerical examination shows that $\cos t$ and $S_t(-1, 1)$, truncated after 80 terms, does not coincide for $t > 30.$ Another problem that could be encountered while processing (51) is the short memory problem. Namely, if we take h to be small enough for large values of m the number of the addends in the fractional-derivative approximation of type as (24), becomes enormously large, which causes some extra technical problems, see [51], p. 203.

Since we do not know the duration of the oscillator transient regime, using (51) we may, or may not, reach the steady state solution of the forced oscillator problem. This increases our interest in finding the alternative algorithms. Thus, in the following we are going to apply the Laplace transform and Post's inversion formula by following the lines as in the above impact problem. Namely, applying the Laplace transform to both (48) and (32), after some calculations we get the images

$$X(s) = \frac{\Omega}{(s^2+\Omega^2)} \frac{(1+\tau_{\sigma\alpha} s^\alpha)}{(\tau_{\sigma\alpha} s^{2+\alpha} + s^2 + \tau_{\varepsilon\alpha} s^\alpha + 1)}, \quad F(s) = \frac{\Omega}{(s^2+\Omega^2)} \frac{(1+\tau_{\varepsilon\alpha} s^\alpha)}{(\tau_{\sigma\alpha} s^{2+\alpha} + s^2 + \tau_{\varepsilon\alpha} s^\alpha + 1)}. \tag{52}$$

In the special case when $\alpha = 1$, corresponding to the Zener model, the direct inversion of (52), easily performed using standard software packages, yields the solutions $x(t)$ and $f(t).$ In the general case $\alpha < 1$ the standard software packages fail to proceed, but one could obtain both $x(t)$ and $f(t)$ using Post's inversion formula (42). The Post result could serve as analytical approximation for $x(t)$ and $f(t)$ provided the computer has enough memory and is fast enough to perform a large amount of symbolic differentiation that is in case of (52) more complex and therefore more time and memory consuming than (41). Note that this procedure avoids the problems connected to the convergence of the series $S_t(-\alpha, \Omega)$ but despite the simplicity of the Post inversion formula the expressions obtained by it are very long and rather difficult to handle. Thus, we turn now to the most elegant solution of the system (48), (32) with (10).

In order to examine the motion of the forced oscillator, the inversion of (52) by complex integration will be done. Following the standard procedure, [29] p. 259, first, we chose the contour with a cut along the negative real axis, say γ, as shown in Fig. 7 (the path $ABDEFGA$). Then we analyze the number of poles of (52) inside γ. The poles $s_1 = j\Omega$ and $s_2 = -\Omega j$, where j stands for the imaginary unit, are obvious. In order to determine the other ones, we apply the Rouché theorem, see [56], p. 287. Namely, rewriting the second multiplicand in (52), as $1/(p + p_1)$ where $p = s^2 + 1$ and $p_1 = (\tau_{\varepsilon\alpha} - \tau_{\sigma\alpha})\, s^\alpha / (1 + \tau_{\sigma\alpha} s^a)$, and noting that for $s = \rho e^{j\theta}$ the condition $|g|/|f| < 1$ is satisfied on γ we conclude that p and $p + p_1$ have the same numbers of zeros inside γ, (in our case 2).

In order to find two more poles of both $X(s)$ and $F(s)$, we split the equation

$$\tau_{\sigma\alpha} s^{2+\alpha} + s^2 + \tau_{\varepsilon\alpha} s^\alpha + 1 = 0, \tag{53}$$

into the system

$$\begin{aligned} \tau_{\sigma\alpha}\rho^{2+\alpha}\cos(2+\alpha)\theta + \rho^2\cos 2\theta + \tau_{\varepsilon\alpha}\rho^\alpha\cos\alpha\theta + 1 = 0, \\ \tau_{\sigma\alpha}\rho^{2+\alpha}\sin(2+\alpha)\theta + \rho^2\sin 2\theta + \tau_{\varepsilon\alpha}\rho^\alpha\sin\alpha\theta = 0. \end{aligned} \tag{54}$$

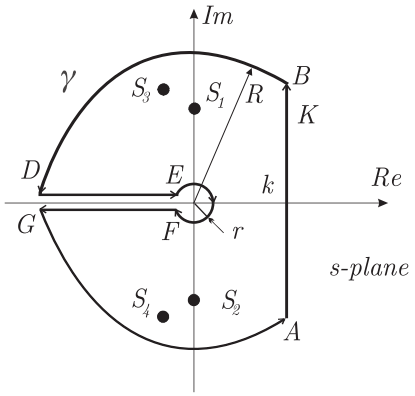

Fig. 7: Contour of integration.

Applying the Newton method we can find the solutions of (54) that correspond to the principal branch, say $\bar{\rho}$ and $\bar{\theta}$, and the remaining poles of $X(s)$ and $F(s)$, say $s_3 = \bar{\rho}e^{\bar{\theta} j}$, $s_4 = \bar{\rho}e^{-\bar{\theta} j}$. With this preparation done, we are ready to find

$$x(t) = \lim_{K \to \infty} (2\pi j)^{-1} \int_{k-Kj}^{k+Kj} e^{st} X(s)\, ds,\ f(t) = \lim_{K \to \infty} (2\pi j)^{-1} \int_{k-Kj}^{k+Kj} e^{st} F(s)\, ds, \tag{55}$$

for $t > 0$, i.e., as the integral along AB, where k is suitably chosen, so all poles lie to the left of the line $s = k$. Let us find the original $x(t)$ first. It remains to explore the residue theorem. According to it, the integral along the closed path γ is $2\pi j$ times the sum of the residues of $e^{st}X(s)$ at the singularities enclosed by γ. Rewriting $e^{st}X(s)$ as $F_1(s, t)/F_2(s)$ with

$$F_1(s, t) = \left(1 + \tau_{\sigma\alpha} s^{\alpha}\right)\ \Omega e^{st},\ F_2(s) = \left(s^2 + \Omega^2\right)\ \left(\tau_{\sigma\alpha} s^{2+\alpha} + s^2 + \tau_{\varepsilon\alpha} s^{\alpha} + 1\right), \tag{56}$$

the residue of $F_1(s, t)/F_2(s)$, at the point s_o, reads $F_1(s_o, t)/F_2'(s_o)$ where prime represents the derivative with respect to s, see [20] p. 161. Referring to Doetcsh once again, we conclude that the integrals along BD, GA and EF vanish (when $R \to \infty$ and $r \to 0$). After calculating the sum of the integrals along DE and FG, we finally obtain the motion of the forced oscillator with fractional type of dissipation as

$$x(t) = \sum_{i=1}^{4} \frac{F_1(s_i, t)}{F_2'(s_i)} + \frac{\Omega(\tau_{\varepsilon\alpha} - \tau_{\sigma a})\ \sin\alpha\pi}{\pi} \times I(t), \tag{57}$$

where

$$I(t) = \int_0^{\infty} \frac{\left(\eta^2 + \Omega^2\right)^{-1} \eta^{\alpha} e^{-rt} d\eta}{\left(1 + \eta^2\right)^2 + \left(\tau_{\sigma\alpha}\eta^{2+\alpha} + \tau_{\varepsilon\alpha}\eta^{\alpha}\right)^2 + 2\left(1 + \eta^2\right)\ \left(\tau_{\sigma\alpha}\eta^{2+\alpha} + \tau_{\varepsilon\alpha}\eta^{\alpha}\right)\ \cos\alpha\pi}. \tag{58}$$

Note that the residuals of $X(s)$ determine the value of $I(0)$. The value $I(t)$ could be easily calculated by standard numerical procedures. Also, note that $\lim_{t \to \infty} I(t) = 0$.

In order to illustrate the above results, we are going to present motions of the forced oscillator for numerical values of constants $0 < \alpha < 1$, $\tau_{\sigma\alpha}$ and $\tau_{\varepsilon\alpha}$ taken from the paper of Fenander where the railpad models were investigated, [24].

Namely, for $\alpha=0.23$, $\tau_{f\alpha}=0.004$, $\tau_{x\alpha}=1.183$ and $\Omega=1$, the solutions of (54) read $\bar{\rho}=1.499$ and $\bar{\theta}=\pm 1.679$. Substituting these values into (57), (58) we obtain the amplitude of the steady state regime to be 0.818. Performing the same type of numerical experiments while increasing Ω we conclude that the system goes towards the vibroisolation area. For example for $\alpha=0.23$, $\tau_{\sigma\alpha}=0.004$, $\tau_{\varepsilon\alpha}=1.183$ and $\Omega=10$ the amplitude of x in the steady state is less than 0.01. When compared to the standard viscoelastic solid $\alpha=1$ the solution for $\alpha<1$ exhibits smaller amplitudes, which agrees with the results presented in [22]. Namely, noting that

$$\lim_{t\to\infty}\sum_{i=3}^{4}F_1(s_i,\,t)\,\left[F_2'(s_i)\right]^{-1}+\pi^{-1}\Omega(\tau_{\varepsilon\alpha}-\tau_{\sigma a})\;I(t)\;\sin\alpha\pi\to 0,$$

we obtain the steady state solution in the form

$$x_s(t)=\sum_{i=1}^{2}\frac{F_1(s_i,\,t)}{F_2'(s_i)}.$$

The amplitude of it reads

$$A_s=\sqrt{\left(\frac{F_1(\Omega j,\,0)}{F_2'(\Omega j)}+\frac{F_1(-\Omega j,\,0)}{F_2'(-\Omega j)}\right)^2+\left(\frac{F_1(\Omega j,\,0)}{F_2'(\Omega j)}-\frac{F_1(-\Omega j,\,0)}{F_2'(-\Omega j)}\right)^2}. \tag{59}$$

Calculating (59) for different values of Ω, α, $\tau_{\varepsilon\alpha}$ and $\tau_{\sigma\alpha}$ we can obtain the magnification factor for the oscillator with fractional type of dissipation. Since the dimension of the parameter space in the introduced model is 4, we omit here the usual graphical presentation of that factor. However, with introduced $\Delta\tau_\alpha$ as in (49) we note that increasing the value $\Delta\tau_\alpha$ the value of A_s decreases. This fact could be very useful in engineering applications.

Finally, we may pose a question how the system under consideration will behave if the second law of thermodynamics is violated. Choosing $\Delta\tau_\alpha<0$ and solving (54) yield $\bar{\theta}<\pi/2$ and thus $\lim_{t\to\infty}\sum_{i=3}^{4}F_1(s_i,\,t)\,\left[F_2'(s_i)\right]^{-1}\to\infty$, which leads to a motion represented by time diverging function. The last comment deals with the possibility of obtaining time diverging functions. Once again, we turn to the equation (53). If the roots of (53) are imaginary and symmetrically displaced about the origin, it is possible to have one of them coincide with Ω in which case we would have a second order pole at $s=\pm\Omega j$ and a diverging time function, [63], p.196. Namely, putting $\bar{\rho}=\Omega$ and $\bar{\theta}=\pi/2$ in (54), for $\alpha>0$ one obtains

$$1-\Omega^2+\Omega^{\alpha}\left(\tau_{\varepsilon\alpha}-\tau_{\sigma\alpha}\Omega^2\right)\;\cos\frac{\alpha\pi}{2}=0,\;\tau_{\varepsilon\alpha}-\tau_{\sigma\alpha}\Omega^2=0, \tag{60}$$

which can be satisfied only if $\Omega=1$ and $\tau_{\varepsilon\alpha}=\tau_{\sigma\alpha}$. Indeed, if we put $\tau_{\varepsilon\alpha}=\tau_{\sigma\alpha}$ into (53) the straightforward inversion yields $x_{\tau_{\varepsilon\alpha}=\tau_{\sigma\alpha}}(t)=(\sin t-t\cos t)/2$. We close by noting that $\tau_{\varepsilon\alpha}=\tau_{\sigma\alpha}$ is never satisfied for thermodynamically well-behaved models and that for such models the resonance may not occur. This agrees with the classical linear theory with spring and dashpot as a model. Thus, as a consequence of the Clausius Duhem inequality, we claim that the time diverging functions are allowed only for the linearly elastic (Hookean) models, which are not realistic, once again see [10], p.155.

So far, both applications were single degree of freedom systems. The presented methods can be applied in the analysis of impact and forced oscillations in systems incorporating elastomers, polymers and, biological tissues since they are well described by the modified Zener model. It should be noted that biosystems are more complex and less well defined. Particularly, the human form is not composed of simple geometrical shapes and besides mechanical functions it has physiological functions too. It has more degrees of freedom to deal with.

Thus, one may pose a problem how much effort is needed for more complex systems. For simplicity reasons, we shall avoid biological problems and consider a seismic response of a column-like structure with ductile regions composed of polymers and elastomers. What distinguishes this particular damping system from those previously tested on the same model structure is the constitutive model of ductile regions that comprises a simple shear deformation pattern, fractional derivatives of shear stress and shear strain and the restrictions on the coefficients that follow from the Clausius-Duhem inequality. In the next section, we shall pose the problem and show that the dynamics of the system lead to coupled differential equations of real order. Then, we are going to solve the problem using the Laplace transform and inversion by complex integral.

6.3.3 A column-like structure under seismic load

In the earthquake-prone regions, the principal problem of structural dynamics is the behavior of structures subjected to earthquake-induced motion of the base of the structure. Various types of viscous dampers have been used for over fifty years in order to reduce seismic response. Conventional seismic design is based on allowing structures to dissipate energy in specially detailed ductile regions. Following a strong earthquake damage to these viscoelastic regions is to be expected, but prevention of structural collapse to ensure the preservation of life-safety maintained, [48]. These regions, usually composed of elastomers and polymers, significantly attenuate the seismic response of the structure and provide a considerable energy dissipation while the main non-ductile structural load carrying elements remain unchanged. The principal objectives of this subsection are twofold: to predict the response characteristics of the column-like structure with such dampers and to estimate the influence of four parameters included in the viscoelastic material description on the solution which can be used in evaluating shear resisting capacity of each story, leading to a new model of either sky-scrapers or antic-columns protection.

We study dynamics of a system of three stone blocks positioned one on another with viscoelastic plates between them. We assume that there is no sliding between the blocks and the plates. Initially, the system was at rest forming a column-like structure. At a certain moment $t=t_0=0$ the lowest block - the base - will start to move along the horizontal line with u as an absolute coordinate of the base movement. This, in turn, causes the horizontal movement of the upper blocks, caused by internal shearing forces through deformable plates of negligible mass, see Fig. 8, representing the plane of material symmetry.

Let m_1 and m_2 be the masses of the blocks, and let x_1 and x_2 be the relative displacements between the block and the base and between the blocks respectively, see Fig. 8. In what follows we use $f_i\ (i=1,2)$ to denote the internal shearing forces in the first and the second deformable plate.

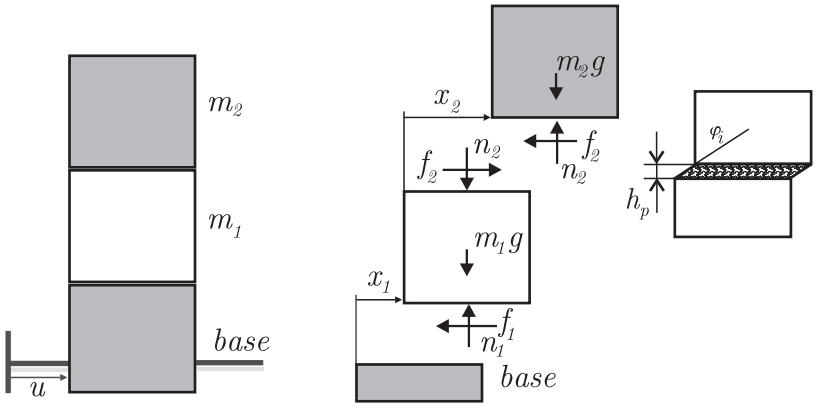

Fig. 8: System under considerations, free-body diagrams and simple shear deformation pattern.

From the free-body diagram presented in Fig. 8 we write the differential equations of motion

$$m_1\left(u^{(2)}+x_1^{(2)}\right)=-f_1+f_2,\ m_2\left(u^{(2)}+x_1^{(2)}+x_2^{(2)}\right)=-f_2, \tag{61}$$

and the corresponding initial conditions

$$x_1(0)=x_2(0)=0,\ x_1^{(1)}(0)=x_2^{(1)}(0)=0,\ f_1(0)=f_2(0)=0. \tag{62}$$

with usual notation $(\cdot)^{(k)}=d^k(\cdot)/dt^k$ for $k-$th derivative with respect to time t.

The model of grounded motion $u^{(2)}$ is represented by $u^{(2)}=\beta(t)\ u_{gst}^{(2)}$, in which $\beta(t)$ is a deterministic non-negative envelope function and $u_{gst}^{(2)}$ is a stationary ground acceleration. The stationary random ground acceleration at a time t is obtained by a summation over all frequency components $u_{gst}^{(2)}=\sum_{i=1}^{N}A_g(\Omega_i)\ \sin\left(2\pi\Omega_i t+\alpha_g\right)$ where α_g is a random phase angle varying between 0 and 2π and A_g is the amplitude of the ground motion acceleration which is related to the modulus of the physical ground acceleration, [33]. In this study, we do not put stress on the earthquake model but on viscoelastic behavior of the structure and thus we use a simplified earthquake model with

$$u^{(2)}=\exp\left(-a_g t\right)\ \left(-\Omega^2 u_0\sin\Omega t\right), \tag{63}$$

with a_g being a positive constant of dimension time to the power -1.

In our analysis, as before, we use the constitutive model of the plates that comprises simple shear deformation pattern, fractional derivatives of shear stress and shear strain taken in the standard Riemann-Liouville form, and the restrictions on the coefficients that follow from Clausius-Duhem inequality. Namely, introducing the shear angles φ_i, and assuming that $\varphi_i=\tan\varphi_i=x_i/h_p$ we can take the constitutive equations in the following form

$$f_1+\tau_{\sigma\alpha}f_1^{(\alpha)}=\frac{G_\alpha A_p}{h_p}\left(x_1+\tau_{\varepsilon\alpha}x_1^{(\alpha)}\right),\ f_2+\tau_{\sigma\beta}f_2^{(\beta)}=\frac{G_\beta A_p}{h_p}\left(x_2+\tau_{\varepsilon\beta}x_2^{(\beta)}\right), \tag{64}$$

where $0<\alpha,\ \beta\le 1$, A_p and h_p stand for the area of the cross-section and thickness of the plates, G_α and $G_\in$ are the shear modules, $\tau_{\sigma\alpha}$ and $\tau_{\varepsilon\alpha}$ are the relaxation constants of dimension $[\text{time}]^\alpha$, and $\tau_{\sigma\beta}$, and $\tau_{\varepsilon\beta}$ are the relaxation constants of dimension $[\text{time}]^\beta$. In equations (64) we use $(\cdot)^{(\psi)}$, $\psi=\alpha,\ \beta$, to denote the $\psi-th$ derivative of a function $(\cdot)$ taken in the standard Riemann-Liouville form. In order to be well behaved, we assume that the model satisfies fundamental restrictions on the coefficients of the model that follow from the second law of thermodynamics

$$G_\alpha>0,\ \tau_{\sigma\alpha}>0,\ \tau_{\varepsilon\alpha}>\tau_{\sigma\alpha},\ G_\beta>0,\ \tau_{\sigma\beta}>0,\ \tau_{\varepsilon\beta}>\tau_{\varepsilon\beta}. \tag{65}$$

For one possible set of these four constants obtained for applications in civil engineering we refer to (20).

Next, we introduce the following dimensionless quantities

$$\phi=\frac{m_1}{m_2},\ \zeta=\frac{G_\beta}{G_\alpha},\ \bar{x}_i=\frac{x_i}{u_0},\ \bar{f}_i=\frac{f_i h_p}{G_\alpha A_p u_0},\ i=1,2;$$

$$\bar{\Omega}=\Omega\sqrt{\frac{m_1 h_p}{G_\alpha A_p}},\ \bar{a}_g=a_g\sqrt{\frac{m_1 h_p}{G_\alpha A_p}},\ \bar{t}=t\sqrt{\frac{G_\alpha A_p}{m_1 h_p}}, \tag{66}$$

$$\bar{\tau}_{\sigma\psi}=\tau_{\sigma\psi}\left(\frac{G_\alpha A_p}{m_1 h_p}\right)^{\psi/2},\ \bar{\tau}_{\varepsilon\psi}=\tau_{\varepsilon\psi}\left(\frac{G_\alpha A_p}{m_1 h_p}\right)^{\psi/2},\ \psi=\alpha,\ \beta;$$

and from (61), (63) and (64) we get the system of equations describing the dynamics of simplified earthquake model of the column-like structure with fractional type of dissipation

$$x_1^{(2)} = \Omega^2 \exp\left(-a_g t\right)\ \sin\Omega t - f_1 + f_2,\ x_1^{(2)} + x_2^{(2)} = \Omega^2 \exp\left(-a_g t\right)\ \sin\Omega t - \phi f_2, \tag{67}$$

$$f_1 + \tau_{\sigma\alpha} f_1^{(\alpha)} = x_1 + \tau_{\varepsilon\alpha} x_1^{(\alpha)},\ f_2 + \tau_{\sigma\beta} f_2^{(\beta)} = \zeta\left(x_2 + \tau_{\varepsilon\beta} x_2^{(\beta)}\right), \tag{68}$$

where the derivatives are taken with respect to dimensionless time and where the bars are suppressed over the dimensionless variables. The equations (67) and (68) together with (62) and (65)$_{2,3,5,6}$ remain unchanged.

Introducing the Laplace transform of $x_i(t)$ and $f_i(t)$ say $X_i = X_i(s) = \mathcal{L}x_i(t) = \int_0^\infty e^{-st} x_i(t)\ dt$ and $F_i = F_i(s) = \mathcal{L}f_i(t) = \int_0^\infty e^{-st} f_i(t)\ dt,\ i = 1,\ 2$ from (67), (68) and (62) we get

$$X_1 = \frac{\Omega^3\left[\left(1+\tau_{\sigma\alpha}s^\alpha\right)\ \left(1+\tau_{\sigma\beta}s^\beta\right)\ s^2 + \zeta(1+\phi)\ \left(1+\tau_{\sigma\alpha}s^\alpha\right)\ \left(1+\tau_{\varepsilon\beta}s^\beta\right)\right]}{\left[\left(s+a_g\right)^2 + \Omega^2\right]\ D(s)}, \tag{69}$$

where

$$\begin{aligned} D(s) = \{&\left(1+\tau_{\sigma\alpha}s^\alpha\right)\ \left(1+\tau_{\sigma\beta}s^\beta\right)\ s^4 + [\left(1+\tau_{\varepsilon\alpha}s^\alpha\right)\ \left(1+\tau_{\sigma\beta}s^\beta\right) + \\ &\zeta(1+\phi)\ \left(1+\tau_{\sigma\alpha}s^\alpha\right)\ \left(1+\tau_{\varepsilon\beta}s^\beta\right)]s^2 + \zeta\phi\left(1+\tau_{\varepsilon\alpha}s^\alpha\right)\ \left(1+\tau_{\varepsilon\beta}s^\beta\right)\} \end{aligned} \tag{70}$$

and

$$X_2 = \frac{\Omega^3\left(1+\tau_{\varepsilon\alpha}s^\alpha\right)\ \left(1+\tau_{\sigma\beta}s^\beta\right)}{\left[\left(s+a_g\right)^2 + \Omega^2\right]\ D(s)} \tag{71}$$

$$F_1 = \frac{\Omega^3\left[\left(1+\tau_{\varepsilon\alpha}s^\alpha\right)\ \left(1+\tau_{\sigma\beta}s^\beta\right)\ s^2 + \zeta(1+\phi)\ \left(1+\tau_{\varepsilon\alpha}s^\alpha\right)\ \left(1+\tau_{\varepsilon\beta}s^\beta\right)\right]}{\left[\left(s+a_g\right)^2 + \Omega^2\right]\ D(s)}, \tag{72}$$

$$F_2 = \frac{\zeta\Omega^3\left(1+\tau_{\varepsilon\alpha}s^\alpha\right)\ \left(1+\tau_{\varepsilon\beta}s^\beta\right)}{\left[\left(s+a_g\right)^2 + \Omega^2\right]\ D(s)}, \tag{73}$$

where we have used the standard expression for the Laplace transform of the $\psi-$th derivative, $0 < \psi = \alpha,\ \beta < 1,$ given by (12).

The functions $X_i(s),\ F_i(s),\ i = 1,\ 2,$ have the originals $x_i(t),\ f_i(t),\ i = 1,\ 2$, respectively. In order to prove that, we use the argument presented in [44], p. 85. Namely, for each $X_i(s),\ F_i(s),\ i = 1,\ 2,$ say $Y(s),$ it has to be shown that $Y(s)$ is analytic in the area $\mathrm{Re}(s) > k_0$ that tends to zero when $\mathrm{Im}(s) \to \infty$ and that $\int_{-\infty}^{\infty} \left|Y(k + j\eta)\right|\ d\eta$ converges. In order to examine the motion of the column-like structure after a major seismic event the inversion of (69), (71) - (73) by complex integration will be done. Following the standard procedure first, we chose the contour with a cut along the negative real axis, say $\gamma = ABDEFGA$ of Fig. 9. Then we analyze the number of poles of (69) inside γ noting that the same analysis will be applied to (71) - (73).

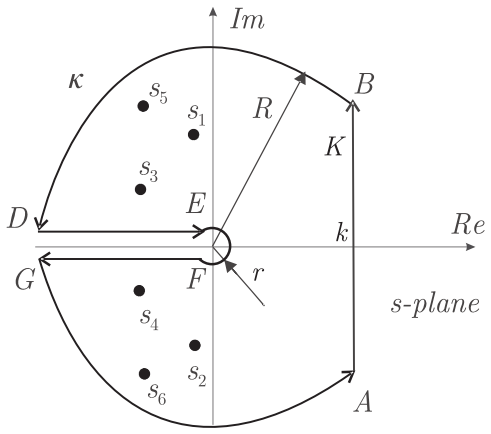

Fig 9: Countour of integration for the column problem.

The poles $s_1 = -a_g + \Omega j$ and $s_2 = -a_g - \Omega j$, where j stands for the imaginary unit are obvious. In order to determine the number of zeros of $D(s)$, given by (70), once again we apply Roushè's theorem. Namely, first we note that $1+\tau_{\sigma\alpha}s^{\alpha}$ and $1+\tau_{\sigma\beta}s^{\beta}$ does not vanish on the first sheet of the Riemann surface. Then, we rewrite the function $D(s)/\left[\left(1+\tau_{\sigma\alpha}s^{\alpha}\right)\left(1+\tau_{\sigma\beta}s^{\beta}\right)\right]$ in the form $u(s)+w(s)$ with

$$u(s) = s^4 + \left[1+\zeta(1+\phi)\right] s^2 + \zeta\phi,$$

$$w(s) = \left[\frac{\zeta(1+\phi)\left(\tau_{\varepsilon\beta}-\tau_{\sigma\beta}\right)s^{\beta}}{1+\tau_{\sigma\beta}s^{\beta}} + \frac{\left(\tau_{\varepsilon\alpha}-\tau_{\sigma\alpha}\right)s^{\alpha}}{1+\tau_{\sigma\alpha}s^{\alpha}}\right] s^2 +$$
$$\zeta\phi\left[\frac{\left(\tau_{\varepsilon\alpha}-\tau_{\sigma\alpha}\right)s^{\alpha}}{1+\tau_{\sigma\alpha}s^{\alpha}} + \frac{\left(\tau_{\varepsilon\beta}-\tau_{\sigma\beta}\right)s^{\beta}}{1+\tau_{\sigma\beta}s^{\beta}} + \frac{\left(\tau_{\varepsilon\alpha}-\tau_{\sigma\alpha}\right)s^{\alpha}}{1+\tau_{\sigma\alpha}s^{\alpha}}\frac{\left(\tau_{\varepsilon\beta}-\tau_{\sigma\beta}\right)s^{\beta}}{1+\tau_{\sigma\beta}s^{\beta}}\right].$$

Finally, we put s in the form $\rho(\cos\theta + j\sin\theta)$, then show that $\lim_{\rho\to 0}|w|/|u|<1$ and $\lim_{\rho\to\infty}|w|/|u|<1$ and conclude that u and $u+w$ have the same number of zeros inside γ, in this case four.

In order to find four more poles of (69), say s_b, $b=3,..,6$, using (70), we split (70) into the system

$$\begin{gathered}
\tau_{\sigma\alpha}\tau_{\sigma\beta}\rho^{4+\alpha+\beta}\cos(4+\alpha+\beta)\theta + \tau_{\sigma\alpha}\rho^{4+\alpha}\cos(4+\alpha)\theta + \\
\tau_{\sigma\beta}\rho^{4+\beta}\cos(4+\beta)\theta + \rho^4\cos 4\theta + \\
\left(\tau_{\sigma\beta}\tau_{\varepsilon\alpha} + \tau_{\sigma\alpha}\tau_{\varepsilon\beta}(1+\phi)\zeta\right)\rho^{2+\alpha+\beta}\cos(2+\alpha+\beta)\theta + \\
\left(\tau_{\sigma\alpha}(1+\phi)\zeta + \tau_{\varepsilon\alpha}\right)\rho^{2+\alpha}\cos(2+\alpha)\theta + \\
\left(\tau_{\sigma\beta} + \tau_{\varepsilon\beta}(1+\phi)\zeta\right)\rho^{2+\beta}\cos(2+\beta)\theta + (1+\zeta+\phi\zeta)\rho^2\cos 2\theta + \\
\zeta\phi\tau_{\varepsilon\alpha}\tau_{\varepsilon\beta}\rho^{\alpha+\beta}\cos(\alpha+\beta)\theta + \zeta\phi\tau_{\varepsilon\alpha}\rho^{\alpha}\cos\alpha\theta + \zeta\phi\tau_{\varepsilon\beta}\rho^{\beta}\cos\beta\theta + \zeta\phi = 0
\end{gathered} \tag{74}$$

$$\begin{gathered}\tau_{\sigma\alpha}\tau_{\sigma\beta}\rho^{4+\alpha+\beta}\sin(4+\alpha+\beta)\theta+\tau_{\sigma\alpha}\rho^{4+\alpha}\sin(4+\alpha)\theta+\\ \tau_{\sigma\beta}\rho^{4+\beta}\sin(4+\beta)\theta+\rho^{4}\sin 4\theta+\\ \left(\tau_{\sigma\beta}\tau_{\varepsilon\alpha}+\tau_{\sigma\alpha}\tau_{\varepsilon\beta}(1+\phi)\zeta\right)\ \rho^{2+\alpha+\beta}\sin(2+\alpha+\beta)\theta+\\ \left(\tau_{\sigma\alpha}(1+\phi)\zeta+\tau_{\varepsilon\alpha}\right)\ \rho^{2+\alpha}\sin(2+\alpha)\theta+\\ \left(\tau_{\sigma\beta}+\tau_{\varepsilon\beta}(1+\phi)\zeta\right)\ \rho^{2+\beta}\sin(2+\beta)\theta+(1+\zeta+\phi\zeta)\ \rho^{2}\sin 2\theta+\\ \zeta\phi\tau_{\varepsilon\alpha}\tau_{\varepsilon\beta}\rho^{\alpha+\beta}\sin(\alpha+\beta)\theta+\zeta\phi\tau_{\varepsilon\alpha}\rho^{\alpha}\sin\alpha\theta+\zeta\phi\tau_{\varepsilon\beta}\rho^{\beta}\sin\beta\theta=0\end{gathered} \tag{75}$$

Applying the Newton method we can find the solutions of (74), (75) that correspond to the principal branch, say $s_3=\rho_3 e^{j\vartheta_3}$, $s_4=\rho_3 e^{-j\vartheta_3}$, $s_5=\rho_5 e^{j\vartheta_5}$ and $s_6=\rho_5 e^{-j\vartheta_5}$. With this preparation done, we are ready to find

$$\begin{gathered}x_i(t)=\frac{1}{2\pi j}\lim_{K\to\infty}\int_{p-Pj}^{k+Kj}X_i(s)\ \exp(ts)\ ds,\\ f_i(t)=\frac{1}{2\pi j}\lim_{K\to\infty}\int_{k-Kj}^{k+Kj}F_i(s)\ \exp(ts)\ ds,\ (t>0),\ i=1,2,\end{gathered} \tag{76}$$

where k is suitably chosen, so all poles lie to the left of the line $s=k$, for each x_i, f_i, $i=1.2$. It remains to explore the residue theorem. According to it, each of the integrals along the closed path γ is $2\pi j$ times the sum of the residues of each of $e^{st}X_i(s)$, $e^{st}F_i(s)$, $i=1,2$, at the singularities enclosed by γ, in Fig. 9. We use (69) - (73) and rewrite

$$X_i(s)\ \exp(ts)=\frac{U_i(t,s)}{C(s)},\ F_i(s)\ \exp(ts)=\frac{W_i(t,s)}{C(s)},\ i=1,2, \tag{77}$$

with $C(s)=\left[\left(s+a_g\right)^2+\Omega^2\right]D(s)$, and where $U_i(t,s)$ and $W_i(t,s)$ are obtained by multiplying the numerators of (69) - (73) by e^{ts} respectively. The residue of $U_i(t,s)/C(s)$, (as well as $W_i(t,s)/C(s)$ $i=1,2$) at the point s_0 reads $U_i(t,s_0)/C'(s_0)$ where prime represents the derivative with respect to s. Referring to Doetcsh once again, we conclude that the integrals along BD, GA and EF vanish (when $R\to\infty$ and $r\to 0$). After calculating the sum of the integrals along DE and FG, we finally obtain the motion and shearing forces as

$$\begin{gathered}x_i(t)=\sum_{b=1}^{6}\frac{U_i(t,s_b)}{C'(s_b)}-\frac{1}{2\pi j}\int_0^{\infty}\left[\frac{U_i(t,\eta\exp(\pi j))}{C(\eta\exp(\pi j))}-\frac{U_i(t,\eta\exp(-\pi j))}{C(\eta\exp(-\pi j))}\right]d\eta,\\ i=1,2,\\ f_i(t)=\sum_{b=1}^{6}\frac{W_i(t,s_b)}{C'(s_b)}-\frac{1}{2\pi j}\int_0^{\infty}\left[\frac{W_i(t,\eta\exp(\pi j))}{C(\eta\exp(\pi j))}-\frac{W_i(t,\eta\exp(-\pi j))}{C(\eta\exp(-\pi j))}\right]d\eta.\end{gathered} \tag{78}$$

The integrals on the right side of (78) could be easily calculated by standard numerical procedures.

In order to illustrate the above results, we are going to present the motions of the system for numerical values of constants $a_g=0.25$, $\Omega=3$ that determine the ground motion in horizontal direction and the values of constants $0<\alpha,\beta<1$, $\tau_{f\alpha}$, $\tau_{f\beta}$ and $\tau_{x\alpha}$, $\tau_{x\beta}$. Namely, for $\alpha=0.23$, $\tau_{\sigma\alpha}=0.004$, $\tau_{\varepsilon\alpha}=1.183$, $\beta=0.49$, $\tau_{\sigma\beta}=5\times10^{-8}$, $\tau_{\varepsilon\beta}=0.886$, in Fig. 10 we present the motion of the blocks and the corresponding shearing

forces. The corresponding solution of the system (74), (75) reads $\rho_3 = 0.886$, $\rho_5 = 2.393$, $\theta_3 = 1.703$, $\theta_5 = 1.784$.

Introducing $\Delta\tau_a = \tau_{\varepsilon\alpha} - \tau_{\sigma\alpha}$ and $\Delta\tau_\beta = \tau_{\varepsilon\beta} - \tau_{\sigma\beta}$ we note that increasing the values α, β, as well as the values $\Delta\tau_a$, $\Delta\tau_\beta$ the more energy will be dissipated and thus the amplitude of the blocks motion decreases as expected, see [60]. This fact could be very useful for sky-scrapers and antic-columns protection. Finally, we may pose a question how the system under consideration will behave if the second law of thermodynamic is violated. Choosing $\Delta\tau_\alpha < 0$, $\Delta\tau_\beta < 0$ and solving (74), (75) yield $\theta_3, \theta_5 < \pi/2$ and thus $\lim_{t\to\infty} \sum_{b=1}^{6}(\cdot)$ in (78) tends to infinity, which leads to the motion represented by the time diverging function. More numerical examples are given in [68].

