# JWKB Asymptotic Matching Rule Via The 1st Order BDE: Normal Form Analysis By Change of Independent Variable

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*Abstract:* The traditional first order JWKB method (=: $(JWKB)_1$) is a conventional semiclassical approximation method mainly used in quantum mechanical systems for accurate solutions. General $(JWKB)_1$ solution of the Time Independent Schroedinger's Equation (TISE) involves application of the conventional asymptotic matching rules to give accurate wavefunction in the Classically Inaccessible Region (CIR) of the related quantum mechanical system, i.e., Bender and Orszag (1999); Deniz (2010) [1, 2]. In this work, Bessel Differential Equation of the first order (=: $(BDE)_1$) is chosen as a mathematical model and its $(JWKB)_1$ solution is obtained by first transforming into the normal form *via the change of independent variable*. General $(JWKB)_1$ solution for appropriately chosen initial values in the normal form representation is being analyzed via generalized $(JWKB)_1$ asymptotic matching rules regarding to the $\widetilde{S}_{ij}$ matrix elements defined in Deniz (2010) [2]. Instead of applying the common $(JWKB)_1$ asymptotic matching rules relying on the physical nature of the quantum mechanical system, i.e., a physically acceptable (normalizable) wavefunction, a pure semiclassical analysis without interfering the physical nature of the system is being studied via the model $(BDE)_1$ to show their match in the normal form representation mathematically.

*Key–Words: JWKB (or WKB), Semiclassical Approximation, Asymptotic Matching, Linear Differential Equations, Bessel's Equation, Normal Form* Typing manuscripts, LaTeX

## 1 Introduction

$(JWKB)_1$ method[1] is conventionally known to be a strong and effective semiclassical approximation method enabling accurate analytical solutions in quantum mechanical systems, i.e., [1–8]. Quantum mechanical systems described by the Time Independent Schroedinger's Equation (TISE), which is in the form of a linear second order homogenous (normal form) differential equation:

$$y''(x) + f(x)y(x) = 0 \tag{1a}$$

$$f(x) = k^2(x) = \frac{2m}{\hbar^2}\left[E - V(x)\right] \tag{1b}$$

where these terms are in usual meanings ($m$ represents mass, $\hbar$ represents Planck's constant divided by $2\pi$, $E$ represents total energy, and $V(x)$ represents potential function, namely) has exact and approximate $(JWKB)_1$ solutions in the following forms:

$$y_{EX}(x) =: y(x) = k_1 y_1(x) + k_2 y_2(x) \tag{2a}$$

$$y_{JWKB}(x) =: \widetilde{y}(x) = \widetilde{k}_1\widetilde{y}_1(x) + \widetilde{k}_2\widetilde{y}_2(x) \tag{2b}$$

where $k_1 \& k_2$ and $\widetilde{k}_1 \& \widetilde{k}_2$ are the arbitrary constants and, $y_1 \& y_2$ and $\widetilde{y}_1 \& \widetilde{y}_2$ are the exact and JWKB complementary solutions, respectively. These constant coefficients in the general exact and JWKB solutions can be found from given initial values. The $(JWKB)_1$ solution has a typical property that both complementary $(JWKB)_1$ solutions (and hence the general $(JWKB)_1$ solution) diverge at a small region around the classical turning point where $f(x) = 0 \Rightarrow E = V(x)$, i.e., [1–3]. Moreover, the general $(JWKB)_1$ solution in (2b) is accurate for the Classically Accessible Region (CAR) but it needs asymptotic matching in the Classically Inaccessible Region (CIR) for accurate $(JWKB)_1$ solutions [1, 2]. CAR is the region where the particle can classically exist since its potential energy is smaller than its total energy: $f(x) > 0 \Rightarrow E > V(x)$, and CIR is the region where it can not classically exist since its potential energy is greater than its total energy: $f(x) < 0 \Rightarrow E < V(x)$. Conventional $(JWKB)_1$ asymptotic matching rules require that either of the complementary solutions in

[1] We simply refer to "$n$th order JWKB (or WKB)" by a simple abbreviation: $(JWKB)_n$ here.