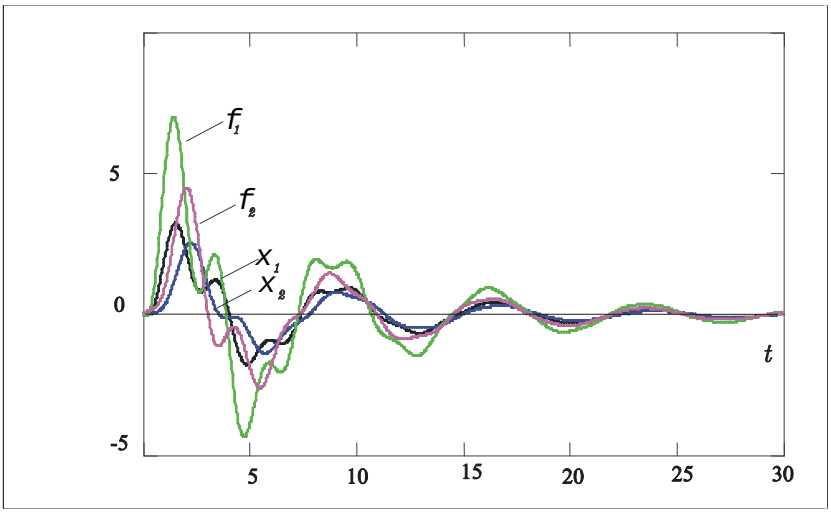

Fig 10: The solution $x_i(t)$, $f_i(t)$, $i=1, 2$ for $\alpha = 0.23$, $\tau_{\sigma\alpha} = 0.004$, $\tau_{\varepsilon\alpha} = 1.183$, $\beta = 0.49$, $\tau_{\sigma\beta} = 5\times10^{-8}$, $\tau_{\varepsilon\beta} = 0.886$,

For the time being we are mainly concerned with linear problems, since the contribution of fractional modeling, given by (9) and (10), to good descriptions of viscoelastic properties of real materials, while still remaining in physically linear theories, has been most valuable. This is due to the fact that the use of fractional derivative constitutive law can be viewed as the limit for linearized models to capture nonlinear response. In other words, it seems that a generalized linear model of a viscoelastic body that contains fractional derivatives of stress and strain is capable of describing viscoelastic behavior of real materials in a more accurate way than nonlinear constitutive models with derivatives of integer order. However, apart from physical considerations nonlinearity can come from purely geometrical arguments. Following this line, one may pose a problem on how to treat finite deformations coupled with so called standard fractional viscoelastic body. This will lead to nonlinear fractional differential equations. Actually, the study of considered fractional standard linear solid models possessas an essential mathematical interest too. In dealing with it, special care is needed since not all properties of integer derivatives are recognized for fractional ones.

In the following subsection the motion of an airplane landing on a smooth straight line and stretching a weightless viscoelastic fiber whose ends are anchored at the points at a given distance from the line will be considered. The dynamics of the problem can be represented by a single integral equation involving the Mittag-Leffler-type function whose solution is ensured by the Contraction Mapping Principle, see [61], and will be obtained numerically using the first-order fractional difference approximation.

6.3.4 A geometrically nonlinear problem

Consider the motion of an airplane landing on a straight line and stretching a weightless viscoelastic fiber whose ends are anchored at the points at a given distance from the line, see Fig. 11. Roughly speaking, the landing script could be as follows. At the time $t = 0$ the airplane of mass m, with velocity v_0, touches the

flight deck and at the same moment, it touches the weightless viscoelastic fiber, of length $2l$, which was perpendicular to the line of landing. It is assumed that the area of the fibre cross-section A remains unchanged during the deformation process. The stretching of the fibre will proceed until the airplane slows dawn. With a very low velocity, say at $t=\overline{t}$, the airplane will release the fibre and in order to stop, if necessary, will use the classical brake. This last part is not the subject here, it will be postponed for the next subsection.

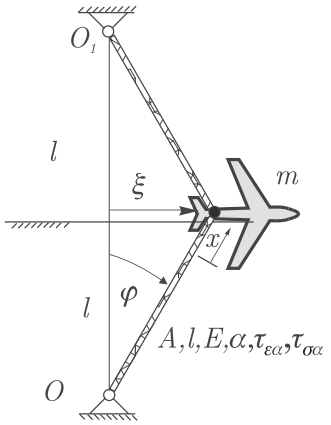

Fig. 11: Landing phase under consideration.

The differential equation of motion of an airplane and the initial conditions read

$$m\,\xi^{(2)}=-2f\sin\varphi,\ \xi^{(1)}(0)=v_0,\ \xi(0)=0,\ f(0)=0, \tag{79}$$

where $(\cdot)^{(k)}=d^k(\cdot)/dt^k$ denotes the $k-$ th derivative with respect to time t, and where $\xi=\xi(t)$ and $f=f(t)$ stand for the coordinate and the contact force between the airplane and the fibre. It should be noted that large values of ξ and φ (the angle describing the fibre deformation) are allowed.

The strain measures are often defined with special requirements in mind, see [3]. Let $x=x(t)$ be the half measure of the isothermal uniaxial deformation of the fibre. Note that with $f=f(t),\ \sigma=f/A$ and $x=x(t),\ \varepsilon=x/l$, a complementary constitutive equation of the deformable fibre (9), may be taken in the usual form (28), (10). Further, it is assumed that

$$x(0)=0. \tag{80}$$

In the following the obvious geometrical relations

$$\sin\varphi=\xi/(l+x),\ \xi^2+l^2=(x+l)^2, \tag{81}$$

are going to be very useful. Namely, introducing the dimensionless quantities

$$\overline{\xi}=\frac{\xi}{l},\ \overline{x}=\frac{x}{l},\ \overline{f}=\frac{f}{E_\alpha A},\ \overline{t}=t\sqrt{\frac{2E_\alpha A}{m\,l}},$$
$$\overline{\tau}_{\varepsilon\alpha}=\tau_{\varepsilon\alpha}\left[\frac{2E_\alpha A}{m\,l}\right]^{\alpha/2},\ \overline{\tau}_{\sigma\alpha}=\tau_{\sigma\alpha}\left[\frac{2E_\alpha A}{m\,l}\right]^{\alpha/2},\ \kappa_1=v_0\sqrt{\frac{m}{2E_\alpha A\,l}}, \tag{82}$$

one gets the following system describing the airplane landing phase

$$\xi^{(2)} = -f\frac{\xi}{\sqrt{1+\xi^2}},\ \xi^{(1)}(0) = \kappa_1,\ \xi(0) = 0,\ f(0) = 0, \tag{83}$$

with (32) and

$$\xi^2 + 1 = (1+x)^2. \tag{84}$$

In equations (83), (84) the bar was omitted and the derivatives are taken with respect to dimensionless time. Once again, note that the thermodynamical restrictions $(10)_{2,3}$ in dimensionless form remains the same, as in the whole Chapter.

The main concern of this work is the solution of (83), (84). Before one proceeds to it, two remarks should be made here. First, by differentiating (84) twice, variable ξ could be eliminated, i.e., the nonlinear equation

$$x^{(2)} - \frac{\left[x^{(1)}\right]^2}{x(1+x)\ (2+x)} + fx\frac{(2+x)}{(1+x)^2} = 0, \tag{85}$$

is to be solved together with (32), with initial conditions

$$x(0) = 0,\ x^{(1)}(0) = 0,\ f(0) = 0, \tag{86}$$

but this form of the problem is not tractable enough, (note that $x^{(2)}(0) = \kappa_1^2 \neq 0$). Secondly, recall that the constitutive equation (32), the so called modified Zener model, is good enough to describe viscoelastic behavior for a wide class of real materials, metals, geological strata, glass, polymers for vibration control, see [54], for example. When dealing with (32), a special attention should be paid to the thermodynamical restrictions $(10)_{2,3}$ that should be observed in determining the parameters of the model from experimental results. However, in some problems, despite the fact it violates thermodynamical constraint $\tau_{\sigma\alpha} > 0$, the term $\tau_{\sigma\alpha}$ is small enough and could be neglected, for example, see [24], where $\tau_{\sigma\alpha}$ reads $0.69\times10^{-9}\ \ \mathrm{sec}^{0.49}$. In such cases the problem (32), (85) reduces to the single nonlinear fractional differential equation

$$x^{(2)} - \frac{\left[x^{(1)}\right]^2}{x(1+x)\ (2+x)} + \frac{\left(x^2 + \tau_{\varepsilon\alpha}xx^{(\alpha)}\right)\ (2+x)}{(1+x)^2} = 0, \tag{87}$$

with initial conditions $(86)_{1,2}$.

In order to solve the airplane landing problem the Laplace transform method will be applied. It will be shown that the dynamics of the problem is governed by a single integral equation involving the Mittag-Leffler-type function, whose solution is ensured by the Contraction Mapping Principle, [36]. In doing so we use (44) and (84). Namely, substituting $x = \sqrt{\xi^2+1} - 1$ in (44) and the obtained expression for $f(t)$ into (83), the landing problem reduces to the following initial data problem

$$\xi^{(2)}=-\frac{\xi}{\sqrt{1+\xi^2}}\left\{\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\left(\sqrt{1+\xi^2}-1\right)+\frac{1}{\tau_{\sigma\alpha}}\left(1-\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\right)\times\right.$$

$$\left.\int_0^t e_{\alpha,\alpha}(t-\eta,\frac{1}{\tau_{\sigma\alpha}})\left[\sqrt{1+\xi^2(\eta)}-1\right]\,d\eta\right\}, \tag{88}$$

$$\xi(0)=0,\ \xi^{(1)}(0)=\kappa_1.$$

Thus, ξ has to satisfy the following integral equation

$$\xi(t)=\int_0^t\left\{\kappa_1-\int_0^s\frac{\xi(u)}{1+\xi^2(u)}\left[\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\left(\sqrt{1+\xi^2(u)}-1\right)+\frac{1}{\tau_{\sigma\alpha}}\left(1-\frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}}\right)\times\right.\right.$$

$$\left.\left.\int_0^u e_{\alpha,\alpha}(u-\eta,\frac{1}{\tau_{\sigma\alpha}})\left(\sqrt{1+\xi^2(\eta)}-1\right)\,d\eta\right]\,du\right\}\,ds=\mathcal{M}(\xi(t)), \tag{89}$$

to which the fixed point theorem can be applied, as in [61].

Finally, using (26) together with the geometrical relation (84) the influence of the four constants describing the fiber properties on the landing track could be examined, see [51] and [61] for details. The obtained algorithm reads

$$\xi_m=2\xi_{m-1}-\xi_{m-2}-\frac{h^2 f_{m-1}\xi_{m-1}}{\sqrt{1+\xi_{m-1}}},\ m=3,4,\ldots$$

$$f_m=\left(\sqrt{\xi_m^2+1}-1\right)\frac{h^\alpha+\tau_{\varepsilon\alpha}}{h^\alpha+\tau_{\sigma\alpha}}+\frac{1}{h^\alpha+\tau_{\sigma\alpha}}\sum_{j=1}^{m}\omega_{j,\alpha}\left[\tau_{\varepsilon\alpha}\left(\sqrt{\xi_{m-j}^2+1}-1\right)-\tau_{\sigma\alpha}f_{m-j}\right], \tag{90}$$

with $\xi_0=0,\ f_0=0,$ and $\xi_1,\ \xi_2$ given as a solution of

$$\xi_2-2\xi_1+h^2\frac{\xi_1\left(\sqrt{\xi_1^2+1}-1\right)}{\sqrt{1+\xi_1^2}}\frac{h^\alpha+\tau_{\varepsilon\alpha}}{h^\alpha+\tau_{\sigma\alpha}}=0,\ \xi_2-4\xi_1+2\kappa_1 h=0,$$

with

$$f_1=\left(\sqrt{\xi_1^2+1}-1\right)\frac{h^\alpha+\tau_{\varepsilon\alpha}}{h^\alpha+\tau_{\sigma\alpha}},$$

and

$$f_2=\left(\sqrt{\xi_2^2+1}-1\right)\frac{h^\alpha+\tau_{\varepsilon\alpha}}{h^\alpha+\tau_{\sigma\alpha}}+\omega_{1,\alpha}\frac{\left[\tau_{\varepsilon\alpha}\left(\sqrt{\xi_1^2+1}-1\right)-\tau_{\sigma\alpha}f_1\right]}{h^\alpha+\tau_{\sigma\alpha}}.$$

In order to illustrate the above results, the numerical solution for $\alpha=0.23$, and dimensionless values $\tau_{f\alpha}=0.004,\ \tau_{x\alpha}=1.183$, $\kappa_1=1$, the motion of the system is presented in Fig. 12.

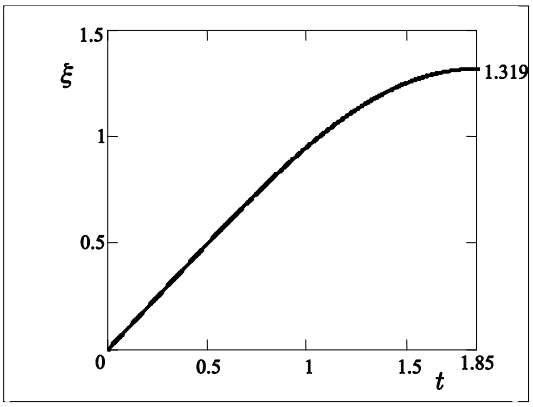

Fig 12: Motion of the airplane in landing phase for $\tau_{f\alpha}=0.004$, $\tau_{x\alpha}=1.183$ and $\kappa_1=1$.

In the aforementioned examples of engineering applications all the constraints were ideal. In reality, it is not so. Dry friction, as a strictly dissipative process, plays an important role in the analysis of energy dissipation. The motion of mechanical systems ceases in a finite time due to the presence of dry friction. In the next section we shall reexamine the impact problem by showing that the constitutive equation describing modified Zener model can be coupled with a set-valued force law describing dry friction phenomena. That law leads to an accurate model of the sliding to stick phase of motion. Moreover, the problem is rather challenging since the coupling of nonlocal operators with nonsmooth multifunctions has to be treated for both qualitative and numerical mathematical analysis. Namely, in the last decade both noninteger differentiation and nonsmooth mechanics have received much interest from the Nonlinear Dynamics community, so both areas were developed to the necessary mathematical and physical consistency. However, while each of the areas has success in models that describe behavior of real systems, well established numerical procedures that merge noninteger derivatives with nonsmooth dynamics approach are still lacking. Namely, increasing number of papers dealing with systems with unilateral constraints, set-valued fractional differential equations or differential inclusions of noninteger type is noticeable but applications of such a model in engineering are still missing. Note that in modeling real practical problems nonsmooth phenomena, caused by kinematic constraints or physical effects like friction, impact or backlash, have been considered as errors and have therefore been neglected for a long time, [52]. Later, these phenomena were considered in an approximate manner by smoothed characteristics often unjustified from the physical point of view and just recently, in refined and more precise models, they have been taken into account correctly as nonsmooth effects.

When a body impinges against the wall, it could either separate or continue to be in contact. Both scripts, rebound and capture, are predicted by the use of the Hertz theory of impact with adhesive forces, see [4]. Despite that, the model used therein is not realistic, since it does not include energy dissipation. With a different approach, as noted in the first example, energy dissipation was taken into account *ab initio*, but only rebound script was considered. In order to get responses that are very close to what experiments show, besides the constitutive model that comprises fractional derivatives of stress and strain and restrictions on the coefficients that follow from Clausius-Duhem inequality, in the following, we intend to include dry friction in the sliding surface, so the body can come to rest in a finite time after the impact. In modeling dry friction phenomena we shall use the Sgn-multifunction but instead of formulation that decomposes it into two unilateral primitives, leading to linear complementarity problems at velocity-impulse level and combinatorial evolution, see [26], as above we shall use a Hertz type theory, together with a slack variable algorithm, used for handling discontinuous model motion phases, see [65]. As a result, the prediction of three different impact scripts: rebound after the impact, capture in the approaching phase and capture in the rebound phase will be confirmed as a real outcome after impact, see [32]. With respect to qualitative analysis, an example of the fractional extension of ordinary differential equations of the Filippov type will be given. We have in mind a mechanical system that contains both the Riemann-Liouville derivatives of arbitrary real order as well as multifunction leading to a so called set-valued fractional differential equation. With respect to numerical analysis, a possible numerical method of dealing with both nonlocal operators and inequality constraints will be explained.

6.3.5 The impact problem in the presence of dry friction

Once again consider a rigid block of mass m, sliding on a dry horizontal surface, as shown in Fig. 1 above. This time there is a dry friction between the block and the surface, so the coefficient of friction μ reads $\mu > 0.$ All other assumptions remain.

The differential equation of motion of the block in the presence of dry friction and the initial conditions read:

$$m\,x^{(2)} = -p + q,\; x(0) = 0,\; x^{(1)}(0) = v_0,\; p(0) = 0, \tag{91}$$

where q is to denote the friction force that resists the motion between the mass and the surface. The complementary relations to (91) are (28) that should be followed by (10). Among all existing dry friction models we assume that the force q obeys Coulomb's frictional law given in the following set-valued form

$$-q \in \mu N\, \mathbf{Sgn}\left(x^{(1)}\right), \tag{92}$$

with μ as the friction coefficient, and where $\mathbf{Sgn}(u)$ stands for a maximal monotone set-valued map (the filled-in relay function) defined as

$$\mathrm{Sgn}(u) = \begin{cases} \{1\}, & \text{if } u > 0 \\ [-1, 1], & \text{if } u = 0 \\ \{-1\}, & \text{if } u < 0. \end{cases} \tag{93}$$

Note that from the free body diagram one obtains that normal reaction of the horizontal surface N equals the weight of the block mg. Also, note that coefficients of friction for impact phenomena μ cannot be accurately determined, and consequently their specification rests upon either pure hypothesis or corresponding values for non-collision processes, [28]. However, during collisions large amounts of energy may be dissipated in a very short period of time and Coulomb friction is not sufficient to account for this loss, [30]. Therefore, extra constitutive laws are necessary for modeling impacts. Here, we use the one that is thermodynamically consistent and taken *ab initio*, by use of (28), (10). On the other hand, note that several regularizations of the set-valued friction law (92) by smooth function are possible, but since they can not describe properly the stick phase, we are not going to consider them here, see [40].

There are many results of nonsmooth dynamics that could be related to this problem, for example, see the papers cited in [32]. For the purpose of the problem at hand we recall that three different scenarios for planar contact problems: sticking $(x^{(1)} = 0 \Rightarrow |q| \le \mu N)$, positive sliding $\left(x^{(1)} > 0 \Rightarrow q = -\mu N\right)$ and negative sliding $\left(x^{(1)} < 0 \Rightarrow q = \mu N\right)$. Small friction leads to a sequence of states of sliding with alternating sign of the velocity after impact. As expected increasing μ the system tends to the state of rest. The sticking case, when motion ceases after the impact motivates the adopted model with the friction force q as a multifunction. Note that q may jump to any point within the set $[-\mu, \mu]$, when transition to stiction occurs, see [27]. Besides, when motion ceases, a constant displacement from initial position is maintained, leading to equilibrium of the forces, in our case p and q. We intuitively understand that, for the case when the body come to rest in a finite time after the impact, the viscoelastic rod is captured between the wall and the asperities of the dry surface below the block, and thus the impact is followed by some kind of stress relaxation due to compression of the rod. We do not treat here this relaxation process since the viscoelastic rod is assumed as massless.

Introducing the duration of impact, say T, we consider that the body is separated when the contact force is equal to zero i.e. $p(T) = 0$, with $x^{(1)}(T) \le 0$, or the body is captured after impact that is $x^{(1)}(T) = 0$, with $p(T) = q(T) \ne 0$. Note that the capture case may take place in either approaching or rebound phase depending

on the amount of dissipated energy during motion. Namely, the amount of initial energy of the system $m v_0^2/2$, dissipated during impact reads

$$\Delta = \int_0^T \left(p x^{(1)} + q \left| x^{(1)} \right| \right) \, dt. \tag{94}$$

The described model of impact belongs to the Hertz's type theories. Its special feature is that it avoids usual problems of the classical nonsmooth dynamics impact theory related to specification of the restitution coefficients that rests upon pure hypothesis, as for example in [26].

Next, we add the dimensionless friction force to (29), as $\bar{q} = q/(m g)$ and get the relations describing the impact of the system presented in Fig. 1 with $\mu > 0$, it reads

$$x^{(2)} = -p + q, \; x(0) = 0, \; x^{(1)}(0) = \xi, \; p(0) = 0, \tag{95}$$

$$-q \in \mu \mathbf{Sgn}\left(x^{(1)} \right), \tag{96}$$

together with (32) and $(10)_{2,3}$, that belong to a class of set-valued fractional differential equations (or multivalued differential equations of arbitrary real order). As usual, the bar was omitted and the derivatives are taken with respect to dimensionless time.

Before one proceeds to the solution of the posed problem, it should be noted that it can be cast in a form that is more tractable for mathematical analysis. Namely, using (44) the dynamics reduces to the Cauchy problem for a single integro-differential inclusion, i.e.

$$\begin{gathered} x^{(2)} + \frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}} x + \frac{1}{\tau_{\sigma\alpha}} \left(1 - \frac{\tau_{\varepsilon\alpha}}{\tau_{\sigma\alpha}} \right) \int_0^t e_{\alpha,\alpha} \left(t - \eta, \frac{1}{\tau_{\sigma\alpha}} \right) x(\eta) \, d\eta \in -\mu \mathbf{Sgn}\left(x^{(1)} \right), \text{ a.e. on } [0, T], \\ x(0) = 0, \; x^{(1)}(0) = \xi. \end{gathered} \tag{97}$$

An existence result for the Cauchy problem given by integrodifferential inclusion (97) could be found as Theorem 2.1, p. 270, of [1].

In order to compute the solution of the problem described by (95), (96), (32) and $(10)_{2,3}$, we apply the numerical method presented in the first Section, [51] together with a slack variable algorithm used for handling discontinuous model motion phases, see [65]. Namely, in dealing with inequality constraints related to stiction and friction effects, included in (96), the unknown in the problem is the time t^* when the equality condition is reached. Referring again to [30] we note that determining the time t^* is a computation-intensive operation and it consumes a major portion of computation time. Thus, we apply the Turner algorithm. Namely as in [65], introducing the slack variable that replaces time as the independent variable, leading to extended state-space, one can calculate the exact value of t^* in just one step. The procedure is as follows.

First, as above, introducing the variable $z(t) = x(t) - \xi t$, we remove the non-homogeneous initial condition and obtain

$$z^{(2)} = -p + q, \; z(0) = 0, \; z^{(1)}(0) = 0, \; p(0) = 0, \tag{98}$$

$$p + \tau_{\sigma\alpha} p^{(\alpha)} = z + \xi t + \tau_{\varepsilon\alpha} z^{(\alpha)} + \frac{\tau_{\varepsilon\alpha} \xi \Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}, \tag{99}$$

$$-q \in \mu \mathbf{Sgn}\left(z^{(1)} + \xi \right), \tag{100}$$

Next, we introduce a time step h, ($t_m = m\,h$, $m = 0, 1, 2, \ldots$), and note that from $(98)_{2-4}$ it follows that $z_0 = p_0 = 0$, $z_1 = 0$, and for $m > 0$ discretization of (99) yields

$$p_m = \frac{1}{1+\tau_{\sigma\alpha} h^{-\alpha}} \left\{ \xi m\, h + \frac{\tau_{\varepsilon\alpha} \xi \Gamma(2) (m\,h)^{1-\alpha}}{\Gamma(2-\alpha)} + z_m \left(1 + \tau_{\varepsilon\alpha} h^{-\alpha}\right) + \frac{1}{h^{\alpha}} \sum_{j=1}^{m} \left[\omega_j \left(\tau_{\varepsilon\alpha} z_{m-j} - \tau_{\sigma\alpha} p_{m-j} \right) \right] \right\}. \quad (101)$$

With this preparation done, we start the integration procedure in the original state-space model

$$z_m^{(2)} = -p_m - \mu, \; m = 1, 2, \ldots \quad (102)$$

together with (101), under the condition

$$z_m^{(1)} + \xi > 0. \quad (103)$$

We increase m until we violate (103). The last value of m before the violation of (103), and the corresponding t, we denote by m_1 and t_{m_1} respectively. Then we introduce the slack variable $\pi_a = z_m^{(1)} + \xi$ and, using the Euler method, in one step, integrate the extended state-space model, see [65], in order to obtain the value of t_a^* that corresponds to equality condition $\pi_a^* = 0$ as well as the end of the approaching phase，that is

$$t_a^* = t_{m_1} + \frac{1}{-p_{m_1} - \mu} (\pi_a^* - \pi_{a0}), \quad (104)$$

with $\pi_{a0} = z_{m_1}^{(1)} + \xi$. Note that the step $h_1 = t_a^* - t_{m_1} < h$, could be used to integrate (102) and obtain the state of the system at $t = t_a^*$. Instead, since the duration of impact is short and computation costs are low, we go back to $t = 0$ and repeat the integration procedure with a new time step $h^* = h + h_1 / m_1$, reaching time t_a^* and the corresponding state after m_1 iterations. Noting that t_a^* corresponds to the switching sign of the velocity component ($x^{(1)}(t_a^*) = 0$) where the model motion phase should change, we examine the value of dissipated energy. If the dissipated energy equals the initial kinetic energy, the motion will stop. This ends the capture script in the approaching phase. Note that this script is followed by instantaneous change of the friction force. Namely, in the approaching phase p and q are of the same direction opposite to the velocity, q equals $-\mu$. When the motion stops q may jump to any point within the set $[-\mu, \mu]$. It jumps to a positive value less than μ and equal to the value of p that is always positive by assumption. Thus the duration of impact ending with the capture in the approaching phase reads $T = t_a^*$, the impact ends with $p(t_a^*) = q(t_a^*)$. If this is not so, the motion will start in opposite direction (the rebound phase with opposite direction of friction force). Then we apply the model

$$z_m^{(2)} = -p_m + \mu, \; m = m_1 + 1, m_1 + 2, \ldots \quad (105)$$

together with (101), and the history identified for $m < m_1$ that is valid under the condition

$$z_m^{(1)} + \xi < 0. \quad (106)$$

In the integration algorithm with the step h^* that follows it could happen that for some instant of time $t = T$ the contact force $p(T)$ could vanish with the corresponding value $x^{(1)}(T) = z^{(1)} + \xi < 0$ which confirms the rebound script when the block and the wall separate with a relative velocity that is less than ξ. This is due to

dissipation introduced with both modified Zener's model and dry friction as strictly dissipating processes. If it is not the case, we continue to increase m unless we violate (106). Then, by m_2 and t_{m_2} we denote the last value of m before the violation of (106), and the corresponding t respectively. As before, we use the slack variable π_r but this time with the corresponding original model (106) and by similar procedure obtain the instant of time t_r^* where the motion in the rebound phase stops, i.e.

$$t_r^* = t_{m_2} + \frac{1}{-p_{m_2} + \mu}\left(\pi_r^* - \pi_{r1}\right), \tag{107}$$

with $\pi_{r1} = z_{m2}^{(1)} + \xi$ and $\pi_r^* = 0$. Noting that $x^{(1)}\left(t_r^*\right) = 0$, the duration of impact reads $T = t_r^*$. The remaining values of the state vector can be obtained using the Euler method and integrating in the extended state space, see [65]. Also, note that in the rebound phase the forces p and q are of opposite direction, q equals μ and when motion ceases it instantaneously jumps to the value equal to $p\left(t_r^*\right)$. Finally, since q was chosen to be multifunction, both denominators in (104) and (107) are well defined.

In Fig. 13 we show the solutions for $\alpha = 0.23$, $\tau_{\sigma\alpha} = 0.004$, $\tau_{\varepsilon\alpha} = 1.183$. For $\mu = 0.2$ the block will separate from the wall after $T = 2.086$ with velocity $x^{(1)}(T) = -0.538$. The capture in rebound phase was obtained for $\mu = 0.6$. The motion ceases after $T = 2.705$, the corresponding values of reaction force and friction force in equilibrium reads 0.381. For $\mu = 0.9$ the block will come to rest in the approaching phase in the finite time $T = 0.667$. The corresponding value of the reaction force reads $p(T) = 0.804$, and the corresponding jump of the friction force was from -0.9 to $q(T) = 0.804$.

From the curves presented in Fig. 13 we conclude that by increasing μ, the maximal values of reaction force $p_{\max}$ and deformation $x_{\max}$ decreases, while the residual deformation of the viscoelastic rod $x(T)$ increases. All the scenarios were confirmed and for $\alpha = 0.49$, $\tau_{\sigma\alpha} = 5\times10^{-8}$, $\tau_{\varepsilon\alpha} = 0.886$. The values of μ were assumed as for noncollision processes. In all the calculations the value of ξ was equal to 1, and the initial time step was 10^{-3}. In order to examine the influence of the initial time step on suggested numerical procedure, it was changed to 10^{-4}. Then all the calculations were repeated yielding the same results as shown in the following tables. As expected for $\mu \to 0$, the results of the first subsection were recovered. For the values $\mu < 0.3$ the mass will rebound from the wall for both sets of viscoelastic constants, see Table 2.

In Table 3 we present the results for $\mu > 0.3$. When compared for the same value of $\mu = 0.4$ the material with $\alpha = 0.23$ will separate from the wall, which was not the case for material with $\alpha = 0.49$. This result was expected since increasing α the dissipated energy in the deformation process increases. For the values $\mu > 0.5$, for both sets of viscoelastic constants, the capture was the predicted impact script. Namely, it takes place in the rebound phase for $\mu = 0.6$ and in the approach phase for $\mu = 0.9$.

Finally, we comment on the calculated values of post-impact velocities $x^{(1)}(T)$ obtained for the realistic choice of constants (10) and μ. These values correspond to the restitution coefficient of the Newton's impact law since pre-impact velocities $\xi = v_0\left[(E_\alpha A)/(m l)\right]^{1/2}/g$ were chosen to be 1. It seems that these values could be used in various engineering applications: systems with impacts and sliding friction, backlash, accident reconstructions, feeding and drilling machines, etc., see [50], [42], [21], [62], and the references therein.

The last engineering application considered belongs to a class of parallel studies of fractional differential equations to the well-known theory of ordinary differential equations, and particularly on set-valued fractional differential equations that had appeared after several well-known papers of Bagley and Torvik published in 80s of the former century. The suggested numerical method was based on the convexication proposed by Filippov in the 1960s that extends a discontinuous fractional differential equation to a fractional inclusion. The physical model composed of the standard fractional linear viscoelastic body and the set valued dry friction force law can be used as an extension of the previously analyzed problems too.

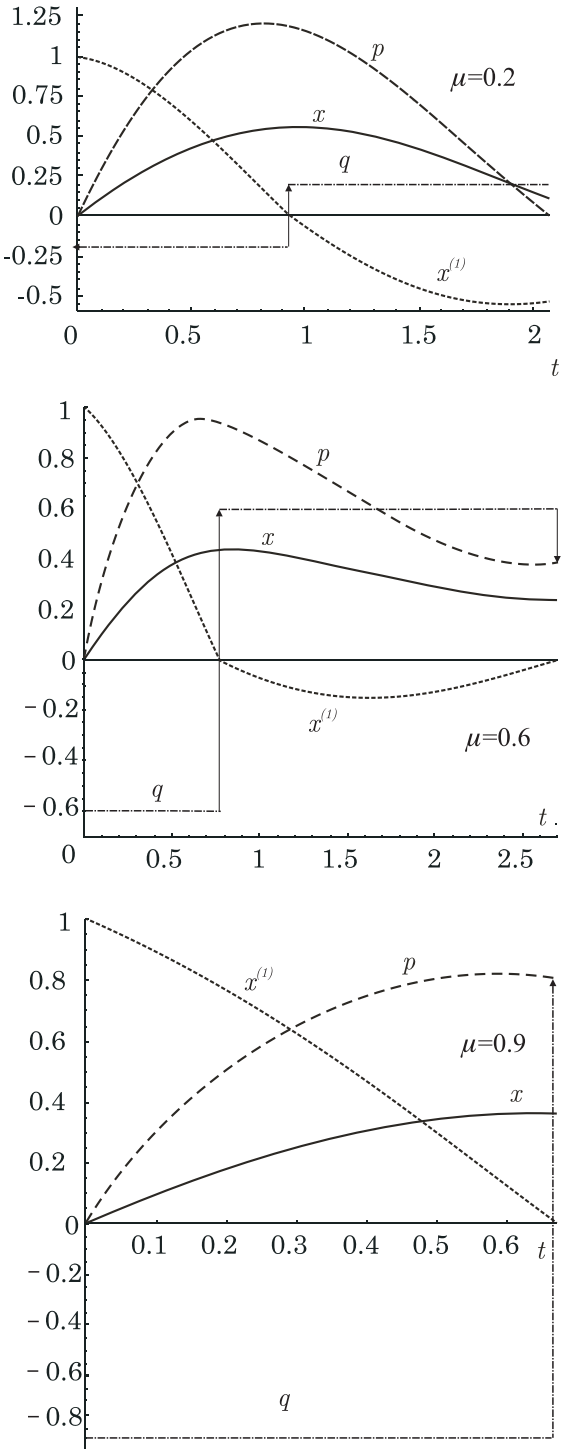

Figure 13: Impact scripts for $\alpha = 0.23$, $\tau_{\sigma\alpha} = 0.004$, $\tau_{\varepsilon\alpha} = 1.183$. and $\mu = 0.2$ (separation), $\mu = 0.6$ (capture in rebound phase) and $\mu = 0.9$ (capture in approaching phase).

Table 2

Friction μ	Viscoelastic constants $\alpha = 0.23$, $\tau_{\sigma\alpha} = 0.004$, $\tau_{\varepsilon\alpha} = 1.183$	Viscoelastic constants $\alpha = 0.49$, $\tau_{\sigma\alpha} = 5\times10^{-8}$, $\tau_{\varepsilon\alpha} = 0.886$
0	$p_{max}(0.91) = 1.356$ $x_{max}(1.034) = 0.632$ $T = 2.024$ $x(T) = 0.135$ $x^{(1)}(T) = -0.771$ $p(T) = 0$ $q(T) = 0$ $\Delta(T) = 0.203$	$p_{max}(0.831) = 1.141$ $x_{max}(1.109) = 0.643$ $T = 2.150$ $x(T) = 0.232$ $x^{(1)}(T) = -0.597$ $p(T) = 0$ $p(T) = 0$ $\Delta(T) = 0.321$
0.01	$p_{max}(0.905) = 1.349$ $x_{max}(1.028) = 0.628$ $T = 2.025$ $x(T) = 0.134$ $x^{(1)}(T) = -0.761$ $p(T) = 0$ $q(T) = 0.01$ $\Delta(T) = 0.209$	$p_{max}(0.826) = 1.134$ $x_{max}(1.103) = 0.638$ $T = 2.153$ $x(T) = 0.230$ $x^{(1)}(T) = -0.586$ $p(T) = 0$ $p(T) = 0.01$ $\Delta(T) = 0.327$
0.2	$p_{max}(0.825) = 1.203$ $x_{max}(0.94) = 0.554$ $T = 2.086$ $x(T) = 0.110$ $x^{(1)}(T) = -0.538$ $p(T) = 0$ $q(T) = 0.2$ $\Delta(T) = 0.355$	$p_{max}(0.735) = 1.014$ $x_{max}(0.992) = 0.557$ $T = 2.318$ $x(T) = 0.173$ $x^{(1)}(T) = -0.361$ $p(T) = 0$ $p(T) = 0.2$ $\Delta(T) = 0.434$

Table 3

Friction μ	Viscoelastic constants $\alpha = 0.23$, $\tau_{\sigma\alpha} = 0.004$, $\tau_{\varepsilon\alpha} = 1.183$	Viscoelastic constants $\alpha = 0.49$, $\tau_{\sigma\alpha} = 5\times10^{-8}$, $\tau_{\varepsilon\alpha} = 0.886$
0.4	$p_{max}(0.746) = 1.070$ $x_{max}(0.852) = 0.487$ $T = 2.456$ $x(T) = 0.076$ $x^{(1)}(T) = -0.203$ $p(T) = 0$ $q(T) = 0.4$ $\Delta(T) = 0.479$	$p_{max}(0.649) = 0.908$ $x_{max}(0.886) = 0.486$ $T = 3.118$ $x(T) = 0.197$ $x^{(1)}(T) = 0$ $p(T) = 0.198$ $p(T) = 0.198$ $\Delta(T) = 0.5$
0.6	$p_{max}(0.674) = 0.957$ $x_{max}(0.772) = 0.431$ $T = 2.705$ $x(T) = 0.235$ $x^{(1)}(T) = 0$ $p(T) = 0.381$ $q(T) = 0.381$ $\Delta(T) = 0.5$	$p_{max}(0.575) = 0.819$ $x_{max}(0.793) = 0.427$ $T = 2.179$ $x(T) = 0.394$ $x^{(1)}(T) = 0$ $p(T) = 0.531$ $p(T) = 0.531$ $\Delta(T) = 0.5$
0.9	$p_{max}(0.58) = 0.819$ $x_{max}(0.667) = 0.362$ $T = 0.667$ $x(T) = 0.362$ $x^{(1)}(T) = 0$ $p(T) = 0.804$ $q(T) = 0.804$ $\Delta(T) = 0.5$	$p_{max}(0.482) = 0.713$ $x_{max}(0.676) = 0.357$ $T = 0.676$ $x(T) = 0.357$ $x^{(1)}(T) = 0$ $p(T) = 0.650$ $p(T) = 0.650$ $\Delta(T) = 0.5$

6.4 Conclusion

We believe that the proposed thermodynamically consistent rheological model is very tractable for engineering applications. It should be included in both analytical and experimental projects *ab initio*, particularly in experiments in which newly developed materials are tested. Also, recent developments in solving fractional differential equations gives us hope that the proposed model has a chance to be suitable for detailed multibody system models of different degree of idealization that are to be generated for further investigations of the physical world. Particularly, when combined with the set-valued dry friction law, it can predict different scenarios and valuable estimation of energy dissipation in various applications for the purpose of computer simulation, analysis and control.

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Modeling Double DNA Helix Main Chains of the Free and Forced Fractional Order Vibrations

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Abstract: DNA transcription process is well described at biochemical level. During transcription, double DNA interacts with transcription proteins; a part of double DNA is unzipped, and only one chain helix is used as a matrix for transcription. Different models of two coupled homogeneous DNA chain vibrations are proposed in the literature. To better understand the DNA transcription process and its behavior through biomechanical point of view, we consider double DNA (dDNA) as an oscillatory system that oscillates in forced regimes. When data from dDNA molecule are not transcripted, we consider that dDNA molecule oscillates in a manner of free fractional order vibrations.On the basis of previous results (DNA mathematical models published by N.Kovaleva, L. Manevich in 2005 and 2007, and multi-pendulum models by Hedrih (Stevanović) and Hedrih) we obtain a corresponding pair of main chain subsystems of the double DNA helix. Analytical expressions of the eigen circular frequencies and eigen fractional order characteristic numbers for the homogeneous model of the double DNA fractional order chain helix are obtained. Also, the corresponding eigen free and forced fractional order vibration modes and possibilities of the appearance of resonant regimes, as well as dynamical absorption under the external forced excitations are considered. Two sets of eigen normal coordinates of the double DNA chain helix for separation of the system into two uncoupled main eigen chains are identified. On the basis of the derived analytical expressions, a transfer of external excitation forced signals is analyzed. There are different cases of the resonant state in one of the main chains, and there are no interactions between main chains for special cases of the external one frequency forced excitation. This may correspond to the base pair order in complementary chains of DNA double helix in a living cell.

Key-Words: Double DNA helix chain, forced vibrations, eigen main chains, resonant state, dynamical absorption, elastic model, fractional order model, transfer of signals.

7.1 Introduction - DNA-structure and function

DNA is a biological polymer which can exist in different forms (A, B, Z, E) but only B form can be found in live organisms. Chemically, DNA consists of two long polymers of simple units called nucleotides, with backbones made of sugars and phosphate groups joined by ester bonds. To each sugar is attached one of four types of molecules called bases (Adenine-A, thymine-T guanine-G and cytosine-C). Two bases on opposite strands are linked via hydrogen bonds holding the two strands of DNA together. It is the sequence of these four bases along the backbone that encodes information.

The basic function of DNA in the cell is to encode the genetic material. For using that information to make proteins, DNA molecule has to interact with other molecules in the cell. DNA molecule is moving, changing its position and shape during the interactions. DNA molecules can be considered to be a mechanical structure on the nanolevel.

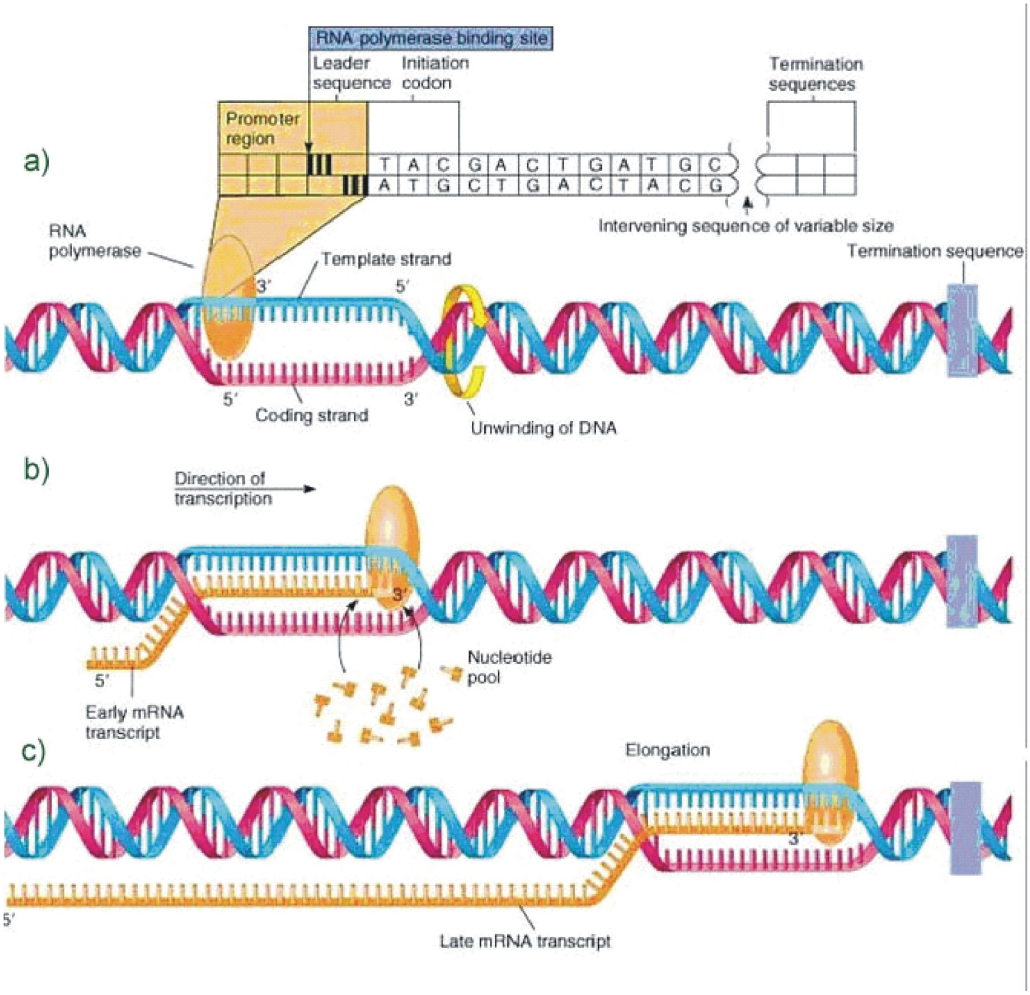


Fig.1. Model of DNA transcription (modified from, **http://www.untiredwithloving.org/satr.html**)

From the biochemical point of view, during the transcription process, genetic information is transcribing from **DNA** to RNA. The transcribed DNA message is used to produce proteins. There are three main steps to the process of DNA transcription: binding of RNA polymerase to DNA, elongation and termination. RNA polymerase is an enzyme that binds to a specific nucleotide sequences that "tell" RNA polymerase where to begin and where to end the transcription. RNA polymerase attaches to the DNA at a specific area called the promoter region. Fig 1. a) Certain proteins, called transcription factors, unwind the DNA strand and allow RNA polymerase to transcribe only a single strand of DNA into a single stranded RNA polymer called messenger RNA (mRNA). The DNA is unwound at the promoter region by RNA polymerase The strand that serves as the template is called the antisense strand. The strand that is not transcribed is called the sense strand. RNA polymerase moves along the DNA until it reaches a terminator sequence, Fig1.b) and c). At that point, RNA polymerase releases the mRNA polymer and detaches from the DNA. The DNA that is been transcribed is rewound into original configuration. Fig.1.

Every process which binds or reads DNA is able to use or modify the mechanical properties of DNA for the purposes of recognition, packaging and modification. It is important to note that DNA found in many cells can be macroscopic in length - a few centimeters long for each human chromosome. Consequently, cells must compact or "package" DNA to carry it within them, [6]. Knowledge of the elastic properties of DNA is required to understand the structural dynamics of cellular processes such as replication and transcription. Binding of proteins and other ligands induces a strong deformation of the DNA structure. The mechanical properties of DNA are closely related to its molecular structure and sequence, particularly the weakness of hydrogen bonds and electronic interactions that hold strands of DNA together compared to the strength of bonds within each strand.

Single-molecule biomechanics of DNA extension, bending and twisting; protein domain motion, deformation and unfolding; the generation of mechanical forces and motions by bimolecular motors is another approach to explain the biological function of DNA in the cell, [4].

Knowledge of the elastic properties of DNA is required to understand the structural dynamics of cellular processes such as replication and transcription. For details for possible movements of DNA molecule, see Appendix E.

There are different approaches to studying the mechanical properties of the DNA molecule (experimental, theoretical modeling).

The aim of our work was to model the DNA dynamics (vibrations of DNA chains) as a biological system in a specific boundary condition that are possible to occur in a live system during regular function of a DNA molecule. The quick review of mechanical properties of DNA achieved experimentally and relevant existed mechanical models of specific dDNA dynamics are given billow. One of soliton existence supporting model of DNA-DNA model by N. Kovaleva and L. Manevich and derived oscillatory models of DNA by Hedrih (Stevanovic) and Hedrih is specially discussed.

7.2 Mechanical properties of DNA achieved experimentally

Experimental evidence suggests that DNA mechanical properties, intrinsic curvature and flexibility in particular, have a role in many relevant biological processes.

For small distortions, DNA overwinds under tension, [18]. Lowering of the temperature does increase the DNA curvature. The DNA double helix is much more resistant to twisting deformations than to bending deformations, and almost all of the supercoiling pressure is normally relieved by writhing, [2]. The twist angle of the helix has been shown to depend on sequence when the molecule is in solution both by the effects on supercoiling parameters when short segments of known sequence are inserted into closed circular DNA [47], [53].

Under low tension, DNA behaves like an isotropic flexible rod. At higher tensions, the behavior of over- and underwound molecules is different. In each case, DNA undergoes a structural change before the twist density necessary for buckling is reached [6]. The environment and its ionic strength have influence on DNA curvature. Mg2+ can induce or enhance curvature in DNA fragments and helps stabilize several types of DNA structures, [5]. DNA length varied in solution with different ionic force. It is significantly longer in solution with lower ionic force, [14].

7.3 Mechanical models of the DNA

A number of mechanical models of the DNA double helix have been proposed until today. Different models are focusing on different aspects of the DNA molecule (biological, physical and chemical processes in which DNA is involved). A number of models have been constructed to describe different kinds of movements in a DNA molecule: asymmetric and symmetric motion; movements of long and short segments; twisting and stretching of dDNA, twist-opening conditions. We are going to mention some of the models that can explain twist-opening conditions. Details for some types of the models are given in the Appendix E.2. (see Ref [23]).

Bryant et al. (see Ref. [6]) have shown that an over- or underwound DNA molecule behaves as a constant-torque wind-up motor capable of repeatedly producing thousands of rotations, and that an overstretched molecule acts as a force–torque converter. The production of continuous directed rotation by molecular devices has potential applications in the construction of nanomechanical systems [4]. Polymer models are used to interpret single-molecule force-extension experiments on ssDNA and dsDNA. They show how combining the elasticity of two single nucleic acid strands with a description of the base-pairing interactions between them explains much of the phenomenology and kinetics of RNA and DNA ‘unzipping’ experiments [8,56]. Eslami-Mossallam and Ejtehadi, [13] proposed the asymmetric elastic rod model for DNA. Their model accounts for the difference between the bending energies of positive and negative rolls, which comes from the asymmetric structure of the DNA molecule. The model can explain the high flexibility of DNA at small length scales, as well as kink formation at high deformation limit. A special type of DNA models are soliton-existence supporting models. One of the first of this kind was the Yakushevich model of DNA and models based on it [15]. The dynamics of topological solitons describing open states in the DNA double helix are studied in the framework of a model that takes into account asymmetry of the helix. Yakushevich, et al [55] investigated

interaction between the solitons, their interactions with the chain inhomogeneities, and stability of the solitons with respect to thermal oscillations, and have shown that three types of topological solitons can occur in the DNA double chain. González and Martín-Landrove, [16] gave a complete qualitative analysis of soliton interaction in DNA torsional equations. The model emphasizes the importance of the solitons for opening of the double DNA helix. The region of the chain where there is a maximum opening is larger for the general case, since the asymptotical behavior for the kink type solitons is smoother than the one corresponding to the solutions in the particular case. There is possibility that an enzyme takes charge for the opening of the chain. The supersonic solutions, since they represent states that are totally open, could contribute significantly to the fusion of the DNA chain to the enzymatic activity. The presence of a propagating soliton along the chain could contribute to its opening through the interaction among different types of open states. The composite model for DNA is also based on the Yakushevich model (Y model). The mechanism for selecting the speed of solitons by tuning the physical parameters of the non-linear medium and the hierarchal separation of the relevant degrees of freedom are described in this model,see [7,11]. In the symmetric twist-opening model of DNA the small amplitude dynamics of the model is shown to be governed by a solution of a set of coupled nonlinear Schrödinger equations. Conditions for modulation instability occurrence are presented and attention is paid to the impact of the backbone elastic constant *K*. It is shown that high values of *K* extend the instability region. This model can be reduced to a set of coupled discrete nonlinear system equations. The growth rate of instability has been evaluated and increases with the coupling constant *K*. The kink-bubble soliton, made of two parts of different size, has been shown to be mobile. Authors supposed that the kink-bubble solution can be used to describe the internal dynamics which usually consists of long-range collective bending and twisting modes of the bases, short-range oscillations of individual bases, and the reorientation of the spin label [52]. Binding of proteins and other ligands on DNA induces a strong deformation of the DNA structure.

The aim of our work was to model the DNA dynamics (vibrations of DNA chains) as a biological system in a specific boundary condition that are possible to occur in a live system during regular function of a DNA molecule. We consider double DNA (dDNA) as an oscillatory system that oscillates in forced regimes during the DNA transcription process. To model the oscillation of dDNA in free and forced regimes we use, as a basic approach, the model of dDNA proposed by N.Kovaleva, L.Manevich, V.Smirnov (see [41], [42]). Basis of the DNA models proposed by Kovaleva and Manevich, will be given in the section bellow.

7.4 DNA models by N. Kovaleva and L. Manevich

One of the soliton-existence supporting models of dDNA is also the model proposed by N.Kovaleva, L.Manevich, V.Smirnov, see [41,42]. They show that in a double DNA helix localized excitation (breather) can exist, which corresponds to predominant rotation of one chain and a small perturbation of the second chain, using a coarse-grained model of the DNA double helix. Each nucleotide is represented by three beads with interaction sites corresponding to phosphate group, group of sugar ring, and the base [41].

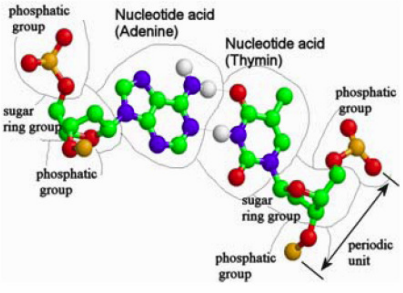


Fig. 2. a

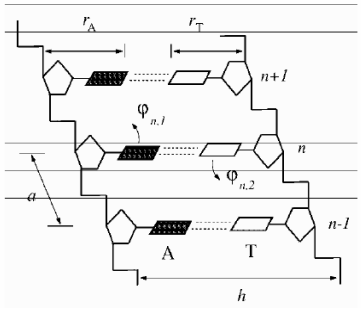


Fig. 2. b

Fig. 2**. a** The model scheme of a double helix on six coarse-grained particles (Kovaleva, Manevich, and Smirnov, 2007). Fig. 2.b Fragment of the DNA double chain consisting of three AT base pairs. Longitudinal pitch of the helix $a = 3.4\,\text{Å}$; transverse pitch $h = 16.15\,\text{Å}$ (see Ref. [42] by Kovaleva, and Manevich, 2005).

Kovaleva, Manevich, and Smirnov, [41] point out that solitons and breathers play a functional role in DNA chains. In a model, the DNA backbone is reduced to the polymeric structure and the base is covalently linked to the center of the sugar ring group, thus a DNA molecule with N nucleotides corresponds to 3N interaction centers. Starting from a coarse-grained off-lattice model of DNA and using cylindrical coordinates, the authors derive simplified continuum equations corresponding to vicinities of gap frequencies in the spectrum of linearized equations of motion. It is shown that obtained nonlinear continuum equations describing modulations of normal modes, admit spatially localized solitons, which can be identified with breathers. The authors formulated conditions of the breathers' existence and estimated their characteristic parameters. The relationship between a derived model and simpler and widely used models is discussed. The analytical results are compared with the data of numerical study of discrete equations of motion. See Fig. 2.a.

Kovaleva, and Manevich, [42] developed the simplest model describing the opening of DNA double helix. The corresponding differential equations are solved analytically using multiple-scale expansions after transition to complex variables. Obtained solution corresponds to localized torsional nonlinear excitation – breather. The stability of breather is also investigated. They consider B form of the DNA molecule, the fragment of which is presented in Fig. 2b. The lines in the figure correspond to the skeleton of the double helix, black and gray rectangles show the bases in pairs (AT and GC).

Let us focus our attention on the rotational motions of bases around the sugar-phosphate chains in the plane perpendicular to the helix axis.

The authors deal with the planar DNA model in which the chains of the macromolecule form two parallel straight lines placed at a distance h from each other, and the bases can make only rotation motions around their own chain, being all the time perpendicular to it. The authors accepted as generalized (independent) coordinates $\varphi_{k,1}$ that are the angular displacement of the k-th base of the first chain, and as generalized (independent) coordinates $\varphi_{k,2}$ the angular displacement of the k-th base of the second chain. Then, using the accepted generalized coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$ for the k-th bases of both chains in the DNA model, the authors derived a system of differential equations describing DNA model vibrations in the following forms:

$$\begin{aligned}
&\mathbf{J}_{\mathbf{k},\mathbf{1}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}} - \frac{K_{\mathbf{k},\mathbf{1}}}{2}\left[\sin\left(\varphi_{\mathbf{k}+\mathbf{1},\mathbf{1}} - \varphi_{\mathbf{k},\mathbf{1}}\right) - \sin\left(\varphi_{\mathbf{k},\mathbf{1}} - \varphi_{\mathbf{k}-\mathbf{1},\mathbf{1}}\right)\right] + K_{\alpha\beta} r_\alpha \left(r_\alpha - r_\beta\right)\sin\varphi_{\mathbf{k},\mathbf{1}} - \\
&\qquad - K_{\alpha\beta}\frac{1}{4}\left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_\alpha - r_\beta\right)^2 \sin\left(\varphi_{\mathbf{k},\mathbf{1}} - \varphi_{\mathbf{k},2}\right) = 0 \\
&\mathbf{J}_{\mathbf{k},2}\ddot{\varphi}_{\mathbf{k},2} - \frac{K_{\mathbf{k},2}}{2}\left[\sin\left(\varphi_{\mathbf{k}+1,2} - \varphi_{\mathbf{k},2}\right) - \sin\left(\varphi_{\mathbf{k},2} - \varphi_{\mathbf{k}-1,2}\right)\right] + K_{\alpha\beta} r_\alpha \left(r_\alpha - r_\beta\right)\sin\varphi_{\mathbf{k},2} + \\
&\qquad + K_{\alpha\beta}\frac{1}{4}\left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_\alpha - r_\beta\right)^2 \sin\left(\varphi_{\mathbf{k},\mathbf{1}} - \varphi_{\mathbf{k},2}\right) = 0
\end{aligned} \tag{1}$$

Here, $\mathbf{J}_{\mathbf{k},\mathbf{1}}$ is the axial moment of mass inertia of the k-th base of the first chain; $\mathbf{J}_{\mathbf{k},2}$ is the axial moment of mass inertia of the k-th base of the second chain, and the point denotes differentiation in time t. For the base pair, the axial moments of mass inertia are equal to $\mathbf{J}_{\mathbf{k},\mathbf{1}} = m_\alpha r_\alpha^2$, $\mathbf{J}_{\mathbf{k},\mathbf{12}} = m_\beta r_\beta^2$. The value of the base mass m_α, the length r_α, and the corresponding axial moment of mass inertia $\mathbf{J}_{\mathbf{k},\mathbf{1}} = m_\alpha r_\alpha^2$ for all possible base pairs the authors accepted as in the [42]. The fourth terms in the previous system of equations describe the interaction of the neighboring bases along each of the macromolecule chain. The parameter $K_{k,i}$, $i = 1,2$ characterizes the energy of interaction of the k-th base with the $(k+1)$-th one along the i-th chain $i = 1,2$. There are different estimations of rigidity. For the calculation, the most appropriate value is close to $K_{k,i} = K = 6\times 10^3 [kJ/mol]$.

7.5 Modified DNA models by N. Kovaleva and L. Manevich for the forced regimes

We have modified the previous model by Kovaleva and Manevich [42] to investigate how the system of dDNA will behave when it is exposed to external excitation. In the process of transcription, the binding of RNA polymerase may correspond with external excitation.

Let's suppose that both coupled chains from the system of the DNA model by Kovaleva and Manevich are excited by the system of external excitation containing two series of the one frequency excitations in the forms $\mathsf{M}_{0,k,.1}\cos\Omega_{k.1}t$ and $\mathsf{M}_{0,k,.2}\cos\Omega_{k.2}t$, $k=1,2,3,.....,n$, where $\mathsf{M}_{0,k,.1}$ and $\mathsf{M}_{0,k,.2}$ are amplitudes, $\Omega_{k.1}$ and $\Omega_{k.2}$ frequencies of the external forced couples each applied to one of the mass particles of the double DNA model coupled chains. Then, the corresponding system of the nonlinear forced vibrations of the double DNA model coupled chains is in the following form:

$$\begin{aligned}
&\mathbf{J}_{\mathbf{k},\mathbf{1}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\frac{K_{\mathbf{k},\mathbf{1}}}{2}\left[\sin\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\sin\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\sin\varphi_{\mathbf{k},\mathbf{1}}-\\
&\qquad-K_{\alpha\beta}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\sin\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)=\mathsf{M}_{0,k,.1}\cos\Omega_{k.1}t\\
&\mathbf{J}_{\mathbf{k},2}\ddot{\varphi}_{\mathbf{k},2}-\frac{K_{\mathbf{k},2}}{2}\left[\sin\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\sin\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\sin\varphi_{\mathbf{k},2}+\\
&\qquad+K_{\alpha\beta}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\sin\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)=\mathsf{M}_{0,k,.2}\cos\Omega_{k.2}t
\end{aligned}\tag{2}$$

7.6 Consideration of the basic DNA model - linearized Kovaleva-Manevich's DNA model

Let us investigate an oscillatory model of DNA considered in the [42] by N.Kovaleva, L. Manevich, (2005) and presented in section 4 by a system of differential equations (1) expressed by generalized (independent) angular coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$ for the k-th bases of both chains in the DNA model.

For a start, it is necessary to consider a corresponding linearized system of the ordinary differential equations of the previous system of differential equations in the following form:

$$\begin{aligned}
&\mathbf{J}_{\mathbf{k},\mathbf{1}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\frac{K_{\mathbf{k},\mathbf{1}}}{2}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\varphi_{\mathbf{k},\mathbf{1}}-\\
&\qquad-K_{\alpha\beta}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)=\mathsf{M}_{0,k,.1}\cos\Omega_{.k,1}t\\
&\mathbf{J}_{\mathbf{k},2}\ddot{\varphi}_{\mathbf{k},2}-\frac{K_{\mathbf{k},2}}{2}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\varphi_{\mathbf{k},2}+\\
&\qquad+K_{\alpha\beta}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)=\mathsf{M}_{0,k,.2}\cos\Omega_{k,2}t
\end{aligned}\tag{3}$$

or in the following form:

$$\begin{aligned}
&\frac{2\mathbf{J}_{\mathbf{k},\mathbf{1}}}{K_{\mathbf{k},\mathbf{1}}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\frac{2K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K_{\mathbf{k},\mathbf{1}}}\varphi_{\mathbf{k},\mathbf{1}}-\\
&\qquad-\frac{K_{\alpha\beta}}{2K_{\mathbf{k},\mathbf{1}}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)=\frac{\mathsf{M}_{0,k,.1}}{K_{\mathbf{k},\mathbf{1}}}\cos\Omega_{k,1}t
\end{aligned}$$

$$\frac{2\mathbf{J}_{\mathbf{k},2}}{K_{\mathbf{k},2}}\ddot{\varphi}_{\mathbf{k},2}-\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\frac{2K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K_{\mathbf{k},2}}\varphi_{\mathbf{k},2}+$$
$$+\frac{K_{\alpha\beta}}{2K_{\mathbf{k},2}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)=\frac{\mathsf{M}_{0,k,.2}}{K_{\mathbf{k},2}}\cos\Omega_{k,2}t \tag{4}$$

For the case of homogeneous systems, we can take into consideration that $\mathbf{J}_{k,\mathbf{1}}=\mathbf{J}_{\mathbf{k},\mathbf{2}}=\mathbf{J}$ and $K_{k,1}=K_{k,2}=K$.

Using change of the generalized independent angular coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$ for the k-th bases of both chains in the linearized DNA model into the following new ξ_k and η_k by the following dependence:

$$\xi_k=\varphi_{k,1}-\varphi_{k,2}\quad\text{and}\quad\eta_k=\varphi_{k,1}+\varphi_{k,2} \tag{5}$$

the previous system of differential equations (3) obtains the following form:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{\mathbf{k}+\mathbf{1}}+2\xi_{\mathbf{k}}\left[1+\frac{K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K}-\frac{K_{\alpha\beta}}{2K}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^{2}\right]-\xi_{\mathbf{k}-1}=\frac{\mathsf{M}_{0,k,.1}}{K}\cos\Omega_{0.k.1}t-\frac{\mathsf{M}_{0,k,2}}{K}\cos\Omega_{0.k.2}t \tag{6}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{\mathbf{k}+\mathbf{1}}+2\eta_{\mathbf{k}}\left(1+\frac{K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K}\right)-\eta_{\mathbf{k}-1}=\frac{\mathsf{M}_{0,k,.1}}{K}\cos\Omega_{0.k.1}t+\frac{\mathsf{M}_{0,k,.2}}{K}\cos\Omega_{0.k.2}t\,,\ k=1,2,3,.....,n \tag{7}$$

The first series of the previous system of ordinary differential equations are decoupled and independent with relations of the second series of the ordinary differential equations. We can conclude then that new coordinates of ξ_k and η_k are the main chain coordinates of original DNA model double chain system and that we obtain two fictive decoupled eigen single chains of the DNA linearized model. *This is the first fundamental conclusion as an important property of the linearized model of vibrations in a double DNA helix.*

The systems of differential equations (6)-(7) contain two separate independent subsystems of no autonomous differential equations expressed by *coordinates of* ξ_k *and* η_k which are the main chain *coordinates of a double DNA chain helix system and separate linear DNA model of forced vibrations into two independent (fictive, mathematical) chains.*

7.6.1. Consideration of the free vibrations of a basic DNA model - linearized Kovaleva-Manevich's DNA model

We assumed that the system of dDNA oscillates with free vibrations when it is not involved in the process of transcription and oscillates with forced vibrations during the process of transcription.

The corresponding systems of autonomous differential equations are joined to the systems of non-autonomous differential equations (6)-(7), which also contain two separate subsystems of non-autonomous differential equations expressed by *coordinates* ξ_k *and* η_k which are the main chain *coordinates of a double DNA chain helix system. The solutions of these corresponding systems of* autonomous differential equations are n*-frequency time functions which correspond to free vibrations with different subsets of* n circular frequencies. The solutions of the non-autonomous differential equations (6)-(7) are multi-frequency time functions with a corresponding subset of eigen circular frequencies containing n eigen circular frequencies of free vibrations and all frequencies of external forced excitations applied to the double DNA helix chain system. So, if external excitation is with $2n$ circular frequencies $\Omega_{k,j}$, $j=1,2$, $k=1,2,3,...,n$, then forced vibrations of the *independent main chain coordinate of the main chains of the linear DNA model a*re $n+2n$*-frequency time functions.*

To prove this conclusion, for a start, it is necessary to express the solutions of the corresponding systems of autonomous differential equations and add the corresponding particular solutions of the systems of non-autonomous differential equations (6)-(7).

Then, for that aim, to obtain the solutions of autonomous system of differential equations, it is possible to apply the trigonometric method (see [48],[49] and [24],[25]) to both series of autonomous differential equations (both subsystems obtained from the system (6)-(7) put $\mathsf{M}_{0,k,j}=0$, $j=1,2$, $k=1,2,3,...,n$) in the form:

$$\xi_k = A_k \cos(\omega t + \alpha) = C \sin k\varphi \cos(\omega t + \alpha),\ k = 1,2,3,.....,n \tag{8}$$

$$A_k = C \sin k\varphi \tag{9}$$

$$\eta_k = \tilde{A}_k \cos(\tilde{\omega} t + \beta) = D \sin k\vartheta \cos(\tilde{\omega} t + \beta),\ k = 1,2,3,.....,n \tag{10}$$

$$\tilde{A}_k = D \sin k\vartheta \tag{11}$$

where A_k and $\tilde{A}_k$ are amplitudes of separate eigen main chain coordinates of main chains of the model *of double DNA chain helix, and* ω *eigen circular frequency of the one, free vibration mode.*

After introducing the proposed solutions into the corresponding autonomous differential equations obtained from the previous separate subsystems (5)-(6) in which we incorporate $\mathsf{M}_{0,k,j} = 0$, $j = 1,2$, $k = 1,2,3,...,n$, we obtain the following separate subsystems of homogeneous algebraic equations along the amplitudes A_k and $\tilde{A}_k$:

$$-A_{\mathbf{k+1}} + 2A_{\mathbf{k}} \left\{ \left[1 + \frac{K_{\alpha\beta} r_\alpha (r_\alpha - r_\beta)}{K} - \frac{K_{\alpha\beta}}{2K} \left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}} \right) (r_\alpha - r_\beta)^2 \right] - \frac{\mathbf{J}}{K} \omega^2 \right\} - A_{\mathbf{k-1}} = 0 \tag{12}$$

$$-A_{\mathbf{k+1}} + 2A_{\mathbf{k}} \left(1 + \frac{K_{\alpha\beta} r_\alpha (r_\alpha - r_\beta)}{K} - \frac{2\mathbf{J}}{K} \tilde{\omega}^2 \right) - A_{\mathbf{k-1}} = 0 \tag{13}$$

After applying the following denotations:

$$\mu - \kappa = \frac{K_{\alpha\beta} r_\alpha (r_\alpha - r_\beta)}{K} - \frac{K_{\alpha\beta}}{2K} \left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}} \right) (r_\alpha - r_\beta)^2 \tag{14}$$

$$\kappa = \frac{K_{\alpha\beta}}{2K} \left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}} \right) (r_\alpha - r_\beta)^2 . \qquad \mu = \frac{K_{\alpha\beta} r_\alpha (r_\alpha - r_\beta)}{K} \tag{15}$$

$$u = \frac{\mathbf{J}}{K} \omega^2 \tag{16}$$

we obtain the following simple forms of the subsystems (12)-(13) in the following separate subsystems of homogeneous algebraic equations along the amplitudes A_k and $\tilde{A}_k$:

$$-A_{\mathbf{k+1}} + 2A_{\mathbf{k}} (1 + \mu - \kappa - u) - A_{\mathbf{k-1}} = 0 \tag{17}$$

$$-\tilde{A}_{\mathbf{k+1}} + 2\tilde{A}_{\mathbf{k}} (1 + \mu - \tilde{u}) - \tilde{A}_{\mathbf{k-1}} = 0 \tag{18}$$

After introducing the proposed solutions (9) and (11), the trigonometric method is applied and we obtain two equations:

$$C \sin k\varphi \left[-2\cos\varphi + 2(1 + \mu - \kappa - u) \right] = 0 \tag{19}$$

$$D \sin k\vartheta \left[-2\cos\vartheta + 2(1 + \mu - \tilde{u}) \right] = 0 \tag{20}$$

From the previous system, we obtain the following eigen characteristic numbers for both separate eigen chains of the model *of double DNA chain helix system free vibrations in the following forms:*

$$u = 2\sin^2 \frac{\varphi}{2} + (\mu - \kappa) \tag{21}$$

$$\tilde{u} = 2\sin^2 \frac{\vartheta}{2} + \mu \tag{22}$$

and the corresponding analytical expressions of the square of ω *- eigen circular frequencies of vibration modes of separate eigen main chains in the following forms:*

$$\omega_s^2 = \frac{K}{J} \left[2\sin^2 \frac{\varphi_s}{2} + (\mu - \kappa) \right] \qquad s = 1,2,3,.....,n \tag{23}$$

$$\tilde{\omega}_r^2 = \frac{K}{J}\left[2\sin^2\frac{\vartheta_r}{2} + \mu\right] \qquad r = 1,2,3,.....,n \tag{24}$$

Solutions of main chains coordinates for free vibrations are in the following forms:

$$\xi_k = \sum_{s=1}^{s=n}\xi_k^{(s)} = \sum_{s=1}^{s=n}A_k^{(s)}\cos(\omega_s t + \alpha_s) = \sum_{s=1}^{s=n}C_s \sin k\varphi_s \cos(\omega_s t + \alpha_s),\ k = 1,2,3,.....,n \tag{25}$$

$$\eta_k = \sum_{r=1}^{r=n}\eta_k^{(s)} = \sum_{r=1}^{r=n}\tilde{A}_k^{(r)}\cos(\tilde{\omega}_r t + \beta_r) = \sum_{r=1}^{r=n}D_r \sin k\vartheta_r \cos(\tilde{\omega}_r t + \beta_r),\ k = 1,2,3,.....,n \tag{26}$$

7.6. 2. Boundary conditions of the double DNA chain helix

Now, it is necessary to consider some boundary conditions (see [48], [49] and [27]) of the double DNA chain helix in accordance with the possible real situations. For that reason, we take into account two cases of double DNA chain helix, when ends of the chains are free and when ends of the chains are fixed. Then, we can write the following boundary conditions of the double DNA chain helix:

a case*: both ends of the double DNA chain helix are free:

For that case, the first and n -th equations from the subsystems are in the form:

$$A_1(1+\mu-\kappa-2u) - A_2 = 0 \tag{27a}$$

$$-A_{n-1} + A_n(1+\mu-\kappa-2u) = 0$$

$$\tilde{A}_1(1+\mu-\kappa-2\tilde{u}) - \tilde{A}_2 = 0$$

$$-\tilde{A}_{n-1} + \tilde{A}_n(1+\mu-2\tilde{u}) = 0 \tag{27b}$$

and after applying the proposed solutions (9) and (11) we obtain:

$$\varphi_s = \frac{s\pi}{n} \text{ and } \vartheta_s = \frac{s\pi}{n} \quad s = 1,2,3,4...,n \tag{27c}$$

b case*: both ends of the of the double DNA chain helix are fixed:

$$A_k = C\sin k\varphi \qquad A_0 = 0 \quad A_{n+1} = 0 \quad A_{m+1} = C\sin(n+1)\varphi = 0 \tag{28a}$$

$$\tilde{A}_k = D\sin k\vartheta \qquad \tilde{A}_0 = 0 \quad \tilde{A}_{n+1} = 0 \quad \tilde{A}_{m+1} = D\sin(n+1)\vartheta = 0 \tag{28b}$$

$$\varphi_s = \frac{s\pi}{(n+1)} \qquad \vartheta_r = \frac{r\pi}{(n+1)},\ s = 1,2,3,4...,n,\ r = 1,2,3,.....,n \tag{28c}$$

Then the analytical expressions of the square of ω_s *- eigen circular frequencies of the vibration modes of the separate main chains in* the double DNA chain helix are [35]*:*

$$\omega_s^2 = \frac{K}{J}\left[2\sin^2\frac{\varphi_s}{2} + \frac{K_{\alpha\beta}r_\alpha(r_\alpha - r_\beta)}{K} - \frac{K_{\alpha\beta}}{2K}\left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)(r_\alpha - r_\beta)^2\right], \quad s = 1,2,3,4...,n \tag{29}$$

$$\tilde{\omega}_r^2 = \frac{K}{J}\left[2\sin^2\frac{\vartheta_r}{2} + \frac{K_{\alpha\beta}r_\alpha(r_\alpha - r_\beta)}{K}\right] \qquad r = 1,2,3,.....,n \tag{30}$$

a case*: both ends of the double DNA chain helix are free (see Fig.3.) :

$$\omega_s^2 = \frac{K}{J}\left[2\sin^2\frac{s\pi}{2n} + \frac{K_{\alpha\beta}r_\alpha(r_\alpha - r_\beta)}{K} - \frac{K_{\alpha\beta}}{2K}\left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)(r_\alpha - r_\beta)^2\right], \quad s = 1,2,3,4...,n \tag{31}$$

$$\tilde{\omega}_r^2 = \frac{K}{J}\left[2\sin^2\frac{r\pi}{2n} + \frac{K_{\alpha\beta}r_\alpha(r_\alpha - r_\beta)}{K}\right], \quad r = 1,2,3,.....,n \tag{32}$$

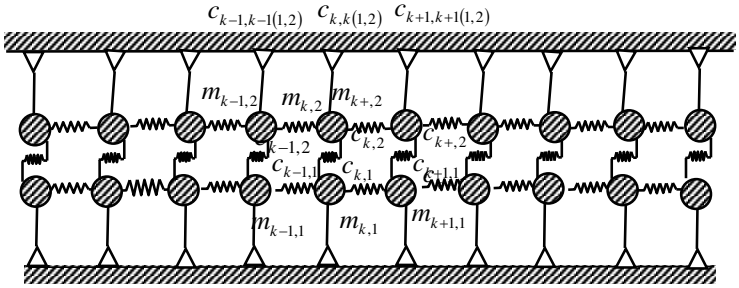


Fig 3. Double DNK Chain helix in the form of multi-pendulum model with free ends

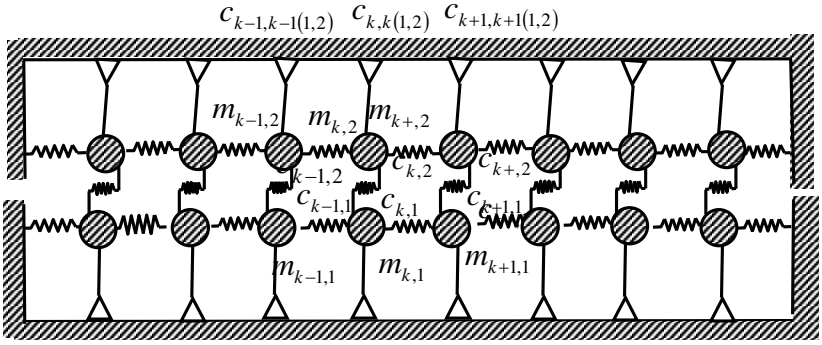


Fig.4. Double DNK Chain helix d model in the form of multi-pendulum system with fixed ends

b case*: both ends of the double DNA chain helix are fixed (see Fig.4.) :

$$\omega_s^2 = \frac{K}{J}\left[2\sin^2\frac{s\pi}{2(n+1)} + \frac{K_{\alpha\beta}r_\alpha(r_\alpha - r_\beta)}{K} - \frac{K_{\alpha\beta}}{2K}\left(1 - \frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)(r_\alpha - r_\beta)^2\right] \quad s = 1,2,3,.....,n \tag{33}$$

$$\tilde{\omega}_r^2 = \frac{K}{J}\left[2\sin^2\frac{r\pi}{2(n+1)} + \frac{K_{\alpha\beta}r_\alpha(r_\alpha - r_\beta)}{K}\right] \qquad r = 1,2,3,.....,n \tag{34}$$

7.6. 3. Consideration of the forced vibrations of a basic DNA model - linearized Kovaleva-Manevich's DNA model

In order to obtain general solutions of both systems (6)-(7) of non-autonomous differential equations corresponding to forced regimes of the main chains vibrations, it is necessary to start with finding the particular solutions for this system. Taking into account the denotation (14)-(16), the previous systems (6)-(7) of non-autonomous differential equations is possible to express in the form:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}} - \xi_{\mathbf{k+1}} + 2\xi_{\mathbf{k}}[1+\mu-\kappa] - \xi_{\mathbf{k-1}} = h_{0,k,1}\cos\Omega_{k.1}t - h_{0,k,2}\cos\Omega_{k.2}t \qquad k = 1,2,3,.....,n \tag{35}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}} - \eta_{\mathbf{k+1}} + 2\eta_{\mathbf{k}}(1+\mu) - \eta_{\mathbf{k-1}} = h_{0,k,1}\cos\Omega_{k.1}t + h_{0,k,2}\cos\Omega_{k.2}t \;, k = 1,2,3,.....,n \tag{36}$$

where $h_{0,k,1} = \frac{\mathsf{M}_{0,k,.1}}{K}$ $h_{0,k,2} = \frac{\mathsf{M}_{0,k,.2}}{K}$, $k = 1,2,3,.....,n$, reduced external excitation amplitudes.