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(2b) should cancel in the CIR as follows [1, 2]:

$$\widetilde{y}^{m\cdot}(x) = \begin{cases} \widetilde{y}(x) \text{ for CAR:} f(x) > 0 \\ \text{either } \widetilde{k}_1\widetilde{y}_1(x) \text{ or } \widetilde{k}_2\widetilde{y}_2(x) \text{ for CIR:} f(x) < 0 \end{cases} \quad (3)$$

and,

$$\widetilde{y}^{m\cdot}(x) \to \begin{cases} \lim_{x\to-\infty} \widetilde{y}(x) = 0 \text{ if CIR lies on the LHS} \\ \lim_{x\to\infty} \widetilde{y}(x) = 0 \text{ if CIR lies on the RHS} \end{cases} \quad (4)$$

where $\widetilde{y}^{m\cdot}(x)$ represents the asymptotically matched general $(JWKB)_1$ solution. In other words, asymptotically diverging term in the CIR should be cancelled in the general solution so that (4) can hold. Formal $(JWKB)_{N\to\infty}$ approximation formula representing both of the complementary functions in (2b) are actually in the form of an infinite series:

$$\widetilde{y}_\infty(x) = \exp\left[\frac{1}{\delta}\sum_{n=0}^{N}\delta^n S_n(x)\right], \begin{pmatrix} \delta \to 0 \\ N \to \infty \end{pmatrix} \quad (5)$$

where $\delta = \hbar/i \to 0$ for the TISE and $S_n$ represents the expansion terms given in [1, 2]. Two-valuedness of these expansion terms give two complementary $(JWKB)_N$ functions. However, only the first two terms (with indices $i = 0$ and 1) are used in the $(JWKB)_1$ approximation, which is known to give accurate-enough solutions for slowly changing potentials in the TISE and a criterion for this is given as follows [1–3]:

$$0 \le g(x) = \left|\frac{\partial_{xx}k(x)}{2k^3(x)} - \frac{3\left[\partial_x k(x)\right]^2}{4k^4(x)}\right| << 1 \quad (6)$$

As the potential in the TISE in (1b) gets sharper (so does function $g(x)$) and (6) fails, some of the higher order terms can no longer be neglected and hence, higher order JWKB terms leading to higher order JWKB approximation, $(JWKB)_{N>1}$, are required for accurate-enough solutions. So, for a general potential $V(x)$ involving both sharp and smooth domains in the TISE, $(JWKB)_1$ approximation should give accurate solutions in some subdomain in the CAR (where the criterion (6) holds) and inaccurate solutions (in need of asymptotical matching) in some other subdomains in the CIR (where the criterion (6) holds). Such a potential with obedient and non-obedient subdomains in the corresponding TISE is being studied semiclassically by the first order Bessel Differential Equation, $(BDE)_1$, as a chosen model differential equation here.

Since $(JWKB)_1$ method is generally applied to the quantum mechanical systems, main principles of the existing asymptotic matching rules rely on the nature of the physical system under study, i.e., a physically acceptable bound state wave function (solution of the TISE) in the CIR should not asymptotically diverge to infinity (≡Eqn. (4)) so that it can be normalized [1–3]. Its semiclassical explanation for the Simple Linear Potential (SLP), as a model potential where the $(JWKB)_1$ applicability criteria is satisfied in the entire domain, was studied in terms of the $(JWKB)_N$ expansion terms in [2]. In this work, similarly, a pure semiclassical analysis of the asymptotic matching rules is being studied for the intentionally chosen $(BDE)_1$ where the $(JWKB)_1$ applicability criteria is now partially satisfied in some subdomains involving both CAR and CIR. In our analysis, appropriately chosen associated initial values are used to compare the general unmatched and matched $(JWKB)_1$ solutions with the related exact solutions. $(JWKB)_1$ solutions of some quantum mechanical systems involving exponential potential decorated TISE, which is associated with the $(BDE)_n$, was obtained by the use of the common asymptotic matching rules given in (4) in the literature [9–11]. Our aim here is rather to search the asymptotic modifications of the $(JWKB)_1$ approximation for the $(BDE)_1$ mathematically via the semiclassical theories, where physical nature of the system regarding the bound and unbound quantum mechanical system analysis is no longer interfered. The $(BDE)_1$ is given in the standard form by:

$$y'' + \frac{1}{x}y' + \frac{(x^2-1)}{x^2}y = 0 \quad (7)$$

whose general exact solution in (2a) is the linear combination of two kinds of 1st order Bessel functions namely:

$$y_1(x) = J_1(x) \text{ and } y_2(x) = Y_1(x) \quad (8)$$

$(JWKB)_1$ solution of the $(BDE)_1$ in (7) can similarly be written (when solved) as a linear combination of two complementary functions as given in (2b). However, to follow this procedure one has to face with the problem arising from the fact that one can not find the general $(JWKB)_1$ solution given in (2b) directly by the conventional methods since the $(JWKB)_1$ technique including the famous $(JWKB)_1$ connection formulas requires (rather than that in (7)) a Linear Differential Equation (LDE) in the normal form given in (1a). So we have to study it in the normal form with a suitable change of variable. Complementary $(JWKB)_1$ functions (solutions) in (2b) can then be easily found by using the famous $(JWKB)_1$ connection formulas given in [1–4]. Once the $(JWKB)_1$

solution of either region (CAR or CIR) is found, the other region can directly be determined via these connection formulas.

So, our interest here can be summarized as follows: *i)* to find the general $(JWKB)_1$ solution of the $(BDE)_1$ whose structure is given in (2b) by using some appropriate change of variable to transform into a normal form (which is not unique), *ii)* to check its accuracy in the (sub)domains of the CAR and the CIR where the $(JWKB)_1$ applicability criterion in (6) holds and, *iii)* to find ways to do the correct asymptotic matching in the necessary (sub)domains by semiclassical analyses mathematically.

# 2 Statement and Re-statement of the Problem

## 2.1 Associated IVP and Statement of The Problem

The process being followed here can be stated by the following proposition:

**Proposition 1** *Once the general $(JWKB)_1$ solution in the form* (2b) *is obtained, one can test the accuracy (or exactness) of this solution by comparing the general exact and $(JWKB)_1$ solutions of the associated Initial Value Problem (IVP) constructed by imposing the following initial values:*

$$y(d_1) = y(x)\Big|_{x=d_1} = \alpha_1(c),\ y'(d_1) = \frac{dy(x)}{dx}\Big|_{x=d_1} = \beta_1(c) \tag{9}$$

*where $d_1$ is some real constant and c is some parameter in the domain $D$ where the $(JWKB)_1$ method is suitable for $(d_1, c) \in D$.*

For the common initial values in (9) general exact solution in (2a) and $(JWKB)_1$ solution in (2b) give:

$$y(c,x) = k_1(c)J_1(x) + k_2(c)Y_1(x) \tag{10}$$

(where $k_1(c), k_2(c)$ are the $c$ dependent coefficients satisfying the initial values given in (9)), and

$$\widetilde{y}(c,x) = \widetilde{k}_1(c)\widetilde{y}_1(x) + \widetilde{k}_2(c)\widetilde{y}_2(x) \tag{11}$$

(where similarly $\widetilde{k}_1(c), \widetilde{k}_2(c)$ are the $c$ dependent coefficients satisfying the same initial values in (9)). Since the common initial values are defined in continuous (or discrete) spectra[2] in the domain of parameter $c$, both general exact and $(JWKB)_1$ solutions span the whole domain of parameter $c$ to enable a successful comparison in two variables according to our proposition (Prop. 1) just as in [2]. But we have to apply an appropriate change of variable and study it in the normal form as explained above.

[2] parameter $c$ is used in continuous spectra here, however, it might be discrete (especially in quantum mechanical bound-state problems).

## 2.2 Change of Independent Variable and Re-statement of the Problem

**Lemma 2** *Although $(BDE)_1$ given in* (7) *is not in the normal form given in* (1a)*, a simple change of variable in the independent variable:*

$$x : (-\infty, \infty) \to (0, \infty)\ ,\ x(\rho) = e^{\frac{c-\rho}{2}} \tag{12}$$

*transforms the $(BDE)_1$ in the standard form in* (7) *to the following desired normal form (just as in* (1a)*) in a new independent variable, $\rho$:*

$$\frac{d^2y(\rho)}{d\rho^2} + \frac{e^{c-\rho} - 1}{4}y(\rho) = 0 \tag{13}$$

**Proof:** Our proof is based on the following neat theorem:

**Theorem 3** *The change of variable:*

$$\rho(x) = \int \exp\left[-\int p(x)dx\right]dx,\ \ u(\rho) = y(x) \tag{14}$$

*transforms the differential equation in the standard form with the independent variable $x$ given by:*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{15}$$