Next, taking into account that this system is linear, for simplifications of the calculation procedure, without loss of generality, we can solve the system of non-autonomous differential equations describing the main chains forced vibrations of the double DNA helix chain system under one frequency external excitation, with frequency $\Omega_{1,1}$ and reduce the amplitude applied $h_{0,k,1} = \frac{\mathsf{M}_{0,k,.1}}{K}$ to one mass particle in the first real chain from the coupled chains. For that reason, we find particular solutions that correspond to forced vibrations with frequency $\Omega_{1,1}$ in the following form (see Fig. 5):

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{\mathbf{k+1}}+2\xi_{\mathbf{k}}\left[1+\mu-\kappa\right]-\xi_{\mathbf{k-1}}=\begin{cases}h_{0,1,1}\cos\Omega_{1.1}t & k=1\\ 0 & k\neq 1\end{cases}\qquad k=1,2,3,.....,n \tag{37}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{\mathbf{k+1}}+2\eta_{\mathbf{k}}\left(1+\mu\right)-\eta_{\mathbf{k-1}}=\begin{cases}h_{0,1,1}\cos\Omega_{1.1}t & k=1\\ 0 & k\neq 1\end{cases},k=1,2,3,.....,n \tag{38}$$

Particular solutions for the first and second system (37)-(38) are proposed in the forms:

$$\xi_{port,k}=N_k\cos\Omega_{1,1}t\qquad k=1,2,3,.....,n \tag{39a}$$

$$\eta_{port,k}=\tilde{N}_k\cos\Omega_{1,1}t\qquad k=1,2,3,.....,n \tag{39b}$$

and introducing the following denotations:

$$u=\frac{\mathbf{J}}{K}\omega^2\quad v_{k,1}=\frac{\mathbf{J}}{K}\Omega_{k,1}^2\qquad v_{k,2}=\frac{\mathbf{J}}{K}\Omega_{k,12}^2 \tag{40}$$

and introducing the proposed particular solutions (39)-(40) into the system (37)-(38), we obtain the following system of algebraic non-homogeneous system:

$$-N_{\mathbf{k+1}}+2N_{\mathbf{k}}\left(1+\mu-\kappa-v_{1,1}\right)-N_{\mathbf{k-1}}=\begin{cases}h_{0,1,1} & k=1\\ 0 & k\neq 1\end{cases}\qquad k=1,2,3,.....,n \tag{41}$$

$$-\tilde{N}_{\mathbf{k+1}}+2\tilde{N}_{\mathbf{k}}\left(1+\mu-\tilde{v}_{1,1}\right)-\tilde{N}_{\mathbf{k-1}}=\begin{cases}h_{0,1,1} & k=1\\ 0 & k\neq 1\end{cases}\qquad k=1,2,3,.....,n \tag{42}$$

where $v_{1,1}=\tilde{v}_{1,1}=\frac{\mathbf{J}}{K}\Omega_{1,1}^2$.

Using the Cramer rule, for the amplitudes of particular solutions, we obtain the following:

$$N_k\left(v_{1,1}\right)=\frac{\Delta_k\left(v_{1,1}\right)}{\Delta\left(v_{1,1}\right)}\qquad k=1,2,3,.....,n \tag{43}$$

$$\tilde{N}_k\left(\tilde{v}_{1,1}\right)=\frac{\tilde{\Delta}_k\left(\tilde{v}_{1,1}\right)}{\tilde{\Delta}\left(\tilde{v}_{1,1}\right)}\qquad k=1,2,3,.....,n \tag{44}$$

where, for example, two-system determinates, $\Delta\left(v_{1,1}\right)$ and $\tilde{\Delta}\left(\tilde{v}_{1,1}\right)$, are in the following forms (for the de-coupled main chains, each with four degrees of freedom):

$$\Delta\left(v_{1,1}\right)=\begin{vmatrix}2\left(1+\mu-\kappa-v_{1,1}\right) & -1 & & & \\ -1 & 2\left(1+\mu-\kappa-v_{1,1}\right) & -1 & & \\ & -1 & 2\left(1+\mu-\kappa-v_{1,1}\right) & -1 & \\ & & -1 & 2\left(1+\mu-\kappa-v_{1,1}\right) & -1\\ & & & -1 & 2\left(1+\mu-\kappa-v_{1,1}\right)\end{vmatrix}\neq 0 \tag{45}$$

$$\tilde{\Delta}\left(\tilde{v}_{1,1}\right)=\begin{vmatrix}2\left(1+\mu-\tilde{v}_{1,1}\right) & -1 & & & \\ -1 & 2\left(1+\mu-\tilde{v}_{1,1}\right) & -1 & & \\ & -1 & 2\left(1+\mu-\tilde{v}_{1,1}\right) & -1 & \\ & & -1 & 2\left(1+\mu-\tilde{v}_{1,1}\right) & -1\\ & & & -1 & 2\left(1+\mu-\tilde{v}_{1,1}\right)\end{vmatrix}\neq 0 \tag{46}$$

For the same example the other determinants $\Delta_k\left(v_{1,1}\right)$ and $\tilde{\Delta}_k\left(\tilde{v}_{1,1}\right)$, $k=1,2,3,.....,n$, are obtained from the corresponding two-system determinates, $\Delta\left(v_{1,1}\right)$ and $\tilde{\Delta}\left(\tilde{v}_{1,1}\right)$ introducing into the corresponding column, the column with free terms from the right sides of the non-homogeneous algebraic equations (41)-(42):

$$\Delta_1\left(v_{1,1}\right)=\begin{vmatrix}h_{0,1,1} & -1 & & & \\ -1 & 2\left(1+\mu-\kappa-v_{1,1}\right) & -1 & & \\ & -1 & 2\left(1+\mu-\kappa-v_{1,1}\right) & -1 & \\ & & -1 & 2\left(1+\mu-\kappa-v_{1,1}\right) & -1\\ & & & -1 & 2\left(1+\mu-\kappa-v_{1,1}\right)\end{vmatrix}=h_{0,1,1}2^{4-1}\prod_{s=1}^{s=3}\left(v_{1,1}-u_s^{(n=3)}\right) \tag{47}$$

$$\tilde{\Delta}_1(\tilde{v}_{1,1})=\begin{vmatrix} h_{0,1,1} & -1 & & & \\ -1 & 2(1+\mu-\tilde{v}_{1,1}) & -1 & & \\ & -1 & 2(1+\mu-\tilde{v}_{1,1}) & -1 & \\ & & -1 & 2(1+\mu-\tilde{v}_{1,1}) & \end{vmatrix} = h_{0,1,1}2^{4-1}\prod_{r=1}^{r=3}\left(\tilde{v}_{1,1}-\tilde{u}_r^{(n=3)}\right) \tag{48}$$

$$\Delta_2(v_{1,1})=\begin{vmatrix} 2(1+\mu-\kappa-v_{1,1}) & h_{0,1,1} & & & \\ -1 & & -1 & & \\ & & 2(1+\mu-\kappa-v_{1,1}) & -1 & \\ & & -1 & 2(1+\mu-\kappa-v_{1,1}) & \end{vmatrix} = h_{0,1,1}2^{4-2}\prod_{s=1}^{s=2}\left(v_{1,1}-u_s^{(n=2)}\right) \tag{49}$$

$$\tilde{\Delta}_2(\tilde{v}_{1,1})=\begin{vmatrix} 2(1+\mu-\tilde{v}_{1,1}) & h_{0,1,1} & & & \\ -1 & & -1 & & \\ & & 2(1+\mu-\tilde{v}_{1,1}) & -1 & \\ & & -1 & 2(1+\mu-\tilde{v}_{1,1}) & \end{vmatrix} = h_{0,1,1}2^{4-2}\prod_{r=1}^{r=2}\left(\tilde{v}_{1,1}-\tilde{u}_r^{(n=2)}\right) \tag{50}$$

$$\Delta_3(v_{1,1})=\begin{vmatrix} 2(1+\mu-\kappa-v_{1,1}) & -1 & h_{0,1,1} & & \\ -1 & 2(1+\mu-\kappa-v_{1,1}) & & & \\ & -1 & & -1 & \\ & & & 2(1+\mu-\kappa-v_{1,1}) & \end{vmatrix} = h_{0,1,1}2\left(\tilde{v}_{1,1}-\tilde{u}_r^{(n=1)}\right) \tag{51}$$

$$\tilde{\Delta}_3(\tilde{v}_{1,1})=\begin{vmatrix} 2(1+\mu-\tilde{v}_{1,1}) & -1 & h_{0,1,1} & & \\ -1 & 2(1+\mu-\tilde{v}_{1,1}) & & & \\ & -1 & & -1 & \\ & & & 2(1+\mu-\tilde{v}_{1,1}) & \end{vmatrix} = h_{0,1,1}2\left(\tilde{v}_{1,1}-\tilde{u}_r^{(n=1)}\right) \tag{52}$$

$$\Delta_4(v_{1,1})=\begin{vmatrix} 2(1+\mu-\kappa-v_{1,1}) & -1 & & h_{0,1,1} \\ -1 & 2(1+\mu-\kappa-v_{1,1}) & -1 & \\ & -1 & 2(1+\mu-\kappa-v_{1,1}) & \\ & & -1 & \end{vmatrix} = h_{0,1,1} \tag{53}$$

$$\tilde{\Delta}_4(\tilde{v}_{1,1})=\begin{vmatrix} 2(1+\mu-\tilde{v}_{1,1}) & -1 & & h_{0,1,1} \\ -1 & 2(1+\mu-\tilde{v}_{1,1}) & -1 & \\ & -1 & 2(1+\mu-\tilde{v}_{1,1}) & \\ & & -1 & \end{vmatrix} = h_{0,1,1} \tag{54}$$

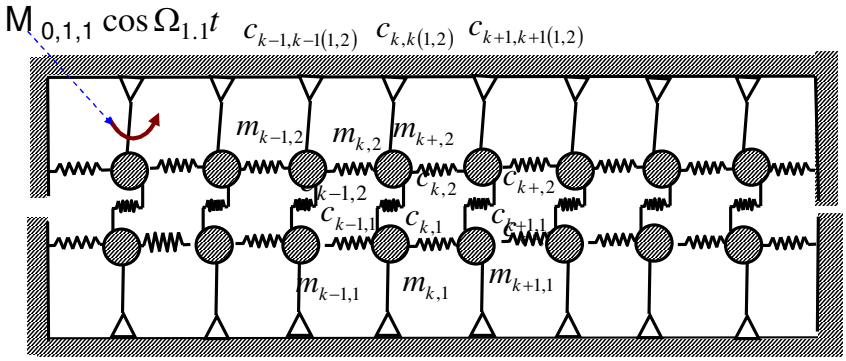


Fig. 5. Double DNK Chain helix model in the form of multi-pendulum system with fixed ends

Particular solutions of the main chains coordinates of considered examples with eight degrees of freedom of double DNA helix chain system containing two coupled chains, each with four degrees of freedom and excited by one frequency external excitation, are in the following forms:

$$\xi_{port,1} = N_1 \cos\Omega_{1,1}t = \frac{h_{0,1,1}\prod_{s=1}^{s=3}\left(v_{1,1} - u_s^{(n=3)}\right)}{2\prod_{s=1}^{s=4}\left(v_{1,1} - u_s^{(4)}\right)}\cos\Omega_{1,1}t \tag{55}$$

$$\eta_{port,1} = \tilde{N}_1 \cos\Omega_{1,1}t = \frac{h_{0,1,1}\prod_{r=1}^{r=3}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(n=3)}\right)}{2\prod_{r=1}^{r=4}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(4)}\right)}\cos\Omega_{1,1}t \tag{56}$$

$$\xi_{port,2} = N_2 \cos\Omega_{1,1}t = \frac{h_{0,1,1}\prod_{s=1}^{s=2}\left(v_{1,1} - u_s^{(n=2)}\right)}{4\prod_{s=1}^{s=4}\left(v_{1,1} - u_s^{(4)}\right)}\cos\Omega_{1,1}t \tag{57}$$

$$\eta_{port,2} = \tilde{N}_2 \cos\Omega_{1,1}t = \frac{h_{0,1,1}\prod_{r=1}^{r=2}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(n=2)}\right)}{4\prod_{4r=1}^{r=4}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(4)}\right)}\cos\Omega_{1,1}t \tag{58}$$

$$\xi_{port,3} = N_3 \cos\Omega_{1,1}t = \frac{h_{0,1,1}\left(v_{1,1} - u_s^{(n=1)}\right)}{8\prod_{s=1}^{s=4}\left(v_{1,1} - u_s^{(n)}\right)}\cos\Omega_{1,1}t \tag{59}$$

$$\eta_{port21} = \tilde{N}_2 \cos\Omega_{1,1}t = \frac{h_{0,1,1}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(n=1)}\right)}{8\prod_{r=1}^{r=4}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(n)}\right)}\cos\Omega_{1,1}t \tag{60}$$

$$\xi_{port,4} = N_4 \cos\Omega_{1,1}t = \frac{h_{0,1,1}}{16\prod_{s=1}^{s=4}\left(v_{1,1} - u_s^{(4)}\right)}\cos\Omega_{1,1}t \tag{61}$$

$$\eta_{port,4} = \tilde{N}_4 \cos\Omega_{1,1}t = \frac{h_{0,1,1}}{16\prod_{r=1}^{r=4}\left(\tilde{v}_{1,1} - \tilde{u}_r^{(4)}\right)}\cos\Omega_{1,1}t \tag{62}$$

Solutions of the main chains coordinates of the homogeneous system for the considered example in free vibration regime are:

$$\xi_{free,k} = \sum_{s=1}^{s=4} C_s \sin k\varphi_s \cos\left(\omega_s t + \alpha_s\right),\ k = 1,2,3,4 \tag{63}$$

$$\eta_{free,k} = \sum_{r=1}^{r=4} D_r \sin k\vartheta_r \cos\left(\tilde{\omega}_r t + \beta_r\right),\ k = 1,2,3,4 \tag{64}$$

General solutions of the main chains coordinates of the homogeneous system for the considered example in coupled free and forced vibration regimes are:

$$\xi_k = \xi_{free,k} + \xi_{part,k} = \sum_{s=1}^{s=4} C_s \sin k\varphi_s \cos\left(\omega_s t + \alpha_s\right) + \xi_{part,k},\ k = 1,2,3,4 \tag{65}$$

$$\eta_k = \eta_{free,k} + \eta_{part,k} = \sum_{r=1}^{r=4} D_r \sin k\vartheta_r \cos\left(\tilde{\omega}_r t + \beta_r\right) + \eta_{part,k},\ k = 1,2,3,4 \tag{66}$$

or in the form

$$\xi_k = \xi_{free,k} + \xi_{part,k} = \sum_{s=1}^{s=4} C_s \sin k\varphi_s \cos(\omega_s t + \alpha_s) + N_k(\nu_{1,1}) \cos\Omega_{1,1} t \ , \ k = 1,2,3,4 \tag{67}$$

$$\eta_k = \eta_{free,k} + \eta_{part,k} = \sum_{r=1}^{r=4} D_r \sin k\vartheta_r \cos(\tilde{\omega}_r t + \beta_r) + \tilde{N}_k(\tilde{\nu}_{1,1}) \cos\Omega_{1,1} t \ , \ k = 1,2,3,4 \tag{68}$$

For the system of double DNA helix chain system with $2n$ degrees of freedom, the previous two-sub-system determinates $\Delta(\nu_{1,1})$ and $\tilde{\Delta}(\tilde{\nu}_{1,1})$ are not difficult to express in similar forms.

Then, taking into account that determinates $\Delta(\nu_{1,1})$ and $\tilde{\Delta}(\tilde{\nu}_{1,1})$ are analogous to determinants describing the frequency equations of the free vibrations of the double DNA helix chain system, which is possible to express in the following forms: $\Delta(u) = 0$ and $\tilde{\Delta}(u) = 0$, and that we have roots of these frequency equations in the forms (23)-(24), then we have roots of the two-system determinates, $\Delta(\nu_{1,1})$ and $\tilde{\Delta}(\tilde{\nu}_{1,1})$ in the forms:

$$\nu_{1,1}^{(s)} = \frac{J}{K}\Omega_{1,1}^{(s)} = u_s^{(n)} = \frac{J}{K}\omega_s^2 = 2\sin^2\frac{\varphi_s}{2} + (\mu - \kappa) \qquad s = 1,2,3,.....,n \tag{69}$$

$$\tilde{\nu}_{1,1}^{(r)} = \frac{J}{K}\tilde{\Omega}_{1,1}^{(r)} = \tilde{u}_r^{(n)} = \frac{J}{K}\tilde{\omega}_r^2 = 2\sin^2\frac{\vartheta_r}{2} + \mu \qquad r = 1,2,3,.....,n \tag{70}$$

Using the previous characteristic numbers of the previous two-sub-system determinates, these determinants, $\Delta_k(\nu_{1,1})$ and $\tilde{\Delta}_k(\tilde{\nu}_{1,1})$, are possible to express in the forms of products:

$$\Delta(\nu_{1,1}) = 2^n \prod_{s=1}^{s=n} \left(\nu_{1,1} - u_s^{(n)}\right) \tag{71}$$

$$\tilde{\Delta}(\tilde{\nu}_{1,1}) = 2^n \prod_{r=1}^{r=n} \left(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n)}\right) \tag{72}$$

It is possible, in the same way, to find the expressions for the amplitude of particular solutions depending on the number of degrees of freedom $2n$. For example, it is obvious without calculations that the amplitude N_1, $\tilde{N}_1$ and N_2, $\tilde{N}_2$ of the particular solutions of the first and second normal coordinates, $\xi_{part,1}$, $\eta_{part,1}$ and $\xi_{part,2}$, $\eta_{part,2}$ of each of the both main chains are in the following forms:

$$N_1 = \frac{h_{0,1,1}\prod_{s=1}^{s=n-1}\left(\nu_{1,1} - u_s^{(n-1)}\right)}{2\prod_{s=1}^{s=n}\left(\nu_{1,1} - u_s^{(n)}\right)} \quad \text{and} \quad \tilde{N}_1 = \frac{h_{0,1,1}\prod_{r=1}^{r=n-1}\left(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n-1)}\right)}{2\prod_{r=1}^{r=n}\left(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n)}\right)} \tag{73}$$

$$N_2 = \frac{h_{0,1,1}\prod_{s=1}^{s=n-2}\left(\nu_{1,1} - u_s^{(n-2)}\right)}{2^2\prod_{s=1}^{s=n}\left(\nu_{1,1} - u_s^{(n)}\right)} \quad \text{and} \quad \tilde{N}_2 = \frac{h_{0,1,1}\prod_{r=1}^{r=n-1}\left(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n-2)}\right)}{2^2\prod_{r=1}^{r=n}\left(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n)}\right)} \tag{74}$$

General solutions of the main chains coordinates of the homogeneous system for the considered example in coupled free and forced vibration regime are in the following forms:

$$\xi_k = \xi_{free,k} + \xi_{part,k} = \sum_{s=1}^{s=n} C_s \sin k\varphi_s \cos(\omega_s t + \alpha_s) + \xi_{part,k} \ , \ k = 1,2,3,.....,n \tag{75}$$

$$\eta_k = \eta_{free,k} + \eta_{part,k} = \sum_{r=1}^{r=n} D_r \sin k\vartheta_r \cos(\tilde{\omega}_r t + \beta_r) + \eta_{part,k} \ , \ k = 1,2,3,.....,n \tag{76}$$

or in the form

$$\xi_k = \xi_{free,k} + \xi_{part,k} = \sum_{s=1}^{s=n} C_s \sin k\varphi_s \cos(\omega_s t + \alpha_s) + N_k(\nu_{1,1}) \cos\Omega_{1,1} t \ , \ k = 1,2,3,.....,n \tag{77}$$

$$\eta_k = \eta_{free,k} + \eta_{part,k} = \sum_{r=1}^{r=n} D_r \sin k\vartheta_r \cos(\tilde{\omega}_r t + \beta_r) + \tilde{N}_k(\tilde{\nu}_{1,1}) \cos\Omega_{1,1} t\,,\ k = 1,2,3,.....,n \tag{78}$$

For the case when one frequency external excitation with reduced amplitude $h_{0,2,1} = \frac{\mathsf{M}_{0,1,.2}}{K}$ with frequency $\Omega_{2,1}$ is applied to the other first material particle n of the other of the coupled real chains, then two subsystems of the main eigen chains are described by the following subsystems of differential equations:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}} - \xi_{\mathbf{k+1}} + 2\xi_{\mathbf{k}}[1+\mu-\kappa] - \xi_{\mathbf{k-1}} = \begin{cases} -h_{0,2,1}\cos\Omega_{2,1}t & k=1 \\ 0 & k\neq 1 \end{cases} \tag{79}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}} - \eta_{\mathbf{k+1}} + 2\eta_{\mathbf{k}}(1+\mu) - \eta_{\mathbf{k-1}} = = \begin{cases} h_{0,2,1}\cos\ \Omega_{2,1}t & k=1 \\ 0 & k\neq 1 \end{cases}, k = 1,2,3,.....,n \tag{80}$$

Particular and general solutions of these previous equations are not difficult to obtain analogously to the previous procedure and by changing the corresponding indices of the kinetic parameters of the main chains.

7.6. 4. Consideration of the forced vibration regimes of a basic DNA model - linearized Kovaleva-Manevich's DNA model-resonance and dynamical absorption

From the expressions (73) and (74), the possibilities of the occurrence of resonant regimes in eigen main chains can be considered.

For the case when the determinants (71) and (72), $\Delta(\nu_{1,1}) = 2^n \prod_{s=1}^{s=n}(\nu_{1,1} - u_s^{(n)}) = 0$ and $\tilde{\Delta}(\tilde{\nu}_{1,1}) = 2^n \prod_{r=1}^{r=n}(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n)}) = 0$ are equal to zero, we obtain two sets of external excitation frequencies for which in the system there appears a resonant regime. But taking into account that eigen main chains have different sets of eigen circular frequencies as well as different sets of resonant circular frequencies of external excitation, then it can be inferred that if in one eigen main chain there appears a resonant regime, then there is no resonance in the other eigen main chain. This is an important fact to consider in the light of the real double DNA helix chain system.

Also, using the expressions for amplitudes of the particular forced solutions, the occurrence of dynamical absorptions at the corresponding main chain coordinate of eigen main chain is possible. To obtain the external excitation frequencies at which the dynamical absorption occurs at the first or second main chain coordinate of the main chains are equal to zero:

$$N_1 = \frac{h_{0,1,1}\prod_{s=1}^{s=n-1}(\nu_{1,1} - u_s^{(n-1)})}{2\prod_{s=1}^{s=n}(\nu_{1,1} - u_s^{(n)})} = 0 \quad \text{or} \quad \tilde{N}_1 = \frac{h_{0,1,1}\prod_{r=1}^{r=n-1}(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n-1)})}{2\prod_{r=1}^{r=n}(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n)})} = 0 \tag{81}$$

$$N_2 = \frac{h_{0,1,1}\prod_{s=1}^{s=n-2}(\nu_{1,1} - u_s^{(n-2)})}{2^2\prod_{s=1}^{s=n}(\nu_{1,1} - u_s^{(n)})} = 0 \quad \text{or} \quad \tilde{N}_2 = \frac{h_{0,1,1}\prod_{r=1}^{r=n-1}(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n-2)})}{2^2\prod_{r=1}^{r=n}(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n)})} = 0 \tag{82}$$

and next

$$\prod_{s=1}^{s=n-1}(\nu_{1,1} - u_s^{(n-1)}) = 0 \quad \text{or} \quad \prod_{r=1}^{r=n-1}(\tilde{\nu}_{1,1} - \tilde{u}_r^{(n-1)}) = 0 \tag{83}$$

$$\prod_{s=1}^{s=n-2}(\nu_{1,1} - u_s^{(n-2)}) = 0 \quad \text{or} \quad \prod_{s=1}^{s=n-2}(\nu_{1,1} - u_s^{(n-2)}) = 0 \tag{84}$$

From the last conditions (83) and (84) we can conclude that:

** Dynamical absorption on the first pair of the main coordinates of the main chains occurs on the resonate circular frequencies of the set of the double DNA helix chain system with one pair of the material particles less compared to the considered real system.*

** Dynamical absorption on the second pair of the main chain coordinates of the main chains occurs on the resonate circular frequencies of the set of the double DNA helix chain system with two pairs of the material particles less compared to the considered system.*

This mathematical fact is important to consider in the light of the interruption or break of the double DNA helix chain system. By choosing the frequency of external excitation force, it is possible to define the conditions of place of the dDNA breakage.

7.7 The double DNA fractional order chain model on the basis of the linearized Kovaleva-Manevich's DNA models for free and forced vibrations

In this section fractional order model of dDNA is considered. Free and forced vibrations of this type of model are discussed. Analytical solutions for main coordinates and eigen frequencies for coupled and decupled system are given. Visualization of the main modes of free vibrations of fractional order dDNA helix chain system and corresponding partial fractional order oscillatory modes are also presented in this section.

7.7.1. Constitutive relation of the standard light fractional order creep element

Basic elements of the multi-mathematical pendulum system or multi-coupled chain system are:

1*Material particles* with mass m_k, with each particle having one degree of motion freedom, defined by the following coordinate φ_k, when k changes by $k = 1,2,3,4,....,N$.

2* *Standard light fractional order coupling element* of negligible mass in the form of axially stressed rod without bending, which has the ability to resist deformation under static and dynamic conditions (see Refs. [12], [20-22], [24-34]). *Standard light creep constraint element* for which the stress-strain relation for the restitution force, as the function of element elongation, is given by fractional order derivatives in the form

$$P(t) = -\left\{c_0 x(t) + c_\alpha \mathrm{D}_t^\alpha [x(t)]\right\} \tag{85}$$

where $\mathrm{D}_t^\alpha[\bullet]$ is operator of the α^{th} derivative with respect to time t in the following form,[19]:

$$\mathrm{D}_t^\alpha \left[x(t)\right] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \tag{86}$$

where c, c_α are rigidity coefficients–momentary and prolonged one, and α a rational number between 0 and 1, $0 < \alpha < 1$.

7.7.2. The double DNA fractional order chain free vibration model on the basis of the linearized Kovaleva-Manevich's DNA model

In this section, we will define the discrete continuum mathematical pendulum chain as a system of material particles inter-coupled by *light standard coupling elements* (*elastic, hereditary* or *creep*) and which are, in natural state, on defined inter-distances (when coupling elements are unstressed) (see Refs. [24-33]).

We used the fractional calculus to model the system that does not have ideally elastic properties. This concept may be incorporated into the theory of aging materials. As DNA molecule is aging it loses ideally elastic properties and expresses more visco-elastic properties.

We define *discrete homogeneous multi-mathematical pendulum chain system* as a system of discrete material particles of the same masses, which can rotate/oscillate along the corresponding circles with the same radius ℓ and centers on the one horizontal line. The entire system is in the vertical plane and the gravitational field (see Fig. 6 and 7).

The number of degrees of freedom of each of these multi-pendulum chains is equal to the number of material particles in it, since we accept the previously defined character of the system.

Furthermore, we introduce the hypotheses about the homogeneity of discrete continual chain, about small deformations of light standard coupling elements, and that displacements of material particles are small.

Also, we introduce the hypothesis that the homogenous discrete continuum, chain, was in natural, non-stressed state, before the initial moment of motion observation i.e. that light standard coupling elements do not have a prehistory or memory of stress-strain state. With these hypotheses, we will direct our research to the dynamics of chain-like homogenous multi-pendulum systems.

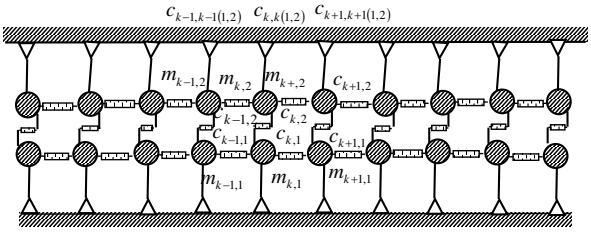


Fig. 6. Double DNK fractional order chain helix in the form of multi-pendulum model with free ends

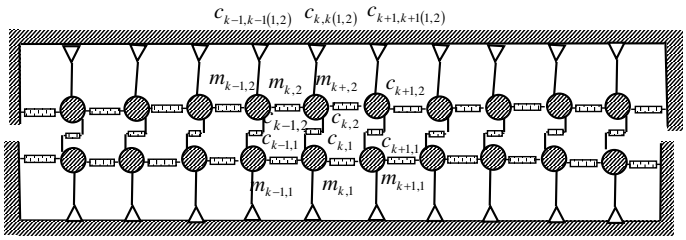


Fig.7. Double DNK fractional order chain helix d model in the form of multi-pendulum system with fixed ends

For the double DNA fractional order chain model on the basis of the linearized Kovaleva-Manevich's DNA model, we accept two chains as presented in Fig. 6 or 7, in the form of the double chain fractional order system containing two coupled multi-pendulum subsystems, in which the corresponding material particles of the corresponding multi-pendulum chains are coupled by one standard light fractional order element with stress-strain constitutive relations in the form (85).

Then, we can use the system (3) of the coupled linear differential equations extended by terms containing fractional order differential operators in the form (85). Then, we can write the corresponding system of the fractional order differential coupled equations for free fractional order vibrations of the double DNA fractional order chain system in the form:

$$\begin{aligned}
&\mathbf{J}_{\mathbf{k},\mathbf{1}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}} - \frac{K_{\mathbf{k},\mathbf{1}}}{2}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]-\frac{K_{\mathbf{k},\mathbf{1},\sigma}}{2}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\\
&\quad + K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\varphi_{\mathbf{k},\mathbf{1}} -\\
&-K_{\alpha\beta}-\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)-K_{\alpha\beta,\sigma}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=0\\
&\mathbf{J}_{\mathbf{k},2}\ddot{\varphi}_{\mathbf{k},2} - \frac{K_{\mathbf{k},2}}{2}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]-\frac{K_{\mathbf{k},2,\sigma}}{2}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\\
&\quad + K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\varphi_{\mathbf{k},2} +\\
&+K_{\alpha\beta}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)+K_{\alpha\beta,\sigma}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=0
\end{aligned} \tag{87}$$

where $K_{\mathbf{k},\mathbf{1},\sigma}$, $K_{\mathbf{k},2,\sigma}, K_{\alpha\beta,\sigma}$ and $K_{\alpha\beta,\sigma}$ are material constants of the double DNA fractional order chain model coupling elements pressing fractional order creep properties. The previous system is possible to rewrite in the following form:

$$\begin{aligned}
&\frac{2\mathbf{J}_{\mathbf{k},\mathbf{1}}}{K_{\mathbf{k},\mathbf{1}}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+-\frac{K_{\mathbf{k},\mathbf{1},\sigma}}{K_{\mathbf{k},\mathbf{1}}}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\\
&\qquad+\frac{2K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K_{\mathbf{k},\mathbf{1}}}\varphi_{\mathbf{k},\mathbf{1}}-\\
&-\frac{K_{\alpha\beta}}{2K_{\mathbf{k},\mathbf{1}}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)-\frac{K_{\alpha\beta,\sigma}}{2K_{\mathbf{k},\mathbf{1}}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=0\\
&\frac{2\mathbf{J}_{\mathbf{k},2}}{K_{\mathbf{k},2}}\ddot{\varphi}_{\mathbf{k},2}-\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]-\frac{K_{\mathbf{k},2,\sigma}}{K_{\mathbf{k},2}}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\\
&\qquad+\frac{2K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K_{\mathbf{k},2}}\varphi_{\mathbf{k},2}+\\
&+\frac{K_{\alpha\beta}}{2K_{\mathbf{k},2}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)+\frac{K_{\alpha\beta,\sigma}}{2K_{\mathbf{k},2}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=0
\end{aligned}\tag{88}$$

As our intention is to use the previous double DNA fractional order chain model for the case of the homogeneous system parameters, we take into account that: $K_{\mathbf{k},\mathbf{1},\sigma}=K_{\mathbf{k},2,\sigma}$=K. and $K_{\alpha\beta,\sigma}=K_{\alpha\beta,\sigma}$. Then, taking into account that we introduce the notation (14), (15) and (16), then the previous system of coupled fractional order differential equations is possible to write in the following form:

$$\begin{aligned}
&\frac{2\mathbf{J}}{K}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]-\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\\
&\qquad+2\mu\varphi_{\mathbf{k},\mathbf{1}}-\kappa\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)-\kappa\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=0\\
&\frac{2\mathbf{J}_{\mathbf{k},2}}{K_{\mathbf{k},2}}\ddot{\varphi}_{\mathbf{k},2}-\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]-\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\\
&\qquad+2\mu\varphi_{\mathbf{k},2}+\kappa\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)+\kappa\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=0
\end{aligned}\tag{89}$$

where $\kappa_{\sigma}=\dfrac{K_{\alpha\beta,\sigma}}{K}$.