*into the normal form with the independent variable $\rho$ as follows:*

$$u''(\rho) + \left\{q(x)\exp\left[2\int p(x)dx\right]\right\}_{x=x(\rho)} u(\rho) = 0 \tag{16}$$

**Proof:** Suppose we have a change of variable $\rho = f(x)$ with $u(\rho) = y(x)$, then the derivatives

$$\frac{d}{dx} = \frac{d\rho}{dx}\frac{d}{d\rho} = f'\frac{d}{d\rho}$$

$$\frac{d^2}{dx^2} = f'^2\frac{d^2}{d\rho^2} + f''\frac{d}{d\rho}$$

in (15) gives:

$$f'^2u'' + f''u' + pf'u' + qu = 0$$

In order to eliminate the $u'$ term, $f$ must satisfy

$$f'' + pf' = 0$$

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which gives:

$$f' = \exp\left(-\int p(x)dx\right)$$
$$\Rightarrow f = \int \exp\left(-\int p(x)dx\right)dx$$

And the differential equation for $u$ is in the form:

$$u'' + \frac{q}{f'^2}u = 0$$

which is equivalent with (16). □ □

Various change of variable applications to transform the $(BDE)_n$ (with $n \to 1$) into any of the normal forms, which is not unique, in a similar fashion are available in the literature, i.e., [12–14]. But, the change of independent variable given in (12) is used here.

Procedure for comparison of the general $(JWKB)_1$ solution with the general exact solution of our IVP here can be achieved as follows: Eq. (13) can be re-written in two variable; $\rho$ and $c$, where $\rho$ represents the changed coordinates (rather than $x$ as explained above), and $c$ (without loss of generality) represents the classical turning point of $f(c,\rho)$ satisfying $f(c,\rho) = 0$ to give:

$$\frac{\partial^2}{\partial\rho^2}y(c,\rho) + f(c,\rho)y(\rho,c) = 0 \tag{17a}$$

where

$$f(c,\rho) = k^2(c,\rho) = \frac{e^{c-\rho}-1}{4}, -\infty < \rho < \infty \tag{17b}$$

whose initial values via Prop. (1) can now be used in the form[3]:

$$y(c,\rho)\bigg|_{\rho=d_2} = \alpha_2(c);\ \frac{\partial y(c,\rho)}{\partial\rho}\bigg|_{\rho=d_2} = \beta_2(c) \tag{18}$$

So, the general exact and $(JWKB)_1$ solutions of the normal form IVP in (17a)-(17b) with (18) takes the desired forms as in the solutions of previous standard form IVP in (10) and (11):

$$y(c,\rho) = c_1(c)y_1(c,\rho) + c_2(c)y_2(c,\rho), -\infty < \rho < \infty \tag{19a}$$

where

$$y_1 = J_1(c,\rho) \text{ and } y_2 = Y_1(c,\rho) \tag{19b}$$

[3]Notation: In this work, we show the initial values in the original (standard form) system (in $x$) by subscript 1 ($d_1, \alpha_1, \beta_1$), and in the transformed (normal form) system (in $\rho$) by 2 ($d_2, \alpha_2, \beta_2$). See also Eqn. (9) for comparison with (18).

and

$$\widetilde{y}(c,\rho) = \widetilde{c}_1(c)\widetilde{y}_1(c,\rho) + \widetilde{c}_2(c)\widetilde{y}_2(c,\rho), -\infty < \rho < \infty \tag{20}$$

In this work, results of the transformed (normal form) representation in $(c,\rho)$ for the associated IVP is being analyzed graphically by means of the semiclassical analysis only. Pure semiclassical asymptotic matching rules (without interfering the physical nature of the system) proposed here are expected to give accurate results in the corresponding (sub)domains for both representations. We study only the normal form representation here.

# 3 Calculations in The Normal Form

## 3.1 Exact and $(JWKB)_1$ Solutions of the $(BDE)_1$ in the Normal Form

There are some important points in the choice of the initial values in this comparison based IVP method and it will soon be shown that the initial values in (18) can safely be chosen as:

$$d_2 = 0\ ;\ \ \alpha_2(c) = 0\ ;\ \ \beta_2(c) = 1 \tag{21}$$

These important points can be summarized as follows:

*i)* Numerical value of $d_2$ (=constant) chosen in (18) should not correspond to the classical turning points of the associated normal form differential equation (where $f(c,\rho)|_{\rho=d_2} = 0$ in the TISE in Eqn. (17a)) at which the $(JWKB)_1$ method typically fails.

*ii)* Similarly, since $(JWKB)_1$ fails also in the CIR, $d_2$ should not be chosen in the CIR, neither.

*iii)* Numerical values of either $\alpha_2(c)$ or $\beta_2(c)$ chosen in (18) should not diverge to infinity for all $c$ in the domain of $f(c,\rho)$.

*iv)* Either $\alpha_2(c)$ or $\beta_2(c)$ in (18) can be chosen as constant functions provided that solutions of the IVP are in the forms of (19a) and (20). Note that initial values in (18) are theoretically defined to be $c$ dependent functions (as a consequence of the calculations via Prop. 1), however, we can choose them as constant functions as a specific case of this generalization (as we do here by (21)), which is not in contradiction with the function theories.

One can simply show that the general exact solution of the $(BDE)_1$ in the transformed normal form in (17a)-(17b) can be written by

$$y(c,\rho) = c_1(c)J_1(e^{\frac{c-\rho}{2}}) + c_2(c)Y_1(e^{\frac{c-\rho}{2}}), -\infty < \rho < \infty \tag{22}$$

and applications of the initial values in (18) give:

$$\begin{cases} y(c,d_2) = c_1(c)y_1(c,d_2) + c_2(c)y_2(c,d_2) = \alpha_2(c) \\ [\partial_\rho y(c,\rho)]_{\rho=d_2} = [c_1(c)\partial_\rho y_1(c,\rho) + c_2(c)\partial_\rho y_2]_{\rho=d_2} = \beta_2(c) \end{cases} \tag{23}$$

whose solutions for the $c$ dependent coefficients via the applications of the initial values in (21) give:

$$c_1(c) = \frac{\alpha_2(c)\frac{\partial}{\partial\rho}y_2(c,\rho)\left|_{\rho=d_2} - \beta_2(c)y_2(c,d_2)\right.}{\triangle(c,d_2)}\Bigg|\left\{\begin{array}{c} d_2 = 0 \\ \alpha_2(c) = 0 \\ \beta_2(c) = 1 \end{array}\right\} \tag{24a}$$

$$= \frac{-4e^{-c/2}Y_1(e^{c/2})}{[Y_2(e^{c/2}) - Y_0(e^{c/2})]J_1(e^{c/2}) + [J_0(e^{c/2}) - J_2(e^{c/2})]Y_1(e^{c/2})}$$

and

$$c_2(c) \tag{24b}$$

$$= \frac{-\alpha_2(c)\frac{\partial}{\partial\rho}y_1(c,\rho)\left|_{\rho=d_2} - \beta_2(c)y_1(c,d_2)\right.}{\triangle(c,d_2)}\Bigg|\left\{\begin{array}{c} d_2 = 0 \\ \alpha_2(c) = 0 \\ \beta_2(c) = 1 \end{array}\right\}$$

$$= \frac{4e^{-c/2}J_1(e^{c/2})}{[Y_2(e^{c/2}) - Y_0(e^{c/2})]J_1(e^{c/2}) + [J_0(e^{c/2}) - J_2(e^{c/2})]Y_1(e^{c/2})}$$

where the discriminant $\triangle(c,d_2)$ (the Wronskian determinant) is defined by

$$\triangle(c,d_2) = \left| \begin{array}{cc} y_1(c,d_2) & y_2(c,d_2) \\ \frac{\partial}{\partial\rho}y_1(c,\rho)\left|_{\rho=d_2}\right. & \frac{\partial}{\partial\rho}y_2(c,\rho)\left|_{\rho=d_2}\right. \end{array} \right| \tag{25}$$