Using change of the generalized independent angular coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$ for the k-th bases of both chains in the double DNA helix chain model into the following new coordinates ξ_k and η_k, as relations (5) in section 6 in the form: $\xi_k=\varphi_{k,1}-\varphi_{k,2}$ and $\eta_k=\varphi_{k,1}+\varphi_{k,2}$, the previous system of ordinary differential fractional order equations (89) obtains the following form:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{k-1}+2\left(1+\mu-\kappa\right)\xi_k-\xi_{k+1}=-\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[-\xi_{k-1,}+2\left(1-\kappa\right)\xi_k-\xi_{k+1}\right]\tag{90}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{k-1,}+2\left(1+\mu\right)\eta_k-\eta_{k+1}=-\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[-\eta_{k-1,}+2\eta_k-\eta_{k+1}\right],\quad k=1,2,3,.....,n\tag{91}$$

The first series (89) of differential fractional order equations of the previous system (90)-(91) is decoupled and independent with relations of the second series (91) of the fractional order differential equations. *Then, we can conclude that new coordinates of* ξ_k *and* η_k *are the main chain coordinates of the double DNA fractional order chains and that we obtain two fictive decoupled and independent single eigen* fractional order *chains of the double DNA fractional order model. This is a fundamental conclusion as an important property of the fractional order homogeneous model of vibrations in a* double DNA fractional order homogeneous helix.

The systems of fractional order differential equations (90)-(91) contain two separate subsystems of fractional order differential equations expressed by *coordinates of* ξ_k *and* η_k which are the main *coordinates of a double DNA fractional order chain helix and separate DNA fractional order model into two independent fractional order chains.*

7.7.3. Analytical solutions of the subsystems of the main chains fractional order differential equations for free fractional order vibrations

We solve the previous subsystems (90) and (91) of fractional order differential equations using the Laplace transformations (for detail see Appendix E.3). After applying the Laplace transformations of the previous systems (90) and (91) of fractional order differential equations with fractional order derivative and having in mind that we introduced notations $\mathsf{L}\{\xi_k(t)\}$ and $\mathsf{L}\{\eta_k(t)\}$ for the Laplace transformations of unknown normal chain coordinates ξ_k *and* η_k, as well as that:

$$\mathsf{L}\left\{\frac{d^{\sigma}\xi_k(t)}{dt\omega}\right\} = p^{\sigma}\mathsf{L}\{\xi_k(t)\} - \left.\frac{d^{\sigma-1}x_k(t)}{dt^{\varpi-1}}\right|_{t=0} = p^{\sigma}\mathsf{L}\{\xi_k(t)\} \tag{92}$$

$$\mathsf{L}\left\{\frac{d^{\sigma}\eta_k(t)}{dt\omega}\right\} = p^{\sigma}\mathsf{L}\{\eta_k(t)\} - \left.\frac{d^{\sigma-1}\eta_k(t)}{dt^{\varpi-1}}\right|_{t=0} = p^{\sigma}\mathsf{L}\{\eta_k(t)\} \tag{93}$$

and also having in mind that we accepted the hypothesis that the initial conditions of fractional order derivatives of the system are given using: $\left.\frac{d^{\sigma-1}\xi_k(t)}{dt^{\sigma-1}}\right|_{t=0} = 0$ and $\left.\frac{d^{\sigma-1}\eta_k(t)}{dt^{\sigma-1}}\right|_{t=0} = 0$ as well as that:

$$\mathsf{L}\left\{\frac{d^{2}\xi_k(t)}{dt^{2}}\right\} = p^{2}\mathsf{L}\{\xi_k(t)\} - \left[p\xi_{0k} + \dot{\xi}_{0k}\right], \tag{94}$$

$$\mathsf{L}\left\{\frac{d^{2}\eta_k(t)}{dt^{2}}\right\} = p^{2}\mathsf{L}\{\eta_k(t)\} - \left[p\eta_{0k} + \dot{\eta}_{0k}\right], \tag{95}$$

where ξ_{0k} and $\dot{\xi}_{0k}$ as well as η_{0k} and $\dot{\eta}_{0k}$ are defined by initial conditions of the system material particle dynamics in the chains, we can write the following system of the algebraic equations according to the unknown Laplace transforms $\mathsf{L}\{\xi_k(t)\}$ and $\mathsf{L}\{\eta_k(t)\}$ of unknown normal chain coordinates ξ_k *and* η_k:

$$\frac{\left[\frac{2\mathbf{J}}{K}p^{2} + 2\mu - 2\kappa\left(1+\kappa_{\sigma}p^{\sigma}\right)\right]}{\left[1+\kappa_{\sigma}p^{\sigma}\right]}\mathrm{L}\{\xi_k(t)\} - \mathrm{L}\{\xi_{k-1}\} + 2\mathrm{L}\{\xi_k(t)\} - \mathrm{L}\{\xi_{k+1}\} = \frac{2\mathbf{J}}{K}\frac{\left[p\xi_{0k} + \dot{\xi}_{0k}\right]}{\left[1+\kappa_{\sigma}p^{\sigma}\right]} \tag{96}$$

$$\frac{\left(p^{2}\frac{2\mathbf{J}}{K} + 2\mu\right)}{\left(1+\kappa_{\sigma}p^{\sigma}\right)}\mathsf{L}\{\eta_{\mathbf{k}}\} - \mathsf{L}\{\eta_{k-1,}\} + 2\mathsf{L}\{\eta_k\} - \mathsf{L}\{\eta_{k+1}\} = \frac{2\mathbf{J}}{K}\frac{\left[p\eta_{0k} + \dot{\eta}_{0k}\right]}{\left(1+\kappa_{\sigma}p^{\sigma}\right)} \tag{97}$$

Now, we have two separate, uncoupled non-homogeneous subsystems of the algebraic equations in the following forms:

$$-\mathsf{L}\{\xi_{k-1}\} + (2+v)\mathsf{L}\{\xi_k(t)\} - \mathsf{L}\{\xi_{k+1}\} = \frac{\left[p\xi_{0k} + \dot{\xi}_{0k}\right]}{\omega_0^{2}\left[1+\kappa_{\sigma}p^{\sigma}\right]} \tag{98}$$

$$-\mathsf{L}\{\eta_{k-1,}\} + (2+u)\mathsf{L}\{\eta_k\} - \mathsf{L}\{\eta_{k+1}\} = \frac{\left[p\eta_{0k} + \dot{\eta}_{0k}\right]}{\omega_0^{2}\left(1+\kappa_{\sigma}p^{\sigma}\right)} \tag{99}$$

Taking into account that at initial moment we have: $\xi_{0k} = 0.k \neq 1$ and $\dot{\xi}_{0k} = 0.k \neq 1$ as well as $\eta_{0k} = 0.k \neq 1$ and $\dot{\eta}_{0k} = 0.k \neq 1$ the previous equations obtain the following form:

$$-\mathsf{L}\{\xi_{k-1}\} + (2+v)\mathsf{L}\{\xi_k(t)\} - \mathsf{L}\{\xi_{k+1}\} = \begin{cases} h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) & k=1 \\ 0 & k\neq 1 \end{cases} \tag{100}$$

$$-\mathsf{L}\{\eta_{k-1}\} + (2+v)\mathsf{L}\{\eta_k(t)\} - \mathsf{L}\{\eta_{k+1}\} = \begin{cases} h_{\eta}\left(p,\eta_{01},\dot{\eta}_{01}\right) & k=1 \\ 0 & k\neq 1 \end{cases} \tag{101}$$

where

$$v = \frac{\left[p^{2} + 2\mu\omega_0^{2}\right]}{\omega_0^{2}\left[1+\kappa_{\sigma}p^{\sigma}\right]} - 2\kappa,\ u = \frac{\left[p^{2} + 2\mu\omega_0^{2}\right]}{\omega_0^{2}\left[1+\kappa_{\sigma}p^{\sigma}\right]},\quad \omega_0^{2} = \frac{K}{2\mathbf{J}} \tag{102}$$

$$h_{\xi}\left(p,\xi_{0k},\dot{\xi}_{0k}\right)=\frac{\left[p\xi_{0k}+\dot{\xi}_{0k}\right]}{\omega_0^2\left[1+\kappa_{\sigma}p^{\sigma}\right]},\quad h_{\eta}\left(p,\eta_{0k},\eta_{0k}\right)=\frac{\left[p\eta_{0k}+\dot{\eta}_{0k}\right]}{\omega_0^2\left(1+\kappa_{\sigma}p^{\sigma}\right)} \tag{103}$$

Both subsystems (100)-(101) are of the same form and it is enough to solve one of the subsystems and applying the analogy it is easy to solve other subsystem of fractional order differential equations. For that reason, we can use the method proposed in the papers [30] and [31].

Determinates of the previous algebraic subsystem (100) as well as (101) are in the same form as presented in the following form:

$$\Delta_N(v)=\begin{vmatrix} 2+v & -1 & & & \\ -1 & 2+v & & & \\ & & 2+v & -1 & \\ & & -1 & 2+v & \end{vmatrix}_{N\times N}\neq 0\ . \tag{104}$$

Introducing the notation (102) and (103), for the determinants $\tilde{\Delta}_{(k)}\left(v,h_{\xi}\right)$, we can write the following forms:

$$\tilde{\Delta}_{(1)}\left(v,h_{\xi}\right)=\begin{vmatrix} h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) & -1 & & & \\ 0 & 2+v & & & \\ & & 2+v & -1 & \\ & & -1 & 2+v & \end{vmatrix}_{N}=h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right)\Delta_{(N-1)}(v)\,; \tag{105}$$

$$\tilde{\Delta}_{(2)}\left(v,h_{\xi}\right)=\begin{vmatrix} 2+v & h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) & & \\ -1 & 0 & -1 & \\ & 0 & 2+v & -1 \\ & & -1 & 2+v \end{vmatrix}_{N}=(-1)^{2+1}h_{\varsigma}\left(p,\xi_{01},\dot{\xi}_{01}\right)\Delta_{N-2}(v)\neq 0 \tag{106}$$

$$\tilde{\Delta}_{(3)}\left(v,h_{\xi}\right)=\begin{vmatrix} 2+v & -1 & h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) & & & \\ -1 & 2+v & 0 & & & \\ & -1 & 0 & -1 & & \\ & & 0 & 2+v & -1 & \\ & & & -1 & 2+v & -1 \\ & & & & -1 & 2+v \end{vmatrix}_{N}=(-1)^{3+1+1}h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right)\Delta_{N-3}(v) \tag{107}$$

$$\tilde{\Delta}_{(4)}\left(v,h_{\xi}\right)=\begin{vmatrix} 2+v & -1 & 0 & h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) & & \\ -1 & 2+v & -1 & & & \\ & -1 & 2+v & 0 & & \\ & & -1 & 0 & -1 & \\ & & & 0 & 2+v & -1 \\ & & & & -1 & 2+v \end{vmatrix}_{N}=(-1)^{4+1+1+1}h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right)\Delta_{N-4}(v) \tag{108}$$

$$\tilde{\Delta}_{(N)}\left(v,h_{\xi}\right)=\begin{vmatrix} 2+v & -1 & & & & & h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) \\ -1 & 2+v & -1 & & & & 0 \\ & -1 & 2+v & -1 & & & 0 \\ & & -1 & 2+v & -1 & & 0 \\ & & & -1 & 2+v & -1 & 0 \\ & & & & -1 & 2+v & 0 \\ & & & & & -1 & 0 \end{vmatrix}_{N}=(-1)^{2N-1}h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right) \tag{109}$$

To solve the system of the algebraic non-homogeneous equations (100) or (101) with respect to unknown Laplace transforms $\mathsf{L}\{\xi_k(t)\}$ or $\mathsf{L}\{\eta_k(t)\}$ of the time function main chain coordinates $\xi_k(t)$ and $\eta_k(t)$,

unknown normal chain coordinates of the system main chains, we can use the Cramer approach and we can write:

$$\mathrm{L}\left\{\xi_k(t)\right\}_N = \frac{\tilde{\Delta}_{(k)}\left(v,h_\varsigma\right)}{\Delta_N(v)} = \frac{(-1)^{2k-1}h_\xi\left(p,\xi_{0k},\dot{\xi}_{0k}\right)\Delta_{N-k}(v)}{\Delta_N(v)}. \tag{110}$$

$$\mathrm{L}\left\{\eta_k(t)\right\}_N = \frac{\tilde{\Delta}_{(k)}\left(u,h_\eta\right)}{\Delta_N(u)} = \frac{(-1)^{2k-1}h_\eta\left(p,\eta_{0k},\eta_{0k}\right)\Delta_{N-k}(u)}{\Delta_N(u)}. \tag{111}$$

Let us first analyze the solution and characteristic equations of the homogenous subsystem as basic subsystems of the algebraic non-homogeneous equations (100) or (101) with respect to unknown Laplace transforms $\mathrm{L}\{\xi_k(t)\}$ or $\mathrm{L}\{\eta_k(t)\}$ of the time function main coordinate $\xi_k(t)$ and $\eta_k(t)$, unknown normal chain coordinates of the system main chains. These two of the homogenous subsystem as basic subsystems of the algebraic non-homogeneous equations (100) or (101) are in the following forms:

$$-\mathrm{L}\left\{\xi_{k-1}\right\}+(2+v)\mathrm{L}\left\{\xi_k(t)\right\}-\mathrm{L}\left\{\xi_{k+1}\right\}=0 \tag{112}$$

$$-\mathrm{L}\left\{\eta_{k-1,}\right\}+(2+u)\mathrm{L}\left\{\eta_k\right\}-\mathrm{L}\left\{\eta_{k+1}\right\}=0 \tag{113}$$

The solution of such a subsystem of algebraic homogenous equations (112) or (113), from which we obtain a series of determinants, can be obtained using the trigonometric method (see [48] or [27])) or by obtaining the recurrent formulas. Let us use the trigonometric method and, for that reason, the solutions are assumed in the following forms:

$$\mathrm{L}\left\{\xi_k(t)\right\} = N\sin k\phi\,, \tag{114}$$

and introducing into the previous subsystems (112) or (113), we have that: $v=2(\cos\varphi-1)$ and $u=2(\cos\varphi-1)$, as well as the following two characteristic equations:

$$v=\frac{\left[p^2+2\mu\omega_0^2\right]}{\omega_0^2\left[1+\kappa_\sigma p^\sigma\right]}-2\kappa=2(\cos\varphi-1), \qquad u=\frac{\left[p^2+2\mu\omega_0^2\right]}{\omega_0^2\left[1+\kappa_\sigma p^\sigma\right]}=2(\cos\varphi-1) \tag{115}$$

or in the forms:

a* for the first main chain with a set of main chain coordinates $\xi_k(t)$:

$$p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)(1-2\kappa-\cos\varphi_s)=0, \quad s=1,2,3,...,n \tag{116}$$

b* for the second main chain with main chain coordinates $\eta_k(t)$:

$$p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)(1-\cos\varphi_s)=0, \quad s=1,2,3,...,n \tag{117}$$

where φ_s depends on the boundary conditions on the ends of the corresponding system main chain.

Based on the previous two characteristic equations (116) and (117), the subsystem characteristic determinants of the system can be written in the following form:

a* for the first main chain with a set of main chain coordinates $\xi_k(t)$:

$$\Delta_N=\prod_{s=1}^{s=N_x}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)(1-\kappa-\cos\varphi_s)\right]\neq 0\,, \tag{118}$$

b* for the second main chain with main chain coordinates $\eta_k(t)$:

$$\Delta_N = \prod_{s=1}^{s=N_x} \left[p^2 + 2\mu\omega_0^2 + 2\omega_0^2 \left(1 + \kappa_\sigma p^\sigma\right)\left(1 - \cos\varphi_s\right)\right] \neq 0 \tag{119}$$

from which we obtain a series of determinants when we replace one of the columns with a column of free terms on the right side of the fractional order differential equations in system (111) as well as (112), as it is shown by the calculus in (91) - (107). Based on that, for given initial conditions for each of the particle coordinates of material particles in the corresponding chain, we can obtain the following determinants corresponding to a certain column (and to an unknown Laplace transformation $\mathsf{L}\{_k(t)\}$ of the coordinate $\xi_k(t)$) in the following forms:

$$\begin{aligned}
&\tilde{\Delta}_{(1)}\left(v, h_\xi\right) = h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{(N-1)}(v);\\
&\tilde{\Delta}_{(2)}\left(v, h_\xi\right) = (-1)^{2+1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{N-2}(v);\\
&\tilde{\Delta}_{(3)}\left(v, h_\xi\right) = (-1)^{3+1+1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{N-3}(v)\\
&\tilde{\Delta}_{(4)}\left(v, h\ \right) = (-1)^{4+1+1+1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{N_x-4}(v)\\
&\tilde{\Delta}_{(5)}\left(v, h_\xi\right) = (-1)^{5+1+1+1+1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{N-5}(v)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\tilde{\Delta}_{(m)}\left(v, h_\xi\right) = (-1)^{2m-1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{N_x-m}\left(v, h_\xi\right); \ldots\ldots\ldots
\end{aligned} \tag{120}$$

1.a* for the first main chain with a set of main chain coordinates $\xi_k(t)$ the determinate of the subsystem is:

$$\Delta_{N-k}(p) = \prod_{s=1}^{s=N-k} \left[p^2 + 2\mu\omega_0^2 + 2\omega_0^2 \left(1 + \kappa_\sigma p^\sigma\right)\left(1 - \kappa - \cos\varphi_s\right)\right] \neq 0 \tag{121}$$

1.b* for the second main chain with main chain coordinates $\eta_k(t)$ the determinate of the subsystem is:

$$\Delta_{N-k}(p) = \prod_{s=1}^{s=N-k} \left[p^2 + 2\mu\omega_0^2 + 2\omega_0^2 \left(1 + \kappa_\sigma p^\sigma\right)\left(1 - \cos\varphi_s\right)\right] \neq 0 \tag{122}$$

2.a* for the first main chain with a set of main chain coordinates $\xi_k(t)$ the particular determinates of the subsystem are:

$$\begin{aligned}
&\tilde{\Delta}_{(1)}\left(p, \xi_{01}, \dot{\xi}_{01}, h_\xi\right) = h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right) \prod_{s=1}^{s=N-1} \left[p^2 + 2\mu\omega_0^2 + 2\omega_0^2 \left(1 + \kappa_\sigma p^\sigma\right)\left(1 - \kappa - \cos\varphi_s\right)\right] \neq 0\\
&\tilde{\Delta}_{(2)}\left(p, \xi_{01}, \dot{\xi}_{01}, h_\xi\right) = (-1)^3 h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right) \prod_{s=1}^{s=N-2} \left[p^2 + 2\mu\omega_0^2 + 2\omega_0^2 \left(1 + \kappa_\sigma p^\sigma\right)\left(1 - \kappa - \cos\varphi_s\right)\right] \neq 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\tilde{\Delta}_{(k)}\left(p, \xi_{01}, \dot{\xi}_{01}, h_\xi\right) = (-1)^{2k-1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right) \prod_{s=1}^{s=N-k} \left[p^2 + 2\mu\omega_0^2 + 2\omega_0^2 \left(1 + \kappa_\sigma p^\sigma\right)\left(1 - \kappa - \cos\varphi_s\right)\right] \neq 0\\
&\tilde{\Delta}_{(N)}\left(p, \xi_{01}, \dot{\xi}_{01}, h_\xi\right) = (-1)^{N_x + N_x - 1} h_\xi\left(p, \xi_{01}, \dot{\xi}_{01}\right)\Delta_{N-N} = (-1)^{2N_x - 1} h_\xi\left(p, \xi_{0k}, \dot{\xi}_{0k}\right)\Delta_0 = (-1)^{2N-1} h_\xi\left(p, \xi_{0k}, \dot{\xi}_{0k}\right)
\end{aligned} \tag{123}$$

2.b* for the second main chain with main chain coordinates $\eta_k(t)$ $\xi_k(t)$ the particular determinates of the subsystem are:

$$\tilde{\Delta}_{(1)}\left(p,\eta_{01},\dot{\eta}_{01},h_\eta\right)=h_\eta\left(p,\eta_{01},\dot{\eta}_{01}\right)\prod_{s=1}^{s=N-1}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]\neq 0$$

$$\tilde{\Delta}_{(2)}\left(p,\eta_{01},\dot{\eta}_{01},h_\eta\right)=\left(-1\right)^3 h_\eta\left(p,\eta_{01},\dot{\eta}_{01}\right)\prod_{s=1}^{s=N-2}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]\neq 0$$

…… (124)

$$\tilde{\Delta}_{(k)}\left(p,\eta_{01},\dot{\eta}_{01},h_\eta\right)=\left(-1\right)^{2k-1} h_\eta\left(p,\eta_{01},\dot{\eta}_{01}\right)\prod_{s=1}^{s=N-k}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]\neq 0$$

$$\tilde{\Delta}_{(N)}\left(p,\eta_{0k},\dot{\eta}_{0k},h_\eta\right)=\left(-1\right)^{N+N_x-1} h_\eta\left(p,\eta_{0k},\dot{\eta}_{0k}\right)\Delta_{N-N}=\left(-1\right)^{2N_x-1} h_\eta\left(p,\eta_{01},\dot{\eta}_{01}\right)\Delta_0=\left(-1\right)^{2N-1} h_\eta\left(p,\eta_{01},\dot{\eta}_{01}\right)$$

Based on these previous discoveries, we can deduce the following expressions for the unknown Laplace transformations $\mathsf{L}\left\{\xi_k\left(t\right)\right\}$ or $\mathsf{L}\left\{\eta_k\left(t\right)\right\}$ of the time function main chain coordinates $\xi_k\left(t\right)$ and $\eta_k\left(t\right)$, unknown normal chain coordinates of the sub-system main chains:

3.a* the Laplace transformations $\mathsf{L}\left\{\xi_k\left(t\right)\right\}$ of the time function main chain coordinate $\xi_k\left(t\right)$ for the first main chain with a set of main chain coordinates $\xi_k\left(t\right)$ are:

$$\mathsf{L}\left\{\xi_k\left(t\right)\right\}=\frac{p\xi_{01}+\dot{\xi}_{01}}{\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)}\cdot\frac{\left(-1\right)^{2k-1}\prod_{s=1}^{s=N-k}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\kappa-\cos\varphi_s\right)\right]}{\prod_{s=1}^{s=N}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\kappa-\cos\varphi_s\right)\right]},\tag{125}$$

3.b* the Laplace transformations $\mathsf{L}\left\{\eta_k\left(t\right)\right\}$ for the second main chain with main chain coordinates $\eta_k\left(t\right)$ for the first main chain with a set of main chain coordinates $\eta_k\left(t\right)$ are:

$$\mathsf{L}\left\{\eta_k\left(t\right)\right\}=\frac{p\eta_{01}+\dot{\eta}_{01}}{\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)}\cdot\frac{\left(-1\right)^{2k-1}\prod_{s=1}^{s=N-k}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]}{\prod_{s=1}^{s=N}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]},k=1,2,3,...,N\tag{126}$$

The previous solutions for the unknown Laplace transformations $\mathsf{L}\left\{\xi_k\left(t\right)\right\}$ or $\mathsf{L}\left\{\eta_k\left(t\right)\right\}$ of the time function main chain coordinates $\xi_k\left(t\right)$ and $\eta_k\left(t\right)$, the unknown normal chain coordinates of the sub-system main chains can be written in the following form:

4.a* the Laplace transformations $\mathsf{L}\left\{\xi_k\left(t\right)\right\}$ of the time function main chain coordinates $\xi_k\left(t\right)$ for the first main chain with a set of main chain coordinates $\xi_k\left(t\right)$ are:

$$\mathsf{L}\left\{\xi_k\left(t\right)\right\}=\frac{1}{\omega_0^2}\left(\xi_{01}+\frac{\dot{\xi}_{01}}{p}\right)\frac{1}{p^{2N+1}}\frac{1}{\left(1+\kappa_\sigma p^\sigma\right)}\cdot\frac{\left(-1\right)^{2k-1}\prod_{s=1}^{s=N-k}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\kappa-\cos\varphi_s\right)\right]}{\prod_{s=1}^{s=N}\left\{1+\frac{2\omega_0^2}{p^2}\left[\mu+\left(1+\kappa_\sigma p^\sigma\right)\left(1-\kappa-\cos\varphi_s\right)\right]\right\}}$$

$$k=1,2,3,...,N\tag{127}$$

4.b* the Laplace transformations $\mathsf{L}\left\{\eta_k\left(t\right)\right\}$ for the second main chain with main chain coordinates $\eta_k\left(t\right)$ for the first main chain with a set of main chain coordinates $\eta_k\left(t\right)$ are:

$$\mathsf{L}\{\eta_k(t)\}=\frac{1}{\omega_0^2}\left(\eta_{01}+\frac{\dot{\eta}_{01}}{p}\right)\frac{1}{p^{2N+1}}\frac{1}{\left(1+\kappa_\sigma p^\sigma\right)}\cdot\frac{(-1)^{2k-1}\prod_{s=1}^{s=N-k}\left[p^2+2\mu\omega_0^2+2\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]}{\prod_{s=1}^{s=N}\left\{1+\frac{2\omega_0^2}{p^2}\left[\mu+\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]\right\}}$$

$$k=1,2,3,...,N \tag{128}$$

5.a* for the first main chain with a set of main chain coordinates $\xi_k(t)$:

$$\mathsf{L}\{\xi_k(t)\}=\frac{1}{\omega_0^2}\left(\xi_{01}+\frac{\dot{\xi}_{01}}{p}\right)\frac{1}{p^{2N-2k+1}}\frac{1}{\left(1+\kappa_\sigma p^\sigma\right)}\cdot\frac{(-1)^{2k-1}}{\prod_{s=N-(k-1)}^{s=N}\left\{1+\frac{2\omega_0^2}{p^2}\left[\mu+\left(1+\kappa_\sigma p^\sigma\right)\left(1-\kappa-\cos\varphi_s\right)\right]\right\}},$$

$$k=1,2,3,...,N \tag{129}$$

5.b* for the second main chain with main chain coordinates $\eta_k(t)$:

$$\mathsf{L}\{\eta_k(t)\}=\frac{1}{\omega_0^2}\left(\eta_{01}+\frac{\dot{\eta}_{01}}{p}\right)\frac{1}{p^{2N-2k+1}}\frac{1}{\left(1+\kappa_\sigma p^\sigma\right)}\cdot\frac{(-1)^{2k-1}}{\prod_{s=N-(k-1)}^{s=N}\left\{1+\frac{2\omega_0^2}{p^2}\left[\mu+\left(1+\kappa_\sigma p^\sigma\right)\left(1-\cos\varphi_s\right)\right]\right\}}$$

$$k=1,2,3,...,N \tag{130}$$

After developing the binomials into series of previous particular solutions, it is easy to obtain time solutions in the form of the series with respect to time (see Appendix E.3-E.4).

7.7.4. Main coordinates of the fractional order double DNA helix chain system and corresponding partial fractional order oscillators

Starting from two subsystems of the main eigen chain fractional order differential equations (90)-(91) and the corresponding basic linear subsystems:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{k-1}+2\xi_k-\xi_{k+1}+2\mu\xi_{\mathbf{k}}-2\kappa\xi_{\mathbf{k}}=0 \tag{131}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{k-1,}+2\eta_k-\eta_{k+1}+2\mu\eta_{\mathbf{k}}=0\,,\quad k=1,2,3,.....,n \tag{132}$$

we can find two subsystems of the corresponding independent partial fractional order oscillators. For that reason, we applied trigonometric method (see [48], and [24-27]) and introduce into the previous subsystem (59) the following assumed solutions:

$$\xi_k(t)=A_k\cos(\omega t+\alpha) \tag{133}$$

where A_k is unknown amplitudes , and ω is frequency, and α is phase. After introducing the notation:

$$u=2\left(\frac{2\mathbf{J}}{K}\omega^2+\kappa-\mu\right) \tag{134}$$

we obtain the following system of the homogeneous algebraic equations:

$$-A_{k-1}+(2-u)A_k-A_{k+1}=0 \tag{135}$$

Now, for the amplitudes, we assume the following:

$$A_k=C\sin k\varphi \tag{136}$$

and after introducing in the system (63) the algebraic equations

$$-C\sin(k-1)\varphi+(2-u)C\sin k\varphi-C\sin(k+1)\varphi=0 \tag{137a}$$

$$\left(2-u-2\cos\varphi\right)C\sin k\varphi=0 \tag{137b}$$

we obtain:

$$u=2\left(1-\cos\varphi\right) \tag{137c}$$

as well as:

$$\omega^2=\frac{K}{2\mathbf{J}}\left(2\sin^2\varphi+\mu-\kappa\right) \tag{138}$$

The solution (133) must satisfy boundary conditions for a different case of the ends of the double DNA fractional order chain model. For that reason, we must put indices s, and two subsets of the eigen circular frequencies of the corresponding main eigen chains are:

$$\omega_{s\xi}^2=\frac{K}{2\mathbf{J}}\left(2\sin^2\frac{\varphi_s}{2}+\mu-\kappa\right),\ s=1,2,3,....,n \tag{139}$$

$$\omega_{s\eta}^2=\frac{K}{2\mathbf{J}}\left(2\sin^2\frac{\varphi_s}{2}+\mu\right),\ s=1,2,3,....,n \tag{140}$$

For the first main chain of the double DNA chain helix, the eigen amplitudes are in the form $A_k^{(s)}=C_s\sin k\varphi_s$ and generalized main chain coordinates $\xi_k\left(t\right)$ of the first main chain is possible to express by a set of the main normal coordinates $\zeta_{s\xi}$ in the following form:

$$\xi_k\left(t\right)=\sum_{s=1}^{n}A_k^{(s)}\cos\left(\omega_{s\xi}t+\alpha_s\right)=\sum_{s=1}^{n}C_s\sin k\varphi_s\cos\left(\omega_{s\xi}t+\alpha_s\right)=\sum_{s=1}^{n}\zeta_{s\xi}\sin k\varphi_s \tag{141}$$

$$k=1,2,3,....,n$$

Normal coordinates $\zeta_{s\xi}$ or normal modes of the first main chain are in the form

$$\zeta_{s\xi}=C_s\cos\left(\omega_{s\xi}t+\alpha_s\right),s=1,2,3,....,n \tag{142}$$

with known frequencies (139) , and unknown amplitudes C_s and phase α_s depending of initial conditions.

The transformation of the subsystem (90) of the fractional order differential equations of the first main chain by introducing (140) yields:

$$\sum_{s=1}^{n}\frac{2\mathbf{J}}{K}\left(\ddot{\zeta}_{s\xi}+\omega_{s\xi}^2\zeta_{s\xi}+\frac{K}{2\mathbf{J}}\kappa_{s\sigma\xi}\mathsf{D}_t^{\sigma}\left[\zeta_{s\xi}\right]\right)\sin k\varphi_s=0\quad,\ s=1,2,3,....,n \tag{143}$$

where characteristic numbers expressing fractional order subsystem properties are in the following form:

$$\kappa_{s\sigma\xi}=\frac{K}{2\mathbf{J}}\left(2\sin^2\frac{\varphi_s}{2}+\mu-\kappa\right)\quad,\ s=1,2,3,....,n \tag{144}$$

Taking into account that $\sin k\varphi_s$ is equal to zero, only for the boundary conditions, then from (141) we can write that is necessary to be:

$$\ddot{\zeta}_{s\xi}+\omega_{s\xi}^2\zeta_{s\xi}+\omega_{s\sigma\xi}^2\mathsf{D}_t^{\sigma}\left[\zeta_{s\xi}\right]=0,\ s=1,2,3,....,n \tag{145}$$

where sets of circular frequencies $\omega_{s\xi}^2$ and a set of characteristic fractional order numbers $\omega_{s\sigma\xi}^2$ are in the following forms:

$$\omega_{s\xi}^2=\frac{K}{2\mathbf{J}}\left(2\sin^2\frac{\varphi_s}{2}+\mu-\kappa\right),\ s=1,2,3,....,n \tag{146}$$

$$\omega_{s\sigma\xi}^2=\kappa_{s\sigma\xi}\frac{K}{2\mathbf{J}}=\kappa_{\sigma}\frac{K}{\mathbf{J}}\left(2\sin^2\frac{\varphi_s}{2}-\kappa\right),s=1,2,3,....,n \tag{147}$$

The previous system (145) of differential fractional order equations is independent containing one main chain coordinate $\zeta_{s\xi}$ and describing creep vibration modes of the n independent partial fractional order oscillators, each of them with one degree of freedom.

On the basis of the previous approach for the second subsystems of the second main chain, for the second main chain of the fractional order double DNA chain helix, the eigen amplitudes are in the form

$B_k^{(s)} = D_s \sin k\varphi_s$ and generalized main chain coordinates $\eta_k(t)$ of the second main chain are possible to express by a set of the corresponding main normal coordinates $\zeta_{s\eta}$ in the following form:

$$\eta_k(t) = \sum_{s=1}^{n} B_k^{(s)} \cos\left(\omega_{s\eta} t + \beta_s\right) = \sum_{s=1}^{n} D_s \sin k\varphi_s \cos\left(\omega_{s\eta} t + \beta_s\right) = \sum_{s=1}^{n} \zeta_{s\eta} \sin k\varphi_s \quad (148)$$
$$k = 1,2,3,....,n$$

Normal coordinates or normal modes $\eta_k(t)$ of the second main chain of the fractional order double DNA helix chain are in the form:

$$\zeta_{s\eta} = B_s \cos\left(\omega_{s\eta} t + \beta_s\right),\ s = 1,2,3,....,n \quad (149)$$

with frequencies expressed by (138), and unknown amplitudes B_s and phase β_s depending of initial conditions.

The transformation of the second subsystem (91) of the differential equations of the second main chain by introducing (147) yields:

$$\sum_{s=1}^{n} \frac{2\mathbf{J}}{K}\left(\ddot{\zeta}_{s\eta} + \omega_{s\eta}^2 \zeta_{s\eta} + \frac{K}{2\mathbf{J}} \kappa_{s\sigma\eta} \mathsf{D}_t^{\sigma}\left[\zeta_{s\eta}\right]\right) \sin k\varphi_s = 0 \quad (150)$$

where

$$\kappa_{s\sigma\eta} = 2\kappa_{\sigma}\left(\frac{2\mathbf{J}}{K}\omega_{s\eta}^2 - \mu\right),\ s = 1,2,3,....,n \quad (151)$$

Taking into account that $\sin k\varphi_s$ is equal to zero, only for the boundary conditions, then from (147), we can write that it is necessary to be:

$$\ddot{\zeta}_{s\eta} + \omega_{s\eta}^2 \eta_{s\eta} + \omega_{s\sigma\eta}^2 \mathsf{D}_t^{\sigma}\left[\eta_{s\eta}\right] = 0;\ \ s = 1,2,3,....,n \quad (152)$$

where sets of eigen circular frequencies $\omega_{s\eta}^2$ and a set of characteristic fractional order numbers $\omega_{s\sigma\eta}^2$ are in the following forms:

$$\omega_{s\eta}^2 = \frac{K}{2\mathbf{J}}\left(2\sin^2\frac{\varphi_s}{2} + \mu\right) \quad (153)$$

$$\omega_{s\sigma\eta}^2 = 4\kappa_{\sigma}\frac{K}{2\mathbf{J}}\sin^2\frac{\varphi_s}{2},\ s = 1,2,3,....,n \quad (154)$$

We can see that previous subsystem of the fractional order differential equations (143) with respect to their main coordinates $\zeta_{s\eta}$, as in the previous case corresponding subsystem of the fractional order differential equations (152) with respect to their main normal coordinates $\zeta_{s\xi}$, are two subsystems with independent describing creep vibration modes of the $2n$ independent partial fractional order oscillators, each of them with one degree of freedom. Both obtained subsystems (143) and (147) of fractional order differential equations contain fractional order differential equations of the same type, each being the fractional order differential equations containing one main normal coordinate, $\zeta_{s\xi}$ or $\zeta_{s\eta}$, and each of them with one degree of freedom. Then, we can conclude that we start with one fractional order double DNA helix chain system which is, in whole, with $2n$ degrees of freedom and $2n$ generalized independent angular coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$. And using two subsystems of the main chain coordinates $\xi_k = \varphi_{k,1} - \varphi_{k,2}$ and $\eta_k = \varphi_{k,1} + \varphi_{k,2}$ we obtained two independent subsystems, each with n degrees of freedom, and with the corresponding subsystem of independent eigen main fractional order oscillators described by the corresponding sets of eigen main chain coordinates $\zeta_{s\xi}$ or $\zeta_{s\eta}$, and by subsets of the n eigen circular frequencies $\omega_{s\xi}^2$ and $\omega_{s\eta}^2$ and the corresponding creep properties parameters $\omega_{s\sigma\xi}^2$ and $\omega_{s\sigma\eta}^2$ for $s = 1,2,3,....,n$.

In order to solve the system of fractional order differential equations (145) and (152), we use an analogy between the obtained fractional order differential equations and the corresponding fractional order differential equation (E.4.1) in the Appendix E.4, and we can write (see [3] or [34]):

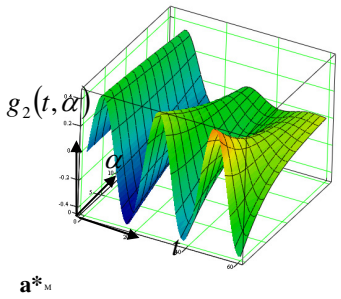
$$\zeta_{s\xi}(t)=\zeta_{s\xi}(0)\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\xi}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\sigma\xi}^{2j}t^{-\alpha j}}{\omega_{s\xi}^{2j}\Gamma(2k+1-\alpha j)}+$$
$$+\dot{\zeta}_{s\xi}(0)\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\xi}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\sigma\xi}^{-2j}t^{-\alpha j}}{\omega_{s\xi}^{2j}\Gamma(2k+2-\alpha j)} \quad (155)$$

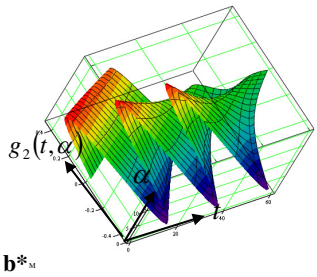
$$\zeta_{s\eta}(t)=\zeta_{s\eta}(0)\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\eta}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\sigma\eta}^{2j}t^{-\alpha j}}{\omega_{s\eta}^{2j}\Gamma(2k+1-\alpha j)}+$$
$$+\dot{\zeta}_{s\eta}(0)\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\eta}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\sigma\eta}^{-2j}t^{-\alpha j}}{\omega_{s\eta}^{2j}\Gamma(2k+2-\alpha j)} \quad (156)$$

7.7.5. Visualization of the main modes of fractional order double DNA helix chain system free vibrations and corresponding partial fractional order oscillator modes

Using the previous expressions (153)-(154) we can separate the following two pairs of the main modes of a fractional order double DNA helix chain system in the following forms:

a* the first pair for the first main chain of a fractional order double DNA helix chain system corresponds to the set of the main coordinate $\zeta_{s\xi}$:





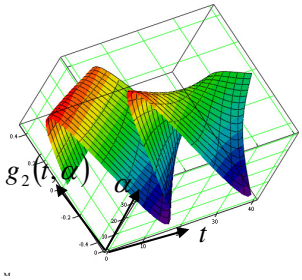
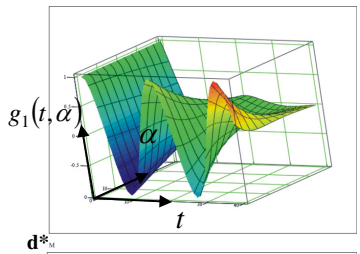


Fig.8.



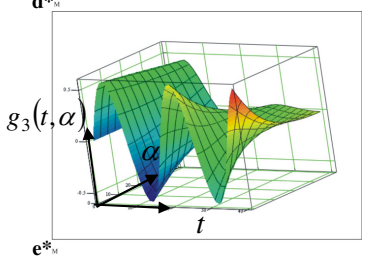
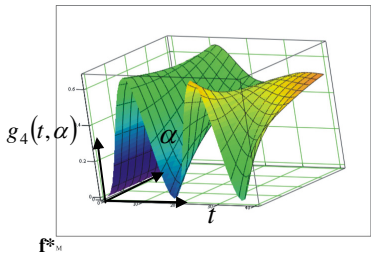


Fig.9.



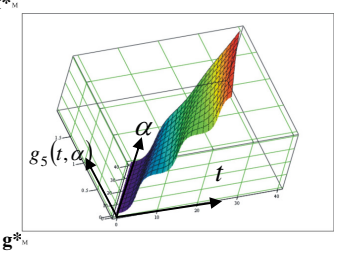


Fig.10.

Fig. 8. Time functions $g_2(t,\alpha)$ surfaces for different multi-plate transversal vibrations kinetic and creep material parameters in the space $(g_2(t,\alpha),t,\alpha)$ for interval $0\le\alpha\le1$.

Fig. 9. Time functions: d* $g_1(t,\alpha)$ and e* $g_3(t,\alpha)$surfaces for the same multi-plate transversal vibrations kinetic and creep material parameters $\omega_0^2=1,\omega_\alpha^2=2$ in the space $(g_1(t,\alpha),t,\alpha)$, as well as in corresponding $(g_1(t,\alpha),t,\alpha)$ for interval $0\le\alpha\le1$.

Fig. 10. Time functions: f* $g_4(t,\alpha)$ and g* $g_5(t,\alpha)$surfaces for the same multi-plate transversal vibrations kinetic and creep material parameters $\omega_0^2=1,\omega_\alpha^2=2$ in the space $(g_4(t,\alpha),t,\alpha)$, as well as in corresponding $(g_5(t,\alpha),t,\alpha)$ for interval $0\le\alpha\le1$.

$$\zeta_{s\xi(\cos)}(t)=\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\xi}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\xi}^{2j}t^{-\alpha j}}{\omega_{s\xi}^{2j}\Gamma(2k+1-\alpha j)} \tag{157}$$

$$\zeta_{s\xi(\sin)}(t)=\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\xi}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\xi}^{-2j}t^{-\alpha j}}{\omega_{s\xi}^{2j}\Gamma(2k+2-\alpha j)} \tag{158}$$

contains two modes like cosine and corresponding sine, with a set of the circular frequencies $\omega_{s\xi}^{2j}$ and fractional order characteristic numbers $\omega_{s\sigma\xi}^{2k}$, with difference in phase analogous to that between cosine $\cos\omega_{s\xi}t$ and sinus $\sin\omega_{s\xi}t$ with the corresponding same circular frequency and difference in phase for $\pi/2$.

b* the second pair for the second main chain of a fractional order double DNA helix chain system corresponding to the set of the main coordinate $\zeta_{s\eta}$:

$$\zeta_{s\eta(\cos)}(t)=\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\eta}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\eta}^{2j}t^{-\alpha j}}{\omega_{s\eta}^{2j}\Gamma(2k+1-\alpha j)} \tag{159}$$

$$\zeta_{s\eta(\sin)}(t)=\sum_{k=0}^{\infty}(-1)^k\,\omega_{s\sigma\eta}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\,\omega_{s\eta}^{-2j}t^{-\alpha j}}{\omega_{s\eta}^{2j}\Gamma(2k+2-\alpha j)} \tag{160}$$

contains two modes like cosine and the corresponding sine, with a set of the circular frequencies $\omega_{s\eta}^{2j}$ and fractional order characteristic numbers $\omega_{s\sigma\eta}^{2k}$, with difference in phase analogous to that between cosinus $\cos\omega_{s\eta}t$ and sinus $\sin\omega_{s\eta}t$ with the corresponding same circular frequency and difference in phase for $\pi/2$.

The previously listed analytical expressions (157)-(158) and (159)-(160) for the corresponding pairs of the first and second main chains fractional order modes of a fractional order double DNA helix chain system also correspond to $2n$ fractional order modes of the partial fractional order oscillators (145) and (152), each for one from the set of $2n$ main coordinates $\zeta_{s\xi}$ and $\zeta_{s\eta}$ of a fractional order double DNA helix chain system.

7.7.6. The double DNA fractional order chain forced vibration model on the basis of the linearized Kovaleva-Manevich's DNA model

For the fractional order forced vibrations of a fractional order double DNA chain model on the basis of the linearized Kovaleva-Manevich's DNA model, we accept two chains, as presented in Fig. 6 or 7, in the form of the double chain fractional order system containing two coupled multi-pendulum subsystems, in which the corresponding material particles of the corresponding multi-pendulum chains are coupled by series of the same standard light fractional order elements.

We assume that both coupled chains from the system of the fractional order DNA model are excited by the system of external excitation containing two series of the one frequency excitations in the forms $\mathsf{M}_{0,k,.1}\cos\Omega_{k.1}t$ and $\mathsf{M}_{0,k,.2}\cos\Omega_{k.2}t$, $k=1,2,3,.....,n$, where $\mathsf{M}_{0,k,.1}$ and $\mathsf{M}_{0,k,.2}$ are amplitudes, $\Omega_{k.1}$ and $\Omega_{k.2}$ are frequencies of the external forced couples, each applied to one of the mass particles of the double DNA model coupled chains.

Then, we can write the corresponding system of the fractional order differential coupled equations in the form:

$$\begin{aligned}
&\mathbf{J}_{\mathbf{k},\mathbf{1}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\frac{K_{\mathbf{k},\mathbf{1}}}{2}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]-\frac{K_{\mathbf{k},\mathbf{1},\sigma}}{2}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\\
&+K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\varphi_{\mathbf{k},\mathbf{1}}--K_{\alpha\beta}-\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)-\\
&-K_{\alpha\beta,\sigma}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=\mathrm{M}_{0,k,,1}\cos\Omega_{k,1}t\\
&\mathbf{J}_{\mathbf{k},2}\ddot{\varphi}_{\mathbf{k},2}-\frac{K_{\mathbf{k},2}}{2}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]-\frac{K_{\mathbf{k},2,\sigma}}{2}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\\
&+K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)\varphi_{\mathbf{k},2}++K_{\alpha\beta}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)+\\
&+K_{\alpha\beta,\sigma}\frac{1}{4}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=\mathrm{M}_{0,k,,2}\cos\Omega_{k,2}t
\end{aligned}\tag{161}$$

The previous system is possible to rewrite in the following form:

$$\begin{aligned}
&\frac{2\mathbf{J}_{\mathbf{k},\mathbf{1}}}{K_{\mathbf{k},\mathbf{1}}}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+-\frac{K_{\mathbf{k},\mathbf{1},\sigma}}{K_{\mathbf{k},\mathbf{1}}}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\\
&+\frac{2K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K_{\mathbf{k},\mathbf{1}}}\varphi_{\mathbf{k},\mathbf{1}}--\frac{K_{\alpha\beta}}{2K_{\mathbf{k},\mathbf{1}}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)-\\
&-\frac{K_{\alpha\beta,\sigma}}{2K_{\mathbf{k},\mathbf{1}}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=\frac{\mathrm{M}_{0,k,,1}}{K_{\mathbf{k},\mathbf{1}}}\cos\Omega_{k,1}t\\
&\frac{2\mathbf{J}_{\mathbf{k},2}}{K_{\mathbf{k},2}}\ddot{\varphi}_{\mathbf{k},2}-\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]-\frac{K_{\mathbf{k},2,\sigma}}{K_{\mathbf{k},2}}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\\
&+\frac{2K_{\alpha\beta}r_{\alpha}\left(r_{\alpha}-r_{\beta}\right)}{K_{\mathbf{k},2}}\varphi_{\mathbf{k},2}++\frac{K_{\alpha\beta}}{2K_{\mathbf{k},2}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)+\\
&+\frac{K_{\alpha\beta,\sigma}}{2K_{\mathbf{k},2}}\left(1-\frac{\omega_{\alpha\beta 2}}{\omega_{\alpha\beta 1}}\right)\left(r_{\alpha}-r_{\beta}\right)^2\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=\frac{\mathrm{M}_{0,k,,2}}{K_{\mathbf{k},2}}\cos\Omega_{k,2}t
\end{aligned}\tag{162}$$

As our intention is to use the previous double DNA fractional order chain model for the case of the homogeneous system parameters, we take into account that: $K_{\mathbf{k},\mathbf{1},\sigma}=K_{\mathbf{k},2,\sigma}$=K. and $K_{\alpha\beta,\sigma}=K_{\alpha\beta,\sigma}$ and taking this into account, we introduce the notation (14) and (15) and then the previous system of coupled fractional order differential equations is possible to write in the following form:

$$\begin{aligned}
&\frac{2\mathbf{J}}{K}\ddot{\varphi}_{\mathbf{k},\mathbf{1}}-\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]-\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,\mathbf{1}}-\varphi_{\mathbf{k},\mathbf{1}}\right)-\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k}-1,\mathbf{1}}\right)\right]+\\
&+2\mu\varphi_{\mathbf{k},\mathbf{1}}-\kappa\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)-\kappa\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=\frac{\mathrm{M}_{0,k,,1}}{K}\cos\Omega_{k,1}t\\
&\frac{2\mathbf{J}_{\mathbf{k},2}}{K}\ddot{\varphi}_{\mathbf{k},2}-\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]-\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k}+1,2}-\varphi_{\mathbf{k},2}\right)-\left(\varphi_{\mathbf{k},2}-\varphi_{\mathbf{k}-1,2}\right)\right]+\\
&+2\mu\varphi_{\mathbf{k},2}+\kappa\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)+\kappa\kappa_{\sigma}\mathrm{D}_t^{\sigma}\left[\left(\varphi_{\mathbf{k},\mathbf{1}}-\varphi_{\mathbf{k},2}\right)\right]=\frac{\mathrm{M}_{0,k,,2}}{K}\cos\Omega_{k,2}t
\end{aligned}\tag{163}$$

where $\kappa_{\sigma}=\dfrac{K_{\alpha\beta,\sigma}}{K}$.

Using change of the generalized independent angular coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$ for the k-th bases of both chains in the DNA model into the following new main chain coordinates ξ_k and η_k by the following dependence-relations: $\xi_k=\varphi_{k,1}-\varphi_{k,2}$ and $\eta_k=\varphi_{k,1}+\varphi_{k,2}$, the previous system of fractional order differential equations (160) obtains the following form:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{k-1}+2\xi_k-\xi_{k+1}+\kappa_\sigma \mathsf{D}_t^\sigma\left[-\xi_{k-1,}+2\xi_k-\xi_{k+1}\right]+2\mu\xi_{\mathbf{k}}-2\kappa\xi_{\mathbf{k}}-2\kappa\kappa_\sigma \mathsf{D}_t^\sigma\left[\xi_{\mathbf{k}}\right]= \\ =\frac{\mathsf{M}_{0,k,.1}}{K}\cos\Omega_{k,1}t-\frac{\mathsf{M}_{0,k,.2}}{K}\cos\Omega_{k,2}t \tag{164}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{k-1,}+2\eta_k-\eta_{k+1}+\kappa_\sigma \mathsf{D}_t^\sigma\left[-\eta_{k-1,}+2\eta_k-\eta_{k+1}\right]+2\mu\eta_{\mathbf{k}}= \\ =\frac{\mathsf{M}_{0,k,.1}}{K}\cos\Omega_{k,1}t+\frac{\mathsf{M}_{0,k,.2}}{K}\cos\Omega_{k,2}t \quad ,\ k=1,2,3,.....,n \tag{165}$$

The first series (161) and second series (163) of the previous system (164)-(165) of the fractional order differential equations are decoupled and independent. *Then, we can conclude that new main chain coordinates* ξ_k *and* η_k *are the main chain coordinates of fractional order double DNA helix chain model system for forced vibration regimes and that we obtain two fictive decoupled eigen single* fractional order *chains of the double DNA fractional order model. This is also one of the fundamental conclusions as an important property of the fractional order homogeneous model of forced vibrations in a* fractional order double DNA homogeneous helix.

The systems of fractional order differential equations (164)-(165) contain two separate subsystems of fractional order differential equations expressed by main chain *coordinates of* ξ_k *and* η_k which are the main chain *coordinates of a fractional order double DNA chain helix forced vibration model and separate DNA fractional order model into two independent fractional order chains.*

7.7.7. Analytical solutions of the subsystems of the main chains fractional order differential equations for forced regime oscillations

We solve the previous subsystems (164) and (165) using the Laplace transformations, as in section 7.3. After applying the Laplace transformations to the previous systems (164) and (165) of differential equation's with fractional order derivative and having in mind that we introduced the notations $\mathsf{L}\{\xi_k(t)\}$ and $\mathsf{L}\{\eta_k(t)\}$ for the Laplace transformations, as well as that: (92), (93), (94) and (95) and also having in mind that we accepted the hypothesis that the initial conditions of fractional order derivatives of the system are given using: $\left.\frac{d^{\sigma-1}\xi_k(t)}{dt^{\sigma-1}}\right|_{t=0}=0$ and $\left.\frac{d^{\sigma-1}\eta_k(t)}{dt^{\sigma-1}}\right|_{t=0}=0$, as well as that:

$$\mathsf{L}\left\{\frac{\mathsf{M}_{0,k,.1}}{K}\cos\Omega_{k,1}t\mp\frac{\mathsf{M}_{0,k,.2}}{K}\cos\Omega_{k,2}t\right\}=\frac{\mathsf{M}_{0,k,.1}}{K}\frac{p}{p^2+\Omega_{k,1}^2}\mp\frac{\mathsf{M}_{0,k,.2}}{K}\frac{p}{p^2+\Omega_{k,2}^2} \tag{166}$$

where ξ_{0k} and $\dot{\xi}_{0k}$ as well as η_{0k} and $\dot{\eta}_{0k}$ are initial angular positions and angular velocities defined by initial conditions of the system material particles dynamics in the chains at initial moment, we can write the following system of the equations with unknown Laplace transforms:

$$\mathsf{L}\left\{\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}\right\}-\mathsf{L}\{\xi_{k-1}\}+2\mathsf{L}\{\xi_k\}-\mathsf{L}\{\xi_{k+1}\}+\kappa_\sigma\mathsf{L}\left\{\mathsf{D}_t^\sigma\left[-\xi_{k-1,}+2\xi_k-\xi_{k+1}\right]\right\}+2\mu\mathsf{L}\{\xi_{\mathbf{k}}\}-2\kappa\mathsf{L}\{\xi_{\mathbf{k}}\}-2\kappa\kappa_\sigma\mathsf{L}\left\{\mathsf{D}_t^\sigma\left[\xi_{\mathbf{k}}\right]\right\}= \\ =+h_{0,k,1}\frac{p}{p^2+\Omega_{k,1}^2}-h_{0,k,2}\frac{p}{p^2+\Omega_{k,2}^2} \tag{167}$$

$$\frac{2\mathbf{J}}{K}\mathsf{L}\{\ddot{\eta}_{\mathbf{k}}\}-\mathsf{L}\{\eta_{k-1,}\}+2\mathsf{L}\{\eta_k\}-\mathsf{L}\{\eta_{k+1}\}+\kappa_\sigma\mathsf{L}\left\{\mathsf{D}_t^\sigma\left[-\eta_{k-1,}+2\eta_k-\eta_{k+1}\right]\right\}+2\mu\mathsf{L}\{\eta_{\mathbf{k}}\}= \\ =+h_{0,k,1}\frac{p}{p^2+\Omega_{k,1}^2}+h_{0,k,2}\frac{p}{p^2+\Omega_{k,2}^2} \quad ,k=1,2,3,.....,n \tag{168}$$

The previous system is possible to rewrite in the following form:

$$\frac{\left[\frac{2\mathbf{J}}{K}p^2+2\mu-2\kappa\left(1+\kappa_\sigma p^\sigma\right)\right]}{\left(1+\kappa_\sigma p^\sigma\right)}\mathsf{L}\{\xi_k(t)\}-\mathsf{L}\{\xi_{k-1}\}+2\mathsf{L}\{\xi_k(t)\}-\mathsf{L}\{\xi_{k+1}\}=\frac{2\mathbf{J}}{K}\frac{\left[p\xi_{0k}+\dot{\xi}_{0k}\right]}{\left(1+\kappa_\sigma p^\sigma\right)}+ \tag{169}$$
$$+h_{0,k,1}\frac{p}{\left(p^2+\Omega_{k,1}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}-h_{0,k,2}\frac{p}{\left(p^2+\Omega_{k,2}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}$$

$$\frac{\left(p^2\frac{2\mathbf{J}}{K}+2\mu\right)}{\left(1+\kappa_\sigma p^\sigma\right)}\mathsf{L}\{\eta_{\mathbf{k}}\}-\mathsf{L}\{\eta_{k-1,}\}+2\mathsf{L}\{\eta_k\}-\mathsf{L}\{\eta_{k+1}\}=\frac{2\mathbf{J}}{K}\frac{\left[p\eta_{0k}+\dot{\eta}_{0k}\right]}{\left(1+\kappa_\sigma p^\sigma\right)}+ \tag{170}$$
$$+h_{0,k,1}\frac{p}{\left(p^2+\Omega_{k,1}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}+h_{0,k,2}\frac{p}{\left(p^2+\Omega_{k,2}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}$$

Now, we have two separate, uncoupled non homogeneous subsystems of the algebraic equations in the following forms:

$$-\mathsf{L}\{\xi_{k-1}\}+(2+v)\mathsf{L}\{\xi_k(t)\}-\mathsf{L}\{\xi_{k+1}\}=\frac{\left[p\xi_{0k}+\dot{\xi}_{0k}\right]}{\omega_0^2\left[1+\kappa_\sigma p^\sigma\right]}+h_{0,k,1}\frac{p}{\left(p^2+\Omega_{k,1}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}- \tag{171}$$
$$-h_{0,k,2}\frac{p}{\left(p^2+\Omega_{k,2}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}$$

$$-\mathsf{L}\{\eta_{k-1,}\}+(2+u)\mathsf{L}\{\eta_k\}-\mathsf{L}\{\eta_{k+1}\}=\frac{\left[p\eta_{0k}+\dot{\eta}_{0k}\right]}{\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)}+h_{0,k,1}\frac{p}{\left(p^2+\Omega_{k,1}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}+ \tag{172}$$
$$+h_{0,k,2}\frac{p}{\left(p^2+\Omega_{k,2}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}$$

or in the following forms:

$$-\mathsf{L}\{\xi_{k-1}\}+(2+v)\mathsf{L}\{\xi_k(t)\}-\mathsf{L}\{\xi_{k+1}\}=h_{\xi hk}\left(p,\xi_{0k},\dot{\xi}_{0k}\right)+h_{\xi pk}\left(p,\Omega_{k,1}^2,\Omega_{k,2}^2,h_{0k,1},h_{0k,2}\right) \tag{174}$$

$$-\mathsf{L}\{\eta_{k-1,}\}+(2+u)\mathsf{L}\{\eta_k\}-\mathsf{L}\{\eta_{k+1}\}=h_{\eta hk}\left(p,\eta_{0k},\eta_{0k}\right)+h_{\eta pk}\left(p,\Omega_{k,1}^2,\Omega_{k,2}^2,h_{0k,1},h_{0k,2}\right) \tag{175}$$

where

$$v=\frac{\left[p^2+2\mu\omega_0^2\right]}{\omega_0^2\left[1+\kappa_\sigma p^\sigma\right]}-2\kappa,\qquad u=\frac{\left[p^2+2\mu\omega_0^2\right]}{\omega_0^2\left[1+\kappa_\sigma p^\sigma\right]},\quad \omega_0^2=\frac{K}{2\mathbf{J}} \tag{176}$$

$$h_{\xi hk}\left(p,\xi_{0k},\dot{\xi}_{0k}\right)=\frac{\left[p\xi_{0k}+\dot{\xi}_{0k}\right]}{\omega_0^2\left[1+\kappa_\sigma p^\sigma\right]},\quad h_{\eta hk}\left(p,\eta_{0k},\eta_{0k}\right)=\frac{\left[p\eta_{0k}+\dot{\eta}_{0k}\right]}{\omega_0^2\left(1+\kappa_\sigma p^\sigma\right)} \tag{177}$$

$$h_{\xi pk}\left(p,\Omega_{k,1}^2,\Omega_{k,2}^2,h_{0k,1},h_{0k,2}\right)=h_{0,k,1}\frac{p}{\left(p^2+\Omega_{k,1}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}-h_{0,k,2}\frac{p}{\left(p^2+\Omega_{k,2}^2\right)\left(1+\kappa_\sigma p^\sigma\right)} \tag{178}$$

$$h_{\xi pk}\left(p,\Omega_{k,1}^2,\Omega_{k,2}^2,h_{0k,1},h_{0k,2}\right)=h_{0,k,1}\frac{p}{\left(p^2+\Omega_{k,1}^2\right)\left(1+\kappa_\sigma p^\sigma\right)}+h_{0,k,2}\frac{p}{\left(p^2+\Omega_{k,2}^2\right)\left(1+\kappa_\sigma p^\sigma\right)} \tag{179}$$

Both subsystems are of the same form and it is necessary to solve one of the subsystems and applying the analogy it is easy to solve the other of the subsystem equations. For that reason, we can use the method proposed in the papers [32] and [31]. The determinate of the previous subsystem (174) as well as (165) is in the following form (104) as for the subsystems of algebraic equations (100) and (101) in section 7.3.

The algebra no homogenous algebra equations (171) and (172) for special case as in section 7.3 are of the same form, as presented in (101) and (102), and we can write:

$$-\mathsf{L}\{\xi_{k-1}\}+(2+v)\mathsf{L}\{\xi_k(t)\}-\mathsf{L}\{\xi_{k+1}\}=\begin{cases}h_{\xi h1}\left(p,\xi_{01},\dot{\xi}_{01}\right)+h_{\xi p1}\left(p,\Omega_{1,1}^2,\Omega_{1,2}^2,h_{01,1},h_{01,2},\right) & k=1\\ 0 & k\neq 1\end{cases} \tag{180}$$

$$-\mathsf{L}\{\eta_{k-1,}\}+(2+u)\mathsf{L}\{\eta_k\}-\mathsf{L}\{\eta_{k+1}\}=\begin{cases}h_{\eta h1}\left(p,\eta_{01},\dot{\eta}_{01}\right)+h_{\eta p1}\left(p,\Omega_{1,1}^2,\Omega_{1,2}^2,h_{01,1},h_{01,2}\right) & k=1\\ 0 & k\neq 1\end{cases} \tag{181}$$

The determinates of the previous algebraic subsystems (180) as well as (181) are of the same form, as presented in (104). Introducing the notation $h_{\xi h1}\left(p,\xi_{01},\dot{\xi}_{01}\right)+h_{\xi}\left(p,\Omega_{1,1}^2,\Omega_{1,2}^2,h_{0k,1},h_{0k,2}\right)$ and $h_{\eta h1}\left(p,\eta_{01},\eta_{01}\right)+h_{\eta p1}\left(p,\Omega_{1,1}^2,\Omega_{1,2}^2,h_{01,1},h_{01,2}\right)$ defined by (178) and (179), for the determinants $\tilde{\Delta}_{(k)}\left(v,h_{\xi}\right)$, we can write similar expressions, as defined by (105)-(109), changing the expressions $h_{\xi}\left(p,\xi_{01},\dot{\xi}_{01}\right)$ by expressions $h_{\xi h1}\left(p,\xi_{01},\dot{\xi}_{01}\right)+h_{\xi}\left(p,\Omega_{1,1}^2,\Omega_{1,2}^2,h_{0k,1},h_{0k,2}\right)$ as well as by $h_{\eta h1}\left(p,\eta_{01},\eta_{01}\right)+h_{\eta p1}\left(p,\Omega_{1,1}^2,\Omega_{1,2}^2,h_{01,1},h_{01,2}\right)$.

To solve the system of the algebraic non-homogeneous equations (180) or (181) with respect to unknown Laplace transforms $\mathsf{L}\left\{\xi_k(t)\right\}$ or $\mathsf{L}\left\{\eta_k(t)\right\}$ of the time function main chain coordinates $\xi_k(t)$ and $\eta_k(t)$-unknown normal chain coordinates of the system main chains for forced vibrations, we can use the Cramer approach in a similar way as in section 7.3.

7.7.8. Forced eigen modes of the subsystems of the main chains of a fractional order double DNA helix chain system forced vibrations

In this part, we start with two subsystems of fractional order differential equations (164) and (165) expressed by eigen a normal chain coordinates $\xi_k=\varphi_{k,1}-\varphi_{k,2}$ and $\eta_k=\varphi_{k,1}+\varphi_{k,2}$, and these subsystems can be rewritten in the following form:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{\mathbf{k+1}}+2\xi_{\mathbf{k}}\left[1+\mu-\kappa\right]-\xi_{\mathbf{k-1}}+\kappa_{\sigma}\mathsf{D}_t^{\sigma}\left[-\xi_{k-1,}+2(1-\kappa)\xi_k-\xi_{k+1}\right]=h_{0,k,1}\cos\Omega_{k.1}t-h_{0,k,2}\cos\Omega_{k.2}t$$
$$k=1,2,3,.....,n \tag{182}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{\mathbf{k+1}}+2\eta_{\mathbf{k}}(1+\mu)-\eta_{\mathbf{k-1}}+\kappa_{\sigma}\mathsf{D}_t^{\sigma}\left[-\eta_{k-1,}+2\eta_k-\eta_{k+1}\right]=h_{0,k,1}\cos\Omega_{k.1}t+h_{0,k,2}\cos\Omega_{k.2}t$$
$$k=1,2,3,.....,n \tag{183}$$

Without loss of generality, our interest was focused next on considering two subsystems of the fractional order differential equations in the following form:

$$\frac{2\mathbf{J}}{K}\ddot{\xi}_{\mathbf{k}}-\xi_{\mathbf{k+1}}+2\xi_{\mathbf{k}}\left[1+\mu-\kappa\right]-\xi_{\mathbf{k-1}}+\kappa_{\sigma}\mathsf{D}_t^{\sigma}\left[-\xi_{k-1,}+2(1-\kappa)\xi_k-\xi_{k+1}\right]=\begin{cases}h_{0,1,1}\cos\Omega_{1.1}t & k=1\\ 0 & k\neq 1\end{cases}$$
$$k=1,2,3,.....,n \tag{184}$$

$$\frac{2\mathbf{J}}{K}\ddot{\eta}_{\mathbf{k}}-\eta_{\mathbf{k+1}}+2\eta_{\mathbf{k}}(1+\mu)-\eta_{\mathbf{k-1}}+\kappa_{\sigma}\mathsf{D}_t^{\sigma}\left[-\eta_{k-1,}+2\eta_k-\eta_{k+1}\right]=\begin{cases}h_{0,1,1}\cos\Omega_{1.1}t & k=1\\ 0 & k\neq 1\end{cases},$$
$$k=1,2,3,.....,n \tag{185}$$

The previous two subsystems are for the case of fractional order forced vibrations of a double DNA helix chain system excited by a single frequency external couple $\mathsf{M}_{0,1,.1}\cos\Omega_{1,1}t$, with amplitude $\mathsf{M}_{0,1,.1}$ and frequency $\Omega_{1,1}$, applied to the first mass particle in the first chain of a double DNA helix chain system.

The first series (184) and second series (185) of the previous system (164)-(165) of the fractional order differential equations for forced vibrations are decoupled and independent. *Then, we can conclude that new coordinates ξ_k and η_k are the main chain coordinates of fractional order double DNA helix chain model system for forced vibration regimes and that we obtain two fictive decoupled eigen single* fractional order *chains of the double DNA fractional order model. This is also one of the fundamental conclusions as an important property of the fractional order homogeneous model of forced vibrations in a* fractional order double DNA homogeneous helix.

The systems of the fractional order differential equations (184)-(185) contain two separate subsystems of fractional order differential equations expressed by *coordinates of ξ_k and η_k* which are the main chain *coordinates of a fractional order double DNA chain helix forced vibration model and separate DNA fractional order chain model into two independent fractional order main chains.*

For the first main chain of the double DNA chain helix (184), the eigen amplitudes for free vibrations are in the form $A_k^{(s)} = C_s \sin k\varphi_s$ and generalized coordinates $\xi_k(t)$ of the first main chain for forced vibrations is possible to express by a set of this eigen main chain main coordinates $\zeta_{s\xi}$ for free vibrations (149) in the following form:

$$\xi_k(t) = \sum_{s=1}^{n} \zeta_{s\xi} \sin k\varphi_s \qquad (186)$$
$$k = 1,2,3,....,n$$

and for the other main chain of the double DNA chain helix (185) generalized coordinates $\eta_k(t)$ of the second main chain for forced vibrations is possible to express by a set of this eigen main chain main coordinates $\zeta_{s\eta}$ for free vibrations (146) in the following form:

$$\eta_k(t) = \sum_{s=1}^{n} \zeta_{s\eta} \sin k\varphi_s \qquad (187)$$

Normal coordinates $\zeta_{s\xi}$ or normal modes of the first main chain for forced vibrations is possible to express in the form similar to that for free vibrations (142), but introducing the assumption that unknown amplitudes C_s and phase α_s depending of initial conditions are not constant, but functions of time, $C_s(t)$ and phase $\alpha_s(t)$, and for fractional order system the main coordinates are in the form

$$\zeta_{s\xi}(t) = C_s(t)\cos\left(\Omega_\xi t + \alpha_s(t)\right), s = 1,2,3,....,n \qquad (188)$$

with known frequencies (139), and unknown time functions - amplitudes $C_s(t)$ and phase $\alpha_s(t)$ depending of time and initial conditions.

Then, we introduce the expressions (186) and (187) and their corresponding second and fractional order derivative into the subsystem of the fractional order differential equations (184) and (185), and we obtain the following sub-systems:

$$\frac{2\mathbf{J}}{K}\sum_{s=1}^{n}\ddot{\zeta}_{s\xi}\sin k\varphi_s - \sum_{s=1}^{n}\zeta_{s\xi}\sin(k-1)\varphi_s + 2[1+\mu-\kappa]\sum_{s=1}^{n}\zeta_{s\xi}\sin k\varphi_s - \sum_{s=1}^{n}\zeta_{s\xi}\sin(k+1)\varphi_s +$$
$$+\kappa_\sigma \mathrm{D}_t^\sigma\left[-\sum_{s=1}^{n}\zeta_{s\xi}\sin(k-1)\varphi_s + 2(1-\kappa)\sum_{s=1}^{n}\zeta_{s\xi}\sin k\varphi_s - \sum_{s=1}^{n}\zeta_{s\xi}\sin(k+1)\varphi_s\right] = \begin{cases} h_{0,1,1}\cos\Omega_{1.1}t & k=1 \\ 0 & k\neq 1 \end{cases}$$
$$k = 1,2,3,.....,n \qquad (189)$$

$$\frac{2\mathbf{J}}{K}\sum_{s=1}^{n}\ddot{\zeta}_{s\eta}\sin k\varphi_s - \sum_{s=1}^{n}\zeta_{s\eta}\sin(k-1)\varphi_s + 2[1+\mu]\sum_{s=1}^{n}\zeta_{s\eta}\sin k\varphi_s - \sum_{s=1}^{n}\zeta_{s\eta}\sin(k+1)\varphi_s +$$
$$+\kappa_\sigma \mathrm{D}_t^\sigma\left[-\sum_{s=1}^{n}\zeta_{s\eta}\sin(k-1)\varphi_s + 2\sum_{s=1}^{n}\zeta_{s\eta}\sin k\varphi_s - \sum_{s=1}^{n}\zeta_{s\eta}\sin(k+1)\varphi_s\right] = \begin{cases} h_{0,1,1}\cos\Omega_{1.1}t & k=1 \\ 0 & k\neq 1 \end{cases}$$
$$k = 1,2,3,.....,n \qquad (190)$$

After making group sublimations of some terms in the previous equations (189), we obtain the following subsystem:

$$\sum_{s=1}^{n}\frac{2\mathbf{J}}{K}\left(\ddot{\zeta}_{s\xi} + 2\frac{K}{2\mathbf{J}}\langle[1+\mu-\kappa]-\cos\varphi_s\rangle + \frac{K}{2\mathbf{J}}2\kappa_\sigma\langle(1-\kappa)-\cos\varphi_s\rangle\mathrm{D}_t^\sigma\left[\zeta_{s\xi}\right]\right)\sin k\varphi_s = \begin{cases} h_{0,1,1}\cos\Omega_{1.1}t & k=1 \\ 0 & k\neq 1 \end{cases} \qquad (191)$$
$$k = 1,2,3,.....,n$$

Then, taking into account the denotations (144), (146) and (145), the previous subsystem of equation is possible to rewrite in the following form:

$$\sum_{s=1}^{n}\frac{2\mathbf{J}}{K}\left(\ddot{\zeta}_{s\xi} + \omega_{s\xi}^2\zeta_{s\xi} + \frac{K}{2\mathbf{J}}\kappa_{s\sigma\xi}\mathrm{D}_t^\sigma\left[\zeta_{s\xi}\right]\right)\sin k\varphi_s = \begin{cases} h_{0,1,1}\cos\Omega_{1.1}t & k=1 \\ 0 & k\neq 1 \end{cases}, k = 1,2,3,.....,n \qquad (192)$$

Taking into account that it is possible to develop, i.e. to express right hand side into series according to $\sin k\varphi_s$ in the following series:

$$\begin{cases} h_{0,1,1}\cos\Omega_{1.1}t & k=1 \\ 0 & k\neq 1 \end{cases} = \begin{cases} \sum_{s=1}^{n}\frac{2\mathbf{J}}{K}h_{0,1,1(s)}\sin k\varphi_s\cos\Omega_{1.1}t & k=1 \\ 0 & k\neq 1 \end{cases} \tag{193}$$

where

$$h_{0,1,1(s)} = \frac{K}{2\mathbf{J}}\frac{\sum_{s=1}^{n}h_{0,1,1}\sin k\varphi_s}{\left\langle\sum_{s=1}^{n}\sum_{r=1}^{n}\sin k\varphi_s\sin k\varphi_r\right\rangle_{r=s}} \qquad k=1$$
$$h_{0,1,1(s)} = 0 \qquad k\neq 1 \tag{194}$$

It is possible to rewrite the equations (192) in the following form:

$$\sum_{s=1}^{n}\frac{2\mathbf{J}}{K}\left(\ddot{\zeta}_{s\xi}+\omega_{s\xi}^2\zeta_{s\xi}+\omega_{s\sigma\xi}^2\mathsf{D}_t^{\sigma}\left[\zeta_{s\xi}\right]-h_{0,1,1(s)}\cos\Omega_{1.1}t\right)\sin k\varphi_s=0,\ k=1,2,3,.....,n \tag{195}$$

Then, taking into account that $\sin k\varphi_s\neq 0$, in a general case, from (195) it is possible to obtain the following subsystem of fractional order differential equations along main chain normal coordinates $\zeta_{s\xi}$ in the following form:

$$\ddot{\zeta}_{s\xi}+\omega_{s\xi}^2\zeta_{s\xi}+\omega_{s\sigma\xi}^2\mathsf{D}_t^{\sigma}\left[\zeta_{s\xi}\right]=h_{0,1,1(s)}\cos\Omega_{1.1}t,\ s=1,2,3,....,n \tag{196}$$

where $\omega_{s\xi}^2$ square of eigen circular frequencies determined by expression (146) and $\omega_{s\sigma\xi}^2$ the corresponding eigen characteristic numbers expressing fractional order subsystem properties, are determined by the expression (147).

Analogously, using (187) and (180) from (190), it is possible to obtain the second subsystem of fractional order differential equations along main chain normal coordinates $\zeta_{s\eta}$ in the following form:

$$\ddot{\zeta}_{s\eta}+\omega_{s\xi}^2\zeta_{s\eta}+\omega_{s\sigma\eta}^2\mathsf{D}_t^{\sigma}\left[\zeta_{s\eta}\right]=h_{0,1,1(s)}\cos\Omega_{1.1}t,\ s=1,2,3,....,n \tag{197}$$

where $\omega_{s\eta}^2$ the square of eigen circular frequencies is determined by the expression (153) and $\omega_{s\sigma\eta}^2$ the corresponding eigen characteristic numbers expressing fractional order subsystem properties are determined by the expression (154).

Then, we have the system (196)-(197) containing two subsets of the main fractional order forced oscillators, each with n fractional order differential equations along one main chain main coordinates $\zeta_{s\xi}$ and $\zeta_{s\eta}$. Each of these $2n$ fractional order differential equations contains only one main eigen coordinate $\zeta_{s\xi}$ or $\zeta_{s\eta}$ of the system.

The system (196)-(197) represents the main fractional order forced oscillators along the independent system of main chain main coordinates $\zeta_{s\xi}$ or $\zeta_{s\eta}$, $s=1,2,3,....,n$ each with one circular frequency of external excitation and one eigen circular frequency and one eigen characteristic number from one of the two sets of: $\omega_{s\xi}$ or $\omega_{s\eta}$ eigen circular frequencies determined by the expression (146) or (153) and $\omega_{s\sigma\xi}^2$ or $\omega_{s\sigma\eta}^2$ the corresponding eigen characteristic numbers expressing fractional order subsystem properties, determined by the expression (147) or (154).

All fractional order differential equations of the system (196)-(197) are the same type and it is possible to solve them in the same way using the Laplace transform $\mathsf{L}\{\zeta_{s\xi}(t)\}$ and $\mathsf{L}\{\zeta_{s\eta}(t)\}$. Applying the Laplace transform to the system (196)-(197) of the fractional order differential equations, we obtain the following two sub-systems of equations:

$$\mathsf{L}\left\{\ddot{\zeta}_{s\xi}\right\}+\omega_{s\xi}^2\mathsf{L}\left\{\zeta_{s\xi}(t)\right\}+\omega_{s\sigma\xi}^2\mathsf{L}\left\{\mathsf{D}_t^{\sigma}\left[\zeta_{s\xi}\right]\right\}=h_{0,1,1(s)}\mathsf{L}\left\{\cos\Omega_{1.1}t\right\},\quad s=1,2,3,....,n \tag{198}$$

$$\mathsf{L}\left\{\ddot{\zeta}_{s\eta}\right\}+\omega_{s\xi}^2\mathsf{L}\left\{\zeta_{s\eta}(t)\right\}+\omega_{s\sigma\eta}^2\mathsf{L}\left\{\mathsf{D}_t^{\sigma}\left[\zeta_{s\eta}\right]\right\}=h_{0,1,1(s)}\mathsf{L}\left\{\cos\Omega_{1.1}t\right\},\quad s=1,2,3,....,n \tag{199}$$

Taking into account that:

$$\mathsf{L}\left\{\frac{d^2\zeta_{s\xi}(t)}{dt^2}\right\}=p^2\mathsf{L}\left\{\zeta_{s\xi}(t)\right\}-\left[p\zeta_{0s\xi}+\dot{\zeta}_{0s\xi}\right],\quad s=1,2,3,....,n \tag{200}$$

$$\mathsf{L}\left\{\frac{d^2\zeta_{s\eta}(t)}{dt^2}\right\}=p^2\mathsf{L}\left\{\zeta_{s\eta}(t)\right\}-\left[p\zeta_{0s\eta}+\dot{\zeta}_{0s\eta}\right],\ s=1,2,3,....,n \tag{201}$$

$$\mathsf{L}\left\{\cos\Omega_{1,1}t\right\}=\frac{p}{p^2+\Omega_{1,1}^2} \tag{202}$$

$$\mathsf{L}\left\{\frac{d^\sigma\zeta_{s\xi}}{dt\omega}\right\}=p^\sigma\mathsf{L}\left\{\zeta_{s\xi}\right\}-\left.\frac{d^{\sigma-1}\zeta_{s\xi}}{dt^{\varpi-1}}\right|_{t=0}=p^\sigma\mathsf{L}\left\{\zeta_{s\xi}\right\},\ s=1,2,3,....,n \tag{203}$$

$$\mathsf{L}\left\{\frac{d^\sigma\zeta_{s\eta}}{dt\omega}\right\}=p^\sigma\mathsf{L}\left\{\zeta_{s\eta}\right\}-\left.\frac{d^{\sigma-1}\zeta_{s\eta}}{dt^{\varpi-1}}\right|_{t=0}=p^\sigma\mathsf{L}\left\{\zeta_{s\eta}\right\},\ s=1,2,3,....,n \tag{204}$$

and after introducing into the sub-systems (198)-(199) for the Laplace transform $\mathsf{L}\left\{\zeta_{s\xi}(t)\right\}$ and $\mathsf{L}\left\{\zeta_{s\eta}(t)\right\}$ of the system double DNA helix chain eigen main coordinates $\zeta_{s\xi}$ and $\zeta_{s\eta}$ for forced regime, we obtain:

$$\mathsf{L}\left\{\zeta_{s\xi}\right\}=\frac{\left[p\zeta_{0s\xi}+\dot{\zeta}_{0s\xi}\right]}{\left(p^2+\omega_{s\xi}^2+\omega_{s\sigma\xi}^2p^\sigma\right)}+\frac{h_{0,1,1(s)}}{\left(p^2+\omega_{s\xi}^2+\omega_{s\sigma\xi}^2p^\sigma\right)}\frac{p}{p^2+\Omega_{1,1}^2},\ s=1,2,3,....,n \tag{205}$$

$$\mathsf{L}\left\{\zeta_{s\eta}\right\}=\frac{\left[p\zeta_{0s\eta}+\dot{\zeta}_{0s\eta}\right]}{\left(p^2+\omega_{s\eta}^2+\omega_{s\sigma\eta}^2p^\sigma\right)}+\frac{h_{0,1,1(s)}}{\left(p^2+\omega_{s\eta}^2+\omega_{s\sigma\eta}^2p^\sigma\right)}\frac{p}{p^2+\Omega_{1,1}^2},\ s=1,2,3,....,n \tag{206}$$

Then, to obtain the system double DNA helix chain eigen main coordinates $\zeta_{s\xi}(t)$ and $\zeta_{s\eta}(t)$ it is necessary to apply the inverse Laplace transform to the expressions (205)-(206). Then we can write the following:

$$\zeta_{s\xi}(t)=\zeta_{s\xi,\mathrm{hom}}(t)+\zeta_{s\xi,part}(t) \tag{207}$$

and

$$\zeta_{s\eta}(t)=\zeta_{s\eta,\mathrm{hom}}(t)+\zeta_{s\eta,part}(t) \tag{208}$$

where

a)* $\zeta_{s\xi,\mathrm{hom}}(t)$ and $\zeta_{s\eta,\mathrm{hom}}(t)$ are terms corresponding to the solutions of the homogeneous fractional order differential equations and the solutions are in the forms (155) and (156) (see Appendix (E.4.1)):

$$\begin{aligned}\zeta_{s\xi,\mathrm{hom}}(t)=&\zeta_{s\xi}(0)\sum_{k=0}^{\infty}(-1)^k\omega_{s\sigma\xi}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega_{s\sigma\xi}^{2j}t^{-\alpha j}}{\omega_{s\xi}^{2j}\Gamma(2k+1-\alpha j)}+\\&+\dot{\zeta}_{s\xi}(0)\sum_{k=0}^{\infty}(-1)^k\omega_{s\sigma\xi}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega_{s\sigma\xi}^{-2j}t^{-\alpha j}}{\omega_{s\xi}^{2j}\Gamma(2k+2-\alpha j)}\end{aligned} \tag{209}$$

$$\begin{aligned}\zeta_{s\eta,\mathrm{hom}}(t)=&\zeta_{s\eta}(0)\sum_{k=0}^{\infty}(-1)^k\omega_{s\sigma\eta}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega_{s\sigma\eta}^{2j}t^{-\alpha j}}{\omega_{s\eta}^{2j}\Gamma(2k+1-\alpha j)}+\\&+\dot{\zeta}_{s\eta}(0)\sum_{k=0}^{\infty}(-1)^k\omega_{s\sigma\eta}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega_{s\sigma\eta}^{-2j}t^{-\alpha j}}{\omega_{s\eta}^{2j}\Gamma(2k+2-\alpha j)}\end{aligned} \tag{210}$$

b)* $\zeta_{s\xi,par}(t)$ and $\zeta_{s\eta,par}(t)$ are terms corresponding to particular solutions of the non-homogeneous fractional order differential equations system (196)-(197) and the solutions must be obtained as an inverse transform of the following expressions:

$$\zeta_{s\xi,par}(\mathrm{t})=\mathsf{L}^{-1}\mathsf{L}\left\{\zeta_{s\xi,par}\right\}=\mathsf{L}^{-1}\left\{\frac{h_{0,1,1(s)}}{\left(p^2+\omega_{s\xi}^2+\omega_{s\sigma\xi}^2p^\sigma\right)}\frac{p}{p^2+\Omega_{1,1}^2}\right\},\ s=1,2,3,....,n \tag{211}$$

$$\zeta_{s\eta,pat}(t)=\mathsf{L}^{-1}\mathsf{L}\left\{\zeta_{s\eta,pat}\right\}=\mathsf{L}^{-1}\left\{\frac{h_{0,1,1(s)}}{\left(p^2+\omega_{s\eta}^2+\omega_{s\sigma\eta}^2p^\sigma\right)}\frac{p}{p^2+\Omega_{1,1}^2}\right\},\ s=1,2,3,....,n \tag{212}$$

or in a developed form

$$\zeta_{s\xi,par}(\mathrm{t}) = \mathsf{L}^{-1}\mathsf{L}\left\{\zeta_{s\xi,par}\right\} = h_{0,1,1(s)}\mathsf{L}^{-1}\left\{\frac{1}{p}\frac{1}{p^2+\Omega_{1,1}^2}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{s\xi}^2}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j p^{\alpha j}\omega_{s\sigma\xi}^{2(j-k)}}{\omega_{s\xi}^{2j}}\right\}, s = 1,2,3,....,n \quad (213)$$

$$\zeta_{s\eta,pat}(t) = \mathsf{L}^{-1}\mathsf{L}\left\{\zeta_{s\eta,pat}\right\} = h_{0,1,1(s)}\mathsf{L}^{-1}\left\{\frac{1}{p}\frac{1}{p^2+\Omega_{1,1}^2}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{s\eta}^2}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j p^{\alpha j}\omega_{s\sigma\eta}^{2(j-k)}}{\omega_{s\eta}^{2j}}\right\}, s = 1,2,3,....,n \quad (214)$$

7.8 Concluding remarks

Finally, we can conclude that new main chain coordinates of ξ_k and η_k, $k = 1,2,3,...,n$ k composed of the generalized independent coordinates in the way $\xi_k = \varphi_{k,1} - \varphi_{k,2}$ and $\eta_k = \varphi_{k,1} + \varphi_{k,2}$, $k = 1,2,3,...,n$ are the main chain coordinates of the main eigen chains of a double DNA helix chain system and that it is possible to obtain two fictive decoupled and separated eigen single chains of the double DNA chain helix linear model as well as fractional order model. This is an important fundamental conclusion as a significant property of the linear model of vibrations in a double DNA helix. Considered as a linear or fractional order mechanical system, DNA molecule as a double helix chain system has its eigen circular frequencies and that is its characteristic. Mathematically, it is possible to decouple it into two chains with their eigen circular frequencies which are different. This may correspond to a different chemical structure (the order of base pairs) of the complementary chains of DNA. We are free to propose that each specific set of base pair order has its eigen circular frequencies and it changes when DNA chains are coupled in the system of double helix. DNA as a double helix in a living cell can be considered as nonlinear system but under certain conditions its behavior can be described by linear dynamics.

Then, analytical expressions of the square of ω_s - eigen circular frequencies of the vibration modes of the separate chains of the homogeneous double DNA chain helix are obtained. By using these results it is easy to consider these values of the system ω_s - eigen circular frequencies of free vibrations as series of resonant frequencies under external multi frequencies excitations, and also possibilities for the appearance of dynamical absorption phenomena and find explanation with real processes in the homogeneous double DNA chain helix.

By using superposition's of these solutions for the case that external excitation are with the same amplitudes and frequencies from the system of differential equations, we can see that for this case of external one frequency excitation in one eigen main chain there appear pure free vibrations with eigen subset of circular frequencies of its free vibrations, and in the other there appear forced vibrations. This conclusion is possible to generalize for the same multi-frequency external excitations applied to each of the material particle pairs in double DNA helix chain system. Eigen main chain in which there occur pure free vibrations with eigen subset of circular frequencies of its free vibrations may correspond with real chain of DNA that is not used as a template in the process of transcription- sense strand. The eigen main chain that oscillates in forced regime may correspond with real antisense strand of DNA –one that is transcribed.

* Dynamical absorption on the first pair of the main coordinates of the main chains appear on the resonate circular frequencies of the set of the double DNA helix chain system with one pair of the material particles less compared to the considered real system.

* Dynamical absorption on the second pair of the main coordinates of the main chains appear on the resonate circular frequencies of the set of the double DNA helix chain system with two pairs of the material particles compared to the considered system.

This mathematical fact is important to consider in the light of the interruption or break of the double DNA helix chain system. In order to transcribe specific DNA sequence, RNA polymerase has to recognize the specific region of DNA where the sequence starts. Promoter regions make the recognition possible. We are free to suggest that, from the mechanical point of view, if specific one frequency excitation caused by RNA polymerase is the same as eigen oscillatory frequency of specific promoter region resonance appears, that is the condition for starting the transcription from the mechanical point of view. This means that every gene has its specific "starting" oscillatory frequency that will correspond with one frequency external excitation. This may also correspond with spatially localized solitons in Soliton –existence supporting models of DNA.

Our next considerations will focus on the small nonlinearity in the double DNA chain helix vibrations and rare nonlinear phenomena such as resonant jumps and energy interactions between nonlinear modes.

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Appendix A

A1. Gamma Function

Euler's gamma function is defined by the so-called *Euler integral of the second kind*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{C} \quad . \tag{A.1.1}$$

Thus

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1, \tag{A.1.2}$$

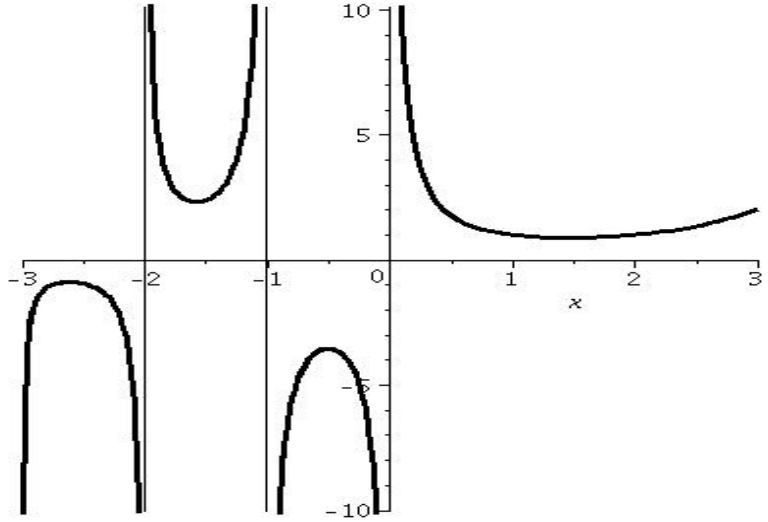

Fig.A.1: The real Gamma function Γ

The integral in the right side of (A.1.1) is convergent for all values of complex argument z with positive real part. However, by means of an analytic continuation it can be extended to the entire complex plane, excluding negative integers and zero.

Gamma function has several well established properties, the first of which is that it can be seen as a generalization of the factorial function. The so called *reduction formula* holds, for $z \in \mathbb{C} \setminus \{0, -1, -2, -3, ...\}$

$$\Gamma(z+1) = z\Gamma(z), \ \Rightarrow \Gamma(n+1) = n(n-1)! = n! \quad n \in \mathbb{N}_0 . \tag{A.1.3}$$

This reduction formula (A.1.2) can easily be proven starting from the integral (A.1.1). The analytic continuation of (A.1.1) is then conducted by application of this formula to arguments with negative real parts. Points at which the Gamma function is not well defined, i.e. negative integers and zero, are its simple poles. Another important relationship for the gamma function is the Legendre formula:

$$\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\, 2^{2z-1} \Gamma(2z), \quad 2z \neq 0, -1, -2, ..., \tag{A.1.4}$$

Taking $z = n + 1/2$ in the previous relation, and utilizing the fact that for integer arguments Gamma function can be evaluated by means of the factorial function, one can obtain a set of particular values of the Gamma function:

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}, \tag{A.1.5}$$

[A1] Podlubny, I. *Fractional Differential Equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press, San Diego, 1999.

A2. Beta Function

Beta function, also known as the *Euler Integral of the First Kind*, is an important special function in general, very widely used in fractional calculus. The importance of Beta function in this context is that its form is similar to the fractional integral and derivative of a number of elementary functions, polynomials in particular, but also Mittag-Leffler function (to be introduced next). Beta integral is defined in the following way

$$B(\alpha,\beta)=\int_0^1(1-u)^{\alpha-1}u^{\beta-1}du=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=B(\beta,\alpha) \tag{A.2.1}$$

Obviously, the Beta-integral is convergent only for and with positive real parts. However, by means of its relation with the Gamma function, it can be continued analytically to the entire complex plane, excluding negative integers and zero.

Appendix B

Table A. A comparison of S-Z approximations of the first order, taken from a broad literature, which represent special cases of a T-integrator [B1]

α	r	$a\equiv\chi_2$	χ_1	$s-z$ approximation	Name of approximation
0	-	-	1	$\frac{1}{s}\approx f_{\mathrm{FD}}(z)=\frac{T}{z-1}$	Euler approximation of first order (FD)
$\alpha\in[0,0.5]$ $\alpha=(1-\chi_1)/2$	-	-	$\chi_1\in[0,1]$	$\frac{1}{s}\approx f_{\mathrm{Le1}}(z,\chi_1)=T\frac{1+\chi_1+(1-\chi_1)z}{2(z-1)}$	parametric FD-BL approximation [45]
$\frac{1}{2}$	1	0	0	$\frac{1}{s}\approx f_{\mathrm{BL}}(z)=\frac{T}{2}\frac{z+1}{z-1}$	Tustin approximation (BL)
0.7736	0.2927	0.5471	0.5471	$\frac{1}{s}\approx=\frac{T}{1.2927}\frac{z+0.2927}{z-1}$	Dostál rule [47]
$\frac{7}{8}$	$\frac{1}{7}$	$\frac{3}{4}$	-	$\frac{1}{s}\approx\frac{7T}{8}\frac{z+1/7}{z-1}$	Al-Alaoui rule [64]
1	0	1	-	$\frac{1}{s}\approx f_{\mathrm{BD}}(z)=\frac{1}{T}\frac{z}{z-1}$	Euler approximation of second order (BD)
$\alpha\in[0,1]$	-	-	-	$\frac{1}{s}\approx f_{\mathrm{Se}}(z,\alpha)=T\frac{1+\alpha(z-1)}{z-1}$	α-approximation of first order (parametric FD-BD approximation) [42-44, 53-55]
$\alpha\in[0.5,1]$ $\alpha=(1+\chi_2)/2$	$r\in[0,1]$ $r=\frac{1-\chi_2}{1+\chi_2}$	$a\equiv\chi_2\in[0,1]$	-	$\frac{1}{s}\approx f_{\mathrm{Le2}}(z,\chi_2)=T\frac{1+\chi_2+(1-\chi_2)z}{2(z-1)}$	parametric BD-BL approximation [46]
$\alpha\in[0.5,1]$ $\alpha=1/(1+r)$ $\alpha=(1+a)/2$	$r\in[0,1]$ $r=\frac{1-a}{1+a}$	$a\in[0,1]$ $a=\frac{1-r}{1+r}$	-	$\frac{1}{s}\approx f_{\mathrm{DP}}(z,r)=\frac{T}{1+r}\frac{z+r}{z-1}$ $\frac{1}{s}\approx f_{\mathrm{AL}}(z,a)=T\frac{1+a+(1-a)z}{2(z-1)}$	Parametric BD-BL approximation [64, 65]

[B1] K. J. Åström, B. Wittenmark, *Computer Controlled Systems: Theory and Design, 3. ed.*, Prentice-Hall, 1997

Appendix C

Appendix C1 *Gronwall-Bellman lemma*

Lemma (*Gronwall-Bellman lemma*), [C.1]

Let $u(t), c(t)$ and $k(t)$ be real continuous functions defined in $[a,b], k(t) \geq 0$ for $t \in [a,b]$. We suppose that on $[a,b]$ we have the inequality

$$u(t) \leq c(t) + \int_a^t k(\tau) u(\tau) d\tau, \tag{C.1.1}$$

then

$$u(t) \leq c(t) + \int_a^t k(\tau) c(\tau) \exp\left[\int_\tau^t k(s) ds\right] d\tau. \tag{C.1.2}$$

in $[a,b]$.

Corollary 1: If $c(t)$ is differentiable, then from (C.1.1) it follows that

$$u(t) \leq c(a)\left(\int_a^t k(s) ds\right) + \int_a^t \exp\left[\int_\tau^t k(s) ds\right] c'(\tau) d\tau. \tag{C.1.3}$$

for all $t \in [a,b]$.

Corollary 2: If $c(t)$ be a positive, monotonic decreasing function, then from (C.1.1) it follows that

$$u(t) \leq c(t) \exp\left(\int_a^t k(\tau) d\tau\right), \tag{C.1.4}$$

for all $t \in [a,b]$.

[C.1] J.K. Hale, *Functional Differential Equations,* Springer, New York,1971

Appendix C.2 Mittag-Leffler function

Mittag-Leffler function appears in two forms and can be considered as a generalization of the exponential function. Mittag-Leffler function generalizes the exponential in two ways. First, its series expansion is a straightforward generalization of the exponential series. Second, solutions to a number of fractional differential equations are expressed in terms of the Mittag-Leffler function just in the same way as the solutions to a number of ordinary differential equations are expressed in terms of the exponential (and its trigonometric relatives). The *single-parameter Mittag-Leffler function* (in fact, the only form actually considered by Mittag-Leffler) is defined as [C2]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \tag{C.2.1}$$

where $\alpha > 0$ and z is an arbitrary complex number. The *two-parameter Mittag-Leffler function* appears most frequently and has the following form

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{C.2.2}$$

where $\alpha>0$, $\beta>0$, and z is an arbitrary complex number.

The importance of the Mittag-Leffler function in a broader concept lays in the fact that for specific values of its parameters, the Mittag-Leffler function reduces to a number of special functions appearing in various fields of engineering. For example, $E_{\alpha,1}(z)=E_{\alpha}(z)$, $E_{1,1}(z)=e^{z}$, $E_{2,1}(z)=\cosh(\sqrt{z})$, $E_{2,1}\left(-z^2\right)=\cos(z)$.

The Laplace transform of the Mittag-Leffler function can be easily found to be

$$\int_0^{\infty}e^{-st}t^{\alpha k-\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm at^{\alpha}\right)dt=\frac{k!s^{\alpha-\beta}}{\left(s^{\alpha}\mp a\right)^{k+1}},\left(\Re(s)>|a|^{1/n}\right) \tag{C.2.3}$$

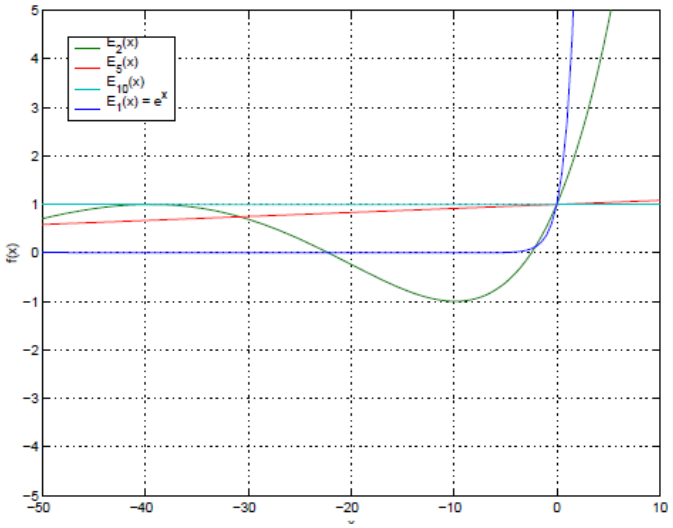

Fig.C.1: The Mittag-Leffler function for different $\alpha,\beta=1$

[C2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo,*Theory and applications of Fractional Differential equations*, edited by J.V. Mill.Elsevier, Amsterdam,2006.

Appendix D

Human Skin

Human skin is the body`s largest organ. It is a major part of integumentary system which comprises between 15-20 % of the total body weight, and covers 1,5-2 m^2 of surface area. Each square centimeter of human skin has $6*10^6$ cells, $5*10^3$ sensory points, 15 sebaceous glands and 100 sweat glands.

Functions of Human Skin

As an organ that interfaces with the environment, skin plays a key role in *protecting* the body from damage and against pathogens (see [D1,D2]). In the human skin Langerhans cells play a protective role since they are a part of the adaptive immune system. Important functions of human skin are insulation and temperature regulation. Skin *heat regulation* is performed by controlling blood supply. Since skin blood supply is far greater than its requirements heat regulation is controlled by vasodilatation and vasoconstriction (constricted blood vessels reduce cutaneous blood flow and conserve heat, while dilated blood vessels increase heat loss).
Another important function of skin is *control of evaporation*. The skin provides semi-impermeable barrier controlling the body fluid loss. Sweat *excretion* containing small amount of urea is accomplished by skin. However, excretion by sweating is at most a secondary function to *temperature regulation*. *Absorption* is also one of the functions of skin. Oxygen (see [D.3]), nutrients or synthetic chemicals, applied topically are adsorbed and/or transported through skin (see [D.4,D.5]). The skin acts also as water resistant barrier so essential nutrients could not be washed out of body. The other important function of human skin is *sensation*. Skin tissue contains variety of nerve endings that react to variety of sensations such as touch, pressure, vibration, heat and cold and tissue injury. Finally, function of human skin is *storage* and synthesis: skin acts as a storage center for lipids and water; synthesis of vitamin D by action of UV.

Structure of Human Skin

Human skin (Figure 1.) is composed of three major layers:

- epidermis
- dermis
- hypodermis

EPIDERMIS

The epidermis (Fig.D.1.) is the top layer of human skin, and therefore it is the first barrier between body and outside world. The thickness of the epidermis is approximately 0,5-1mm. The epidermis contains four types of cells: Merkel cells, keratinocites, melanocites and Langerhans cells. Epidermis consists of five sub layers which are usually named as five strata (Fig.D.1.):

- stratum corneum,
-stratum lucidum,
-stratum granulosum,
-stratum spinosum,
-stratum germinativum (stratum basale)

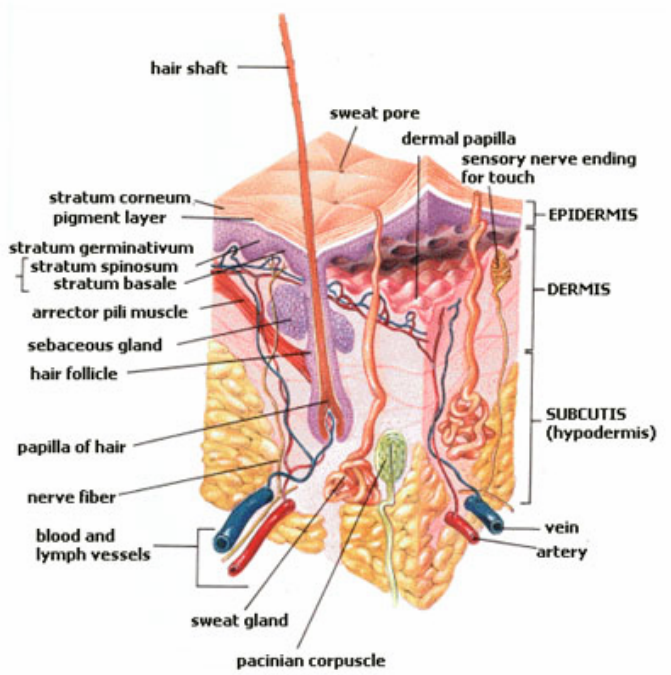

Fig D.1. Structure of human skin (see [D.6])

Keratinocites are formed through mitotic cell division at the lowermost portion of epidermis-stratum basale. The daughter cells move upwards. As they mature, these cells lose water, flatten out and at the end of their life cycle they reach the uppermost layer of the epidermis – stratum corneum. Therefore, stratum corneum contains mainly dead keratinocites, keratin proteins and lipids forming a protective barrier. Dead cells from stratum corneum permanently slough of and are replaced by new cells coming from below. Human skin completely renews itself for approximately 3-5 weeks.
Melanocites are another significant group of cells. These cells produce melanin the pigment responsible for skin color. Langerhans cells are the front door of the immune system and they prevent foreign substances from penetrating the skin. As shown in Fig. D.1. epidermis consists of many layers. Stratum corneum is outer layer made of flattened epithelial dead cells organized in multiple sub layers. Next layer below is translucent and transitional thin layer of cells which might be visible in thick skin. However, nuclei and/or other organelles are not visible. Cytoplasm of these cells is mostly made of keratin filaments. Next three to five layers (supra basal layers) consist of flattened polygonal cells that have granules in cytoplasm. Below them is a layer of cells that contains bundles of keratin filaments and these cells are cube-shaped. The last layer positioned above the basement membrane and the dermis is basal or cell division layer. It is a single layer of cells that undergo mitosis to renew the epidermal upper layers.

DERMIS
The dermis is the middle layer of the skin positioned between the epidermis and hypodermis. It is the thickest skin layer. The major components of the dermis are collagen and elastin fibers. These macromolecular proteins are important for resilience of the skin tissue. The dermis fibroblasts synthesize collagen and elastin as well as other structural molecules. Lymph nodes and capillaries are also important constituents of the dermis. Capillaries are essential for nourishing, oxygenating and temperature regulation of the human skin. They are also very important for protecting the skin from invading microorganisms. Sweat glands, hair follicles, sebaceous glands as well as small number of muscle and nerve cells are also present in the dermis. Sebaceous glands are of particular importance for skin health since they produce sebum oily protective substance that lubricates and water proofs the skin.

HYPODERMIS
The hypodermis is located below the dermis. The purpose of this layer is to attach the skin to bone and muscle and to supply it with nerves and blood vessels. It contains a loose connective tissue and elastin. Hypodermis contains fibroblasts, adipocytes and macrophagues as a main cell types. In the subcutaneous tissue the

predominant type of cells are fat cells. The fat acts as heat insulator protecting underlying tissue from cold and mechanical trauma.

D.1 Proksch, E; Brandner, JM; Jensen, JM (2008). "The skin: an indispensable barrier.". *Experimental Dermatology* **17** (12): 1063–72.

D.2 Madison, KC. (2003). "Barrier function of the skin: "la raison d'être" of the epidermis",*J Invest Dermatol* **121** (2): 231–41.

D.3 Stücker, M., A. Struk, P. Altmeyer, M. Herde, H. Baumgärtl & D.W. Lübbers (2002). The cutaneous uptake of atmospheric oxygen contributes significantly to the oxygen supply of human dermis and epidermis., *Journal of Physiology* **538**(3): 985–994.

D.4 Felipe, P., Silva, J.N., Silva, R., Cirne de Castro, J.L., Gomes, M., Alves, L.C., et al. Stratum Corneum Is an Effective Carrier to TiO2 and ZnO Nanoparticle Percutaneous Absorption. Skin Pharmacology and Physiology 2009;22:266-275

D.5 Vogt, A., Combadiere, B., Hadam, S., Stieler, K., Lademann, J., Schaefer, H., et al. 40nm, but not 750 or 1,500 nm, Nanoparticles Enter Epidermal CD1a+ Cells after Transcutaneous Application on Human Skin. Journal of Investigative Dermatology

D.6 Wikipedia: http://en.wikipedia.org/wiki/Skin

APPENDIX E

APPENDIX E.1 NOMENCLATURE

DNA – Deoxyribonucleic acid (DNA)

$\varphi_{k,1}[\,rad\,]$ - generalized coordinate – angles of the k-th base of the first chain of the double DNA chain helix;

$\varphi_{k,2}[\,rad\,]$ - generalized coordinate – angles of the k-th base of the second chain of the double DNA chain helix;

$\mathbf{J_{k,1}}$ $[\,kgm^2\,]$- is the axial moment of mass inertia of the k-th base of the first chain of the double DNA chain helix;

$\mathbf{J_{k,2}}[\,kgm^2\,]$- is the axial moment of mass inertia of the k-th base of the second chain of the double DNA chain helix;

$\dot{\varphi}_{k,1}$[rads^{-1}] - angular velocity of the k-th base of the first chain of the double DNA chain helix;

$\mathbf{J_{k,1}} = m_\alpha r_\alpha^2$, $\mathbf{J_{k,2}} = m_\beta r_\beta^2$ $[\,kgm^2\,]$ - the base pair of the axial moments of mass inertia;

m_α $[\,kg\,]$- the value of the base mass

$r_\alpha[\,m\,]$ - the length

$\mathbf{J_{k,1}} = m_\alpha r_\alpha^2$ $[\,kgm^2\,]$ - the corresponding axial moment of mass inertia for all possible base pair authors accepted as in the Reference [17].

$K_{k,i}$, $i = 1,2$ [KJmol^{-1}]- parameters characterize the energy of interaction of the k-th base with the $(k+1)$-th one along the i-th chain $i = 1,2$.

$K_{k,i} = K = 6\times10^3[\text{KJmol-1}]$- for the calculation that the most appropriate value is close /

ξ_k, η_k $[\,rad\,]$, $k = 1,2,3,.....,n$ - main orthogonal coordinates of the eigen main chains of the double DNA chain helix;

$\xi_k = \varphi_{k,1} - \varphi_{k,2}$ and $\eta_k = \varphi_{k,1} + \varphi_{k,2}$, $k = 1,2,3,.....,n$ - functional dependence between main orthogonal coordinates ξ_k and η_k of the eigen main chains and generalized coordinates $\varphi_{k,1}$ and $\varphi_{k,2}$ $[\,rad\,]$ of the double DNA chain helix;

$\omega_{\alpha\beta 2}[\,\sec^{-1}\,]$ - frequencies of rotational motions of the bases, in similar and opposite directions accordingly, of the k-th base of the first chain of the double DNA chain helix;

$\omega_{\alpha\beta 1}$ $[\,\sec^{-1}\,]$ - are frequencies of rotational motions of the bases, in similar and opposite directions accordingly, of the k-th base of the first chain of the double DNA chain helix;

$K_{k,1} = K_{k,2} = K$ - for the case of homogeneous double DNA chain helix;

$\mathbf{J}_{k,\mathbf{1}} = \mathbf{J_{k,2}} = \mathbf{J}$ $[\,kgm^2\,]$ - for the case of homogeneous double DNA chain helix;

A_k - amplitude

u=JK$^{-1}\omega^2$- eigen characteristic number of the homogeneous double DNA chain helix;

k= $K_{\alpha\beta}$ 2K^{-1} {1- $\omega_{\alpha\beta 2}$ $\omega_{\alpha\beta 1}$ $^{-1}$) $\left(r_\alpha - r_\beta\right)^2$ - parameter of the homogeneous double DNA chain helix;

μ= $K_{\alpha\beta} r_\alpha \left(r_\alpha - r_\beta\right)$K^{-1} - parameter of the homogeneous double DNA chain helix;

$\omega_{s\xi}^2[\,\sec^{-2}\,]$, $s = 1,2,3,4...,n$ - set of the n eigen circular frequencies of the first eigen main chain of the homogeneous double DNA chain helix;

$\omega_{s\eta}^2[\,\sec^{-2}\,], s = 1,2,3,4...,n$ - set of the n eigen circular frequencies of the first eigen main chain of the homogeneous double DNA chain helix;

APPENDIX E.2

dDNA is biological molecule involved in two very important processes in the living cell: replication (doubling the DNA molecule and consequently the genetic material) and transcription (transcribing the information from DNA to RNA). Replication is a part of the cell division process when one cell is dividing into two new cells but with same genetic information. Translation occurs in all metabolically active cells and it is necessary for one cell to live. As dDNA is placed in nucleus (in 46 chromosomes in somatic cells) and cell organelles called mitochondria in the form of very long polymers it is packed and condensed and in that form it is inactive. Only parts of dDNA that are currently in function are decondensed, unzipped and transcribed. The condensed form of dDNA is possible gratefully to the special type of proteins called histons and special elastic properties of dDNA that are also relevant for DNA's physiological function." The elastic properties of a molecule of duplex DNA are strongly dependent on nucleotide sequence." (see Ref [9] by Coleman et al. 2003). They proposed the theoretical model where the sequence dependence of elastic properties of dDNA is determined by the kinematical variables, called tilt, roll, twist, shift, slide, and rise, (see Fig E.2.1) that describe the relative orientation and displacement of the *n*th and (*n*11)th base pairs. Very important are the symmetry imposed on previous mentioned kinematic variables by the complementarities of bases, i.e., of A to T and C to G, the antiparallel nature of the DNA sugar–phosphate chains, and the requirement that kinematic variables are independent of the choice of the direction of increasing *n,*(see Ref [9] by Coleman at al. 2003).

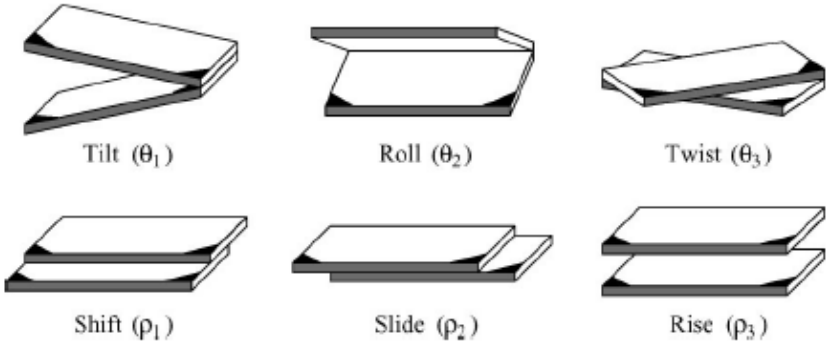

Fig E2.1. Relative orientation and displacement of successive base pairs. From Coleman B.D., Olson W.K., and Swigon, 2003. (Ref [9])

The special enzymes, named topoisomerases, unzipped and zipped dDNA in process of replication and transcription." Topoisomerases relieve the torsional strain in DNA that is built up during replication and transcription." (See [40] by Koster et al, 2005). Using real-time single-molecule observation, Koster et al. (See [40] by Koster et al, 2005) show that Topoisomeras B releases supercoils by a swivel mechanism that involves friction between the rotating DNA and the enzyme cavity: the DNA does not freely rotate. TopIB does not release all the supercoils at once, but in multiple steps. "The number of supercoils removed per step follows an exponential distribution. The enzyme is found to be torque-sensitive, as the mean number of supercoils per step increases with the torque stored in the DNA." (See [40] by Koster et al, 2005).

DNA mechanical models

A number of models have been constructed to describe different kinds of movements in a DNA molecule: asymmetric and symmetric motion; movements of long and short segments; twisting and stretching of dsDNA, twist-opening conditions. Some models have, for example, been made for circular double-stranded DNA molecules in viral capsids. We are discussing here *polymer models, elastic rod models, network models, torsional springs models, soliton -existence supporting models***,**(see [23] by Hedrih, 2011).

Polymer models

In polymer models DNA molecule is considered to be a polymer and calculations are done as for other polymers.There are several DNA polymer models:

- Gaussian polymer (GP) model

- Freely Jointed Chains (FJC) model,
- Worm –Like Chain (WLC) model
- Worm –Like Rod Chain (WLRC) model (see Ref [8] by Cocco et al, 2002).see Fig. E.2.2.
- bead–spring model

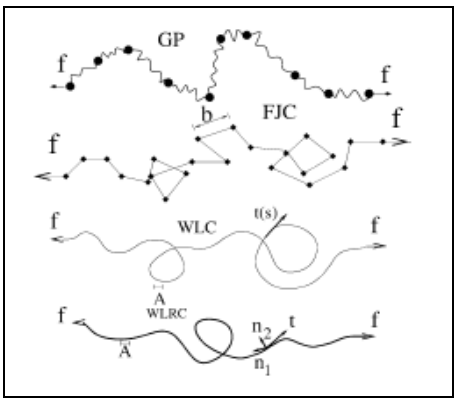

Fig E.2.2 Polimer models. From: Cocco,et al. 2002. (Ref [8])

Double strained DNA differs from simple polymers because it exhibits torsional and bending stiffness and under tension DNA supercoiles. These models are used to interpret single-molecule force-extension experiments on single strand DNA and dDNA. They show how combining the elasticity of two single nucleic acid strands with a description of the base-pairing interactions between them explains much of the phenomenology and kinetics of RNA and DNA 'unzipping' experiments" (see [8] by Cocco et al, 2002 and [56] by Zhou and Lai, 2001).

"A limitation of the initial WLC model was the assumption of intrinsically straight homogeneous polymers whose thermal fluctuations are quantified as deviations from the straight line. DNA almost always contains curved regions, which can strongly affect the persistence length", (see [1] by Anselmi et al, 2005).

"For better fitting the experimental data in to the WLC polymer model Seol et al, (see [50] by Seol et al, 2007) develop finite wormlike chain (FWLC) model. FWLC models are suitable for both short (a few hundred nanometers in contour length) and very long (microns in contour length) molecule." (See [23] by Hedrih, 2011).

In the *bead–spring model* the DNA chain is modeled as a bead–rod system with the first-order effective bead–spring integration scheme. The proposed effective bead–spring model may help simulate the dynamic behaviors of many types of polymer chains with different chain elasticity via an efficient unified integration scheme with large time steps. Combining with angular springs, this model can also be used to simulate the bending behaviors of semiflexible polymers, (see ref [44] by Liu et al, 2008).

Elastic rod models

DNA can be modeled as an ideally elastic rod, or as an anisotropic rod. There are several types of elastic rod models:

- simple elastic rod models
- sequence-dependent anisotropic elastic model
- asymmetric elastic rod model

"In *simple elastic rod models* rod is ideally elastic which means that it will return to its original shape after deformation; one can compute the energy necessary for bending, stretching or torsional deformation. The model is not sequence dependent (i.e. all segments are equal) and the model is equally bendable (deformable) in all directions. The phenomena that can be described using such simplified elastic model include gross shape changes in DNA, such as supercoiling, the response of plasmids to stress, etc. A technique of finite element analysis has been applied successfully for small DNA deformations. Model is suitable for DNA in plasmids (see [46] by Munteanu et al, 1998). If DNA is modeled as a homogeneous and isotropic elastic rod, the DNA chain is characterized by three parameters: The bending rigidity, the torsional rigidity, and the DNA effective diameter,(see [45] by Maxim and Kamenetskii, 1997).

In *sequence-dependent anisotropic elastic model* DNA is considered to be an initially straight, segmented, elastic rod, in which the flexibility of each segment is greater towards the major groove than it is in other directions. This model can predict local bending phenomena and explains phenomena as the affinity of protein binding and kinking (see [46] by Munteanu et al, 1998). Goyal and Perkins (see [17] by Goyal and Perkins, 2008) extend a computational rod model that captures the non-linear dynamics of hyperelastic, isotropic rods to accommodate large and discontinuous variations in bending and torsional stiffness.

Modeling DNA, as an isotropic rod can't explain some mechanical properties of DNA molecule achieved experimentally (twist-strech coupling in single DNA measured by rotor bead tracking technique) like overwinding of DNA molecule under tensions. DNA molecule reaches the maximum twist level at a tension of 30 pN, as tension is increased above this critical value, the DNA begins to unwind. Elastic rod model can explain these unusual mechanical properties.

DNA is modeled as an elastic rod wrapped helically by a stiff wire. The inner core of radius R is assumed to have a Poisson's ratio v= 0.5. "The outer wire is affixed to the inner rod helically with a pitch of 3.4 nm, and contributes to the overall mechanical properties because it resists stretching and compression. The outer helix increases the torsional rigidity and yields a twist–stretch coupling that depends upon the helix angle. Stretching generates an overwinding of the helix because the inner rod decreases in diameter as it is stretched. The outer helix is then able to wrap a larger number of times over the length of the molecule."…"These results have implications for the action of DNA-binding proteins that must stretch and twist DNA to compensate for variability in the lengths of their binding sites." (See Ref [18] by Gore et al, 2006). Linear isotropic rod model has some limitations in predicting DNA supercoiling of long molecules under applied tension and twist. There is coupling between bending and stretching (See Ref [51] by Smith and Healey 2008). Bend-twist coupling is important in predicting the stability boundary. Eslami-Mossallam and Ejtehadi, (See Ref [13] by Eslami-Mossallam and Ejtehadi, 2009) proposed the asymmetric elastic rod model for DNA.

Model of Leburn and Lavery (See Ref [43] by Lebrun and Lavery, 1996) show how the double helix can be extended to twice its normal length before its base pairs break. Results correlate well with nanomanipulation experiments." (See Ref [23] by Hedrih, 2011).

The network model

Double-stranded DNA is treated as a network of coupled oscillators incorporating essential microscopic degrees of freedom of DNA and the inherent interactions between them. The constituents of the oscillator network model represent the nucleotides, which are regarded as single non-deformable entities.

No inner dynamical degrees of freedom of the nucleotides are taken into account that is justified by the time scale separation between the small-amplitude and fast vibrational motions of the individual atoms and the slower and relatively large-amplitude motions of the atom groups constituting the nucleotides. The nucleotides are considered as identical objects of fixed mass all of four types of bases are treated as equal. The network model can explain the mechanical stability and elasticity properties of dsDNA molecules. This model is suitable for studying the opening-closing dynamics of dsDNA molecules that are forced into non-equilibrium conformations, which are relevant for bimolecular processes. Hennig and Archilla (see Ref [39] by Hening and Archilla, 2004) show that the attainment of a quasi-equilibrium regime proceeds faster in the case of the twisted DNA form, than for its thus less flexible ladder counterpart (see Ref [23] by Hedrih, 2011).

Torsional spring model

In torsional springs model Tung et al (see Ref [53] by Tung and Harvey, 1984) made a distinction between purine bases (adenine and guanine) and also between pyrimidines, (cytosine and thymine), using the atomic resolution of conformational energy calculations. See Fig E.2.3.

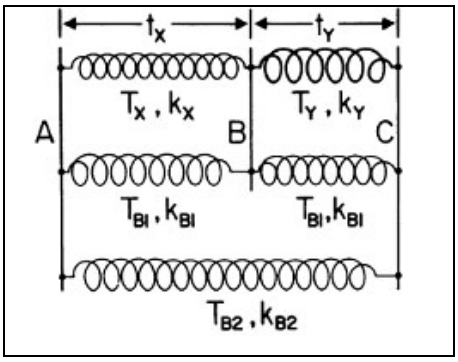

Fig E.2.3 Torsional spring model- Torsional spring model for helix twist angles of a trimer. Distances in this figure correspond to angels not to lengths. From : Tung, CS, and S.C. Harvey, 1984. (ref [53]).

The model predicts a macroscopic torsional stiffness and a longitudinal compressibility (Young's modulus) that are both in good agreement with experiment. They use conformational energy calculations to determine the parameters of the model, and can quantitatively predict helix twist angles (see [53] by Tung and Harvey, 1984), (see [23] by Hedrih, 2011**).**

Soliton -existence supporting models

There several types and its modifications of soliton- existence supporting models:

- *Yakushevich model (Y model)*
- *The composite model (Y based model)*
- *Dynamic plane–base rotator model (See [54] byVasumathi and Daniel, 2008).*
- *Symmetric twist-opening model of DNA. (See f [52] Tabi et al, 2009),*
- *Nonlinear Volkov-Kosevich DNA model*
- *Peyrard–Bishop–Dauxois (PBD)*
- *Cross-grained model by Kovaleva and L. Manevich* (*Y based model) (see [41,42] by Kovaleva et al, 2007; Kovaleva and Manevich, 2005)*

One of the first of this kind was *Yakushevich model (Y model) of DNA* and models based on it (see [15] by Gaeta, 1992, as well as [16] by González and Martín-Landrove,1994)). Dynamics of topological solitons describing open states in the DNA double helix are studied in the framework of a model that takes into account asymmetry of the helix. Yakushevich, et al (see [55] by Yakushevich, et al, 2002) investigated interaction between the solitons, their interactions with the chain inhomogeneities and stability of the solitons with respect to thermal oscillations and have shown that three types of topological solitons can occur in the DNA double chain. The composite model for DNA is also based on Y model. The sugar-phosphate group and the base are described by separate degrees of freedom. The composite model fits experimental data better than the simple Y model. DNA nucleotides are represented as two distinct discs, one still centered about the backbone axis and representing the sugar–phosphate group and the second rigidly rotating about a fixed point of the former. The composite model supports solitonic solutions, qualitatively and quantitatively very similar to the Yakushevich solitons. (See [11] by De Leo and Demelio, and [7] by 2008; Cadoni et al, 2008).

Dynamic plane–base rotator model is suitable for study the nonlinear dynamics of the inhomogeneous dDNA especially angular rotation of bases in a plane normal to the helical axis. (See [10] by Daniel and Vasumathi, 2007). This model is also used to study the effect of phonon interaction on base pair opening in the dsDNA. The velocity of the soliton increases or decreases or remains uniform or even the soliton stops depending on the values of the coupling strengths. There is no change in the topological character of the soliton in the asymptotic region. (See [54] by Vasumathi and Daniel, 2008).

DNA model by N. Kovaleva and L. Manevich

In this model, three beads represent each nucleotide with interaction sites corresponding to phosphate group, group of sugar ring, and the base, see Fig 2a and 2b in the main text. (See [41] by Kovaleva et al, 2007). It is soliton-supporting model.

It is planer DNA model in which the chains of the macromolecule from two parallel strait lines place at a distance *h* from each other, the bases can make only rotation motions around their own chain, being all the time perpendicular to it. N. Kovaleva et al. (See Ref [41,42] by Kovaleva et al, 2007; Kovaleva and Manevich, 2005) point out that solitons and breathers play a functional role in DNA chains. They show that a localized excitation (breather) can exist in a double DNA helix. Authors formulated conditions of the breathers' existence and estimate their characteristic parameters describing opening of DNA double helix. They investigated also the stability of breathers.

Multi-chain/multi-pendulum system model of double DNA

Hedrih and Hedrih, (See Ref [35-38] by Hedrih (Stevanović) and Hedrih, 2010; 2009a, 2009b, 2009c) gave several mechanical models of double DNA. In their models DNA is in a form of homogenous multi-chain/ multi-pendulum system which oscillatory signals can be considered trough a system with fixed and with free ends. See Fig. 3 and 4 in the main text. These figures are for the ideal elastic model and different boundary conditions. The dynamics of oscillatory signals in multi-chain systems are represented in ref [27] (Hedrih (Stevanovic) 2006). Their basic model is linearization of the model propose by Kovaleva and Manevich (See Ref [42] by Kovaleva and Manevich, 2005). System is considered as homogeneous, which means that all the masses are equal; rigidities of the linear-elastic spring elements are equal. The models differ in the way of coupling between the material (mass) particles. If the martial (mass) particles are coupled with standard light hereditary elements the model has viscoelastic properties; if the martial particles are coupled with standard light fractional order elements it has viscoelastic and creep properties. They are several types of these models:

- *Model with ideally elastic properties (linear or nonlinear),*
- *Model with hereditary properties and*
- *Fractional order model*.

For each of multi-pendulum/multi-chain models, it is possible to calculate main coordinates of eigen main chains when system is mathematically decoupled, set of eigen circular frequencies and set of characteristic numbers describing hereditary or fractional order system properties. Calculations and analytical analysis show that there is no energy transfer between eigen main chains of double chain DNA helix system.

Using eigen main normal coordinates of the system dynamics it is possible, for each of three models, to determine independent oscillators: for linear elastic model with harmonics with constant amplitudes and corresponding eigen circular frequencies; for model with hereditary properties *n* partial hereditary oscillators with hereditary properties can be found. Each is defined by one eigen circular frequency and one characteristic number. Characteristic number depends on stiffness and relaxation time and determines hereditary properties of the corresponding single eigen oscillator. For fractional order model, *n* independent fractional order oscillators can be found, each with one eigen circular frequency and corresponding characteristic number expressing fractional order properties of the DNA helix chain system. Characteristic number determines the properties of the fractional order system. There are full mathematical analogy and phenomenological mapping between two models: a double DNA fractional order chain helix model and a double DNA hereditary chain helix.

- **Model with ideally linear elastic properties**

Authors of the multi-pendulum double DNA model suggest existence of new phenomena named eigen main chains of the homogeneous double DNA chain helix. These main eigen chains are partial n-frequency oscillators, and each, with n degrees of freedom. Each of these eigen main chains of the homogeneous double DNA chain helix (with 2n degree of freedom) is an independent partial n-frequency oscillator which oscillates with a subset of n eigen circular frequencies: first with frequencies from the set $\omega^2_{s\xi}$, $s=1,2,3,4...,n$- set described by characteristic equation of the subsystem of differential equations

$$\frac{1}{\omega_0^2}\ddot{\xi}_{\mathbf{k}} - \xi_{\mathbf{k+1}} + 2\xi_{\mathbf{k}}\left[1+\mu-\kappa\right] - \xi_{\mathbf{k-1}} = 0\,,\ k=1,2,3,.....,n \tag{E.2.1}$$

of the n eigen circular frequencies of the first eigen main chain of the homogeneous double DNA chain helix; and second with frequencies from the set $\omega_{s\eta}^2, s=1,2,3,4...,n$ - set described by characteristic equation of the subsystem of differential equations

$$\frac{1}{\omega_0^2}\ddot{\eta}_{\mathbf{k}} - \eta_{\mathbf{k+1}} + 2\eta_{\mathbf{k}}(1+\mu) - \eta_{\mathbf{k-1}} = 0\,, \qquad k=1,2,3,.....,n \tag{E.2.2}$$

of the n eigen circular frequencies of the second eigen main chain of the homogeneous double DNA chain helix. In the case of forced oscillatory regimes, using the obtained sets of eigen circular frequencies obtained from subsystems (E.2.1)-(E.2.2), it is possible that resonant regimes on the corresponding circular frequencies are present only on one eigen main chain, while in the other eigen main chain forced oscillatory regimes are normal, without resonance. Also, in forced frequency regimes it is possible to identify phenomena of dynamical absorptions, without losing mechanical energy of the double DNA chain helix (for detail see [35] Hedrih (Stevanović) and Hedrih, 2010). Also, we obtained two subsets of the eigen circular frequencies (ω^2_s) of the vibration signal modes of separate eigen main chains *in* the double DNA chain helix using the trigonometric method (See [48,49,27,28] by Rašković and Hedrih (Stevanović)), as well as amplitudes. Two subsets of eigen circular frequencies are obtained in the following forms (for detail see [36,37,35], Hedrih (Stevanović) and Hedrih):

$$\omega_s^2 = 2\omega_0^2\left[2\sin^2\frac{\varphi_s}{2} + (\mu-\kappa)\right],\ \ \omega_r^2 = 2\omega_0^2\left[2\sin^2\frac{\vartheta_r}{2} + \mu\right],\quad s,r=1,2,3,4...,n \tag{E.2.3}$$

where φ_s and ϑ_r are characteristic numbers depending of boundary conditions of the model of the double DNA linear order chain helix. By use corresponding boundary conditions of the obtained eigen main chains of the considered double DNA linear order chain helix it is easy to obtain corresponding values of these characteristic numbers in a case when both ends of the of the double DNA chain helix are free as well as fixed.

- **Model with Hereditary Properties**

This model of double DNA is considered as a homogeneous system containing two coupled multi pendulum subsystems. Corresponding material particles of the corresponding multi-pendulum chains are each two inter coupled by one standard light hereditary element (see [35,38,20,22] by Hedrih (Stevanović) and Hedrih, 2009a, 2009c, Goroško and Hedrih (Stevanović), 2001, 2008). Standard light hereditary element has relaxation time *n.* Biological materials are changing their mechanical properties during aging. Biomaterials during aging may express relaxation properties and delay elasticity. We propose model with hereditary properties because it may be suitable for explaining this behavior.

Two subsets of the kinetic parameters corresponding to first eigen main chain $\delta_{0(\xi,s)}, \delta_{(\xi,s)}, \omega_{(\xi,s)}$ and to second eigen main chain $\delta_{0(\eta,s)}, \delta_{(\eta,s)}, \omega_{(\eta,s)}$ of the eigen main hereditary oscillators of double DNA hereditary chain helix vibrations like one frequency oscillation modes in the second approximation are obtained in the following forms:

a* first subset:

$$\delta_{0(\xi,s)} = \frac{k}{n}\left[1+\frac{(1-k)k}{n^2\omega_{\xi,s}^2}\right],\quad \delta_{(\xi,s)} = \frac{1-k}{2n}\left[1-k^2\frac{1}{n^2\omega_{\xi,s}^2}\right] \text{ and } \tilde{\omega}_{(\xi,s)}^2 = \omega_{\xi,s}^2\left[1-(1-k)\frac{1+3k}{4}\frac{1}{4n^2\omega_{\xi,s}^2}\right] \tag{E.2.4}$$

$s=1,2,3,4...,n$

b* Second subset:

$$\delta_{0(\eta,r)} = \frac{k}{n}\left[1+\frac{(1-k)k}{n^2\omega_{\eta,r}^2}\right],\quad \delta_{(\eta,r)} = \frac{1-k}{2n}\left[1-k^2\frac{1}{n^2\omega_{\eta,r}^2}\right] \text{ and } \tilde{\omega}_{(\eta,r)}^2 = \omega_{\eta,r}^2\left[1-(1-k)\frac{1+3k}{4}\frac{1}{4n^2\omega_{\eta,r}^2}\right] \tag{E.2.5}$$

$r=1,2,3,4...,n$

APPENDIX E.3

Expansion of the Laplace transform into series (for details see [19] by Gorenflo and Mainardi (2000); [3] by Bačlić and Atanacković (2000) and [34] by Hedrih (Stevanović) and Filipovski (2002)):

$$\mathsf{L}\{\xi_2\}=\frac{p\varphi_{0i}+\dot{\varphi}_{0i}}{2\left(p^2+\omega_{0\alpha}^2p^\alpha+\widetilde{\omega}_0^2+\omega_{00}^2\right)} \tag{E.3.1}$$

$$\mathsf{L}\{\xi_2(t)\}=\frac{p\varphi_{01}+\dot{\varphi}_{01}}{2p^2\left[1+\frac{\omega_{0\alpha}^2}{p^2}\left(p^\alpha+\frac{\widetilde{\omega}_0^2+\omega_{00}^2}{\omega_{0\alpha}^2}\right)\right]}=\left(\varphi_{01}+\frac{\dot{\varphi}_{01}}{p}\right)\frac{1}{2p}\frac{1}{\left[1+\frac{\omega_{0\alpha}^2}{p^2}\left(p^\alpha+\frac{\widetilde{\omega}_0^2+\omega_{00}^2}{\omega_{0\alpha}^2}\right)\right]} \tag{E.3.2}$$

$$\mathsf{L}\{\xi_2(t)\}=\left(\varphi_{01}+\frac{\dot{\varphi}_{01}}{p}\right)\frac{1}{2p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{0\alpha}^{2k}}{p^{2k}}\left(p^\alpha+\frac{\widetilde{\omega}_0^2+\omega_{00}^2}{\omega_{0\alpha}^{\ 2}}\right)^k \tag{E.3.3}$$

$$\mathsf{L}\{\xi_2(t)\}=\left(\varphi_{01}+\frac{\dot{\varphi}_{01}}{p}\right)\frac{1}{2p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{0\alpha}^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{p^{\alpha j}\omega_{0\alpha}^{2(j-k)}}{\left(\widetilde{\omega}_0^2+\omega_{00}^2\right)^j} \tag{E.3.4}$$

APPENDIX E.4

Solution of a fractional order differential equation of a fractional order creep oscillator with single degree of freedom.The fractional order differential equations obtained and the considered cases of eigen fractional order partial oscillators of the hybrid fractional order multichain system are in mathematical analogy to the same fractional order differential equation with corresponding unknown time-functions. We can use the notation $T(t)$ and all previously derived fractional order differential equations of eigen fractional order partial oscillators with one degree of freedom correspond to the hybrid fractional order multichain system dynamics with sixth degree of freedom, and can be rewritten in the following form:

$$\ddot{T}(t)\pm\omega_\alpha^2T^{(\alpha)}(t)+\omega_0^2T(t)=0 \tag{E.4.1}$$

This fractional order differential equation (E.4.1) on unknown time-function $T(t)$, can be solved applying the Laplace transforms (see [19] by Gorenflo, R., Mainardi, F., (2000); [3] by Bačlić, B. S., Atanacković, T., (2000) or [34] by Hedrih (Stevanović) K. and Filipovski A.,(2002)). Upon that fact, the Laplace transform of solution is in the form:

$$\mathsf{T}(p)=\mathsf{L}[T(t)]=\frac{pT(0)+\dot{T}(0)}{p^2+\omega_0^2\left[1\pm\frac{\omega_\alpha^2}{\omega_0^2}\mathbf{R}(p)\right]} \tag{E.4.2}$$

where $\mathsf{L}\left[D_t^{\ \alpha}[T(t)]\right]=\mathbf{R}(p)\mathsf{L}[\mathrm{T(t)}]$ is the Laplace transform of a fractional derivative $\frac{d^\alpha T(t)}{dt^\alpha}$ for $0\le\alpha\le1$. For creep rheological material those Laplace transforms have the form:

$$\mathsf{L}\left[D_t^{\ \alpha}[T(t)]\right]=\mathbf{R}(p)\mathsf{L}[T(t)]-\frac{d^{\alpha-1}}{dt^{\alpha-1}}T(0)=p^\alpha\mathsf{L}[T(t)]-\frac{d^{\alpha-1}}{dt^{\alpha-1}}T(0) \tag{E.4.3}$$

where the initial values are:

$$\left.\frac{d^{\alpha-1}T(t)}{dt^{\alpha-1}}\right|_{t=0}=0 \tag{E.4.4}$$

so, in that case the Laplace transform of time-function is given by the following expression:

$$\mathsf{L}\{T(t)\}=\frac{pT_0+\dot{T}_0}{\left[p^2\pm\omega_\alpha^2p^\alpha+\omega_0^2\right]} \tag{E.4.5}$$

For boundary cases, when material parameters α take the following values: $\alpha=0$ i $\alpha=1$ we have the two special simple cases, whose corresponding fractional-differential equations and solutions are known. In these cases the fractional-differential equations are:

$$1^*\quad \ddot{T}(t)\pm\widetilde{\omega}_{0\alpha}^2T^{(0)}(t)+\omega_0^2T(t)=0 \text{ for } \alpha=0 \tag{E.4.6}$$

where $T^{(0)}(t)=T(t)$, and

$$2^*\quad \ddot{T}(t)\pm\omega_{1\alpha}^2T^{(1)}(t)+\omega_0^2T(t)=0 \text{ for } \alpha=1 \tag{E.4.7}$$

where $T^{(1)}(t)=\dot{T}(t)$.

The solutions to equations (E.4.6) and (E.4.7) are:

$$1^*\ T(t)=T_0\cos t\sqrt{\omega_0^2\pm\widetilde{\omega}_{0\alpha}^2}+\frac{\dot{T}_0}{\sqrt{\omega_0^2\pm\widetilde{\omega}_{0\alpha}^2}}\sin t\sqrt{\omega_0^2\pm\widetilde{\omega}_{0\alpha}^2} \tag{E.4.8}$$

for $\alpha=0$.

2* a.

$$T(t)=e^{\mp\frac{\omega_1^2}{2}t}\left\{T_0\cos t\sqrt{\omega_0^2-\frac{\omega_{1\alpha}^4}{4}}+\frac{\dot{T}_0}{\sqrt{\omega_0^2-\frac{\omega_{1\alpha}^4}{4}}}\sin t\sqrt{\omega_0^2-\frac{\omega_{1\alpha}^4}{4}}\right\} \tag{E.4.9}$$

for $\alpha=1$ and for $\omega_0>\frac{1}{2}\omega_{1\alpha}^2$. (for soft creep) or for strong creep:

2* b.

$$T(t)=e^{\mp\frac{\omega_{1\alpha}^2}{2}t}\left\{T_0 Ch\, t\sqrt{\frac{\omega_{1\alpha}^4}{4}-\omega_0^2}+\frac{\dot{T}_0}{\sqrt{\frac{\omega_{1\alpha}^4}{4}-\omega_0^2}}Sh\, t\sqrt{\frac{\omega_{1\alpha}^4}{4}-\omega_0^2}\right\} \tag{E.4.10}$$

for $\alpha=1$ and for $\omega_0<\frac{1}{2}\omega_{1\alpha}^2$.

For the critical case:

$$2^*\text{ c.}\qquad T(t)=e^{\mp\frac{\omega_{1\alpha}^2}{2}t}\left\{T_0+\frac{2\dot{T}_0}{\omega_{1\alpha}^2}t\right\}\text{ for }\alpha=1\quad\text{and}\quad\omega_0=\frac{1}{2}\omega_{1\alpha}^2. \tag{E.4.11}$$

Fractional-differential equation (E.1.1) for the general case, when α is real number from the interval $0<\alpha<1$ can be solved by using Laplace's transformation. Using:

$$\mathsf{L}\left\{\frac{d^\alpha T(t)}{dt^\alpha}\right\}=p^\alpha\mathsf{L}\{T(t)\}-\left.\frac{d^{\alpha-1}T(t)}{dt^{\alpha-1}}\right|_{t=0}=p^\alpha\mathsf{L}\{T(t)\} \tag{E.4.12}$$

and introducing for initial conditions of fractional derivatives in the form (E.4.3), and after taking Laplace's transform of the equation (E.4.1), we obtain a corresponding equation. *Analyzing the previous Laplace transform (E.4.12) of solution we can conclude that two cases can be considered.*

For the case when $\omega_0^2\neq0$, the Laplace transform solution can be developed into series in the following way:

$$\mathsf{L}\{T(t)\}=\frac{pT_0+\dot{T}_0}{p^2\left[1+\frac{\omega_\alpha^2}{p^2}\left(\pm p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)\right]}=\left(T_0+\frac{\dot{T}_0}{p}\right)\frac{1}{p}\frac{1}{1+\frac{\omega_\alpha^2}{p^2}\left(\pm p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)} \tag{E.4.13}$$

$$\mathsf{L}\{T(t)\}=\left(T_0+\frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_\alpha^{2k}}{p^{2k}}\left(\pm p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)^k \tag{E.4.14}$$

$$\mathsf{L}\{T(t)\}=\left(T_0+\frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_\alpha^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j p^{\alpha j}\omega_\alpha^{2(j-k)}}{\omega_o^{2j}} \tag{E.4.15}$$

In writing (E.4.15) it is assumed that expansion leads to convergent series. The inverse Laplace transform of previous Laplace transform of the solution (E.4.15) in term-by-term steps is based on known theorem, and yields the following solution of differential equation (E.4.1) of time function in the following form of time series: (E.4.16)

$$T(t)=\mathsf{L}^{-1}\mathsf{L}\{T(t)\}=T_0\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+1-\alpha j)}+\dot{T}_0\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega_\alpha^{-2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+2-\alpha j)}$$

SUBJECT INDEX

Mihailo P. Lazarević is a Professor at the University of Belgrade, Faculty of Mechanical Engineering, Department of Mechanics in Serbia.

Mihailo P. Lazarević was born in Belgrade (Serbia) in 1964. He received four degrees from Belgrade University.
- *B.Sc in Mechanical Engineering (Aerospace),*
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He worked at the University of Belgrade at the Faculty of Mechanical Engineering, Department of Mechanics as an Assistant from 1996-2001. He became an Assistant Professor in 2001 and Associate Professor in 2005 and since 2009 he is a full Professor at the same Faculty.

He is a member of the Serbian Society of Mechanics and the Secretary of the Serbian Society of Mechanics since July 2013. During the years of 2009 to 2012 he was the Chief of the Laboratory of Applied Mechanics of the Faculty of Mechanical Engineering in Belgrade University.

He published 4 National Monographs, 4 Chapters in International Monographs, 7 articles in Leading International Journals, 8 articles in International Journals, 12 articles in National Journals, 80 articles in the proceedings of International Meetings, Conferences and Symposiums, 1 Book and 2 Handbooks as collections of solutions and solved problems. Also, he participated in several research projects in Serbia and abroad (EUREKA) related to the field of (bio)robotics and applications of Fractional Calculus.

He is engaged as a reviewer by several International Journals and organizer of many International/National Conferences. According to Scopus, he has over 120 citations and his H index is 7.

His scientific research can be divided into several thematic areas among them the Mathematical Modeling and Control of Rigid Bodies by Electrical Devices (Robotics and Biomechatronics) which has been his first research interest. He has also been working on extension of ideas of Fractional Calculus to Modeling of Mechatronic Systems and Biomechanical Tissues.

In addition, a wide range of topics of his interest are in Different Classes of Time Delay Systems, in Theory of Electroviscoelasticity, in Applications of Fractional PID, in Fractional Iterative Learning Control and in Applications of Fractional Wavelet Transform in Signal and Image Processing.

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