As to the general $(JWKB)_1$ solutions, Let us first see how well the $(JWKB)_1$ applicability criteria given in (6) is satisfied via function $g(c,\rho)$. Following inequality should hold if the $(JWKB)_1$ method is applicable with a good-enough accuracy for a given $k(c,\rho)$ as in our case with (17b) here [2, 3]:

$$0 \leq g(c,\rho) = \left|\frac{1}{2k^3(c,\rho)}\frac{\partial^2 k(c,\rho)}{\partial\rho^2} - \frac{3}{4k^4(c,\rho)}\left[\frac{\partial k(c,\rho)}{\partial\rho}\right]^2\right| << 1 \tag{26}$$

In other words, one can not expect to have accurate $(JWKB)_1$ solutions in the region(s) where the inequality condition (26) does not hold (since the potential in the TISE gets sharper and higher order JWKB approximation is required, [1–3]). Calculation of $g(c,\rho)$ in (26) for the $(BDE)_1$ with (17b) gives:

$$0 \leq g(c,\rho) = \frac{1}{4}e^{Re(\rho+c)}\left|\frac{4e^\rho + e^c}{e^\rho - e^c}\right| << 1 \tag{27}$$

whose graph with the graph of $f(c,\rho)$ for some $c$ values are given in Fig. 1, from which we see that; classical turning point where $f(c,\rho) = 0$ is at $\rho_t = c$, and the CIR where $f(c,\rho) < 0$ lies on the right-hand-side of this turning point (similarly, the CAR where $f(c,\rho) > 0$ lies on the left-hand-side of it). So, we know in advance from (3)-(4) that we should make the asymptotic matching (modification) in the CIR (since located at the RHS of the turning point ($\rho_t = c$)) as follows [1, 2]:

$$y^{m.}(c,\rho) \to \lim_{\rho\to\infty} y(c,\rho) = 0 \tag{28a}$$

$$\widetilde{y}^{m.}(c,\rho) \to \lim_{\rho\to\infty} \widetilde{y}(c,\rho) = 0 \tag{28b}$$

Moreover, a narrow subregion not obeying (27) for each $c$ value in the domain can not be expected to give accurate results by the $(JWKB)_1$ method since lying just about the turning points (as a typical property of the $(JWKB)_1$ method, as mentioned above). Analysis of such regions in need of higher order JWKB approximation ($(JWKB)_{N>1}$) are beyond the scope of our study here. The width of this narrow sub-region

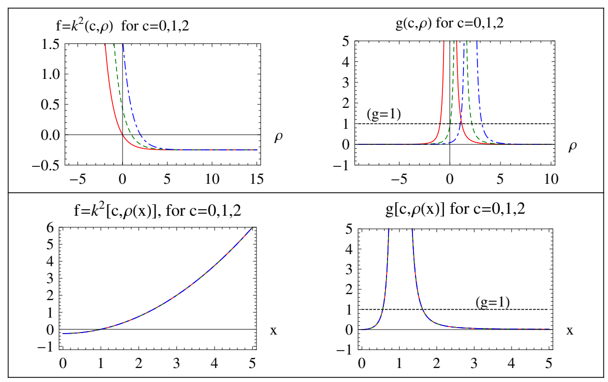


Figure 1: Graph of $f$ and $g$ functions for some $c$ values (solid-red curve: $c = 0$, dotted-green: $c = 1$, dashed-blue: $c = 2$).

can be found from the solution of $g(c,\rho)$ in (27) for real $\rho$ and $c$ values as follows:

$$\rho \in (c - 0.968889, c + 1.17808) \tag{29a}$$

and the remaining wide region:

$$-\infty < \rho << (c - 0.968889) \cup (c + 1.17808) << \rho < \infty \tag{29b}$$

is our main concern for a good-enough $(JWKB)_1$ general solution here.

In the $(JWKB)_1$ calculations, the entire domain can be considered as a unification of two neighboring regions (CAR-CIR) and if we start with the CAR located at the left-hand-side of the turning point and connect it to the CIR by using the conventional $(JWKB)_1$ connection formulas given in [1–3] in the reverse direction, we find the same formulas for

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$y_L(c,\rho)$ and $y_R(c,\rho)$ as in [2, see example 1] (But in $(c,\rho)$ here rather than $(c,x)$) to give:

$$\widetilde{y}(c,\rho) = \begin{cases} \widetilde{y}_L\,(c,\rho), \text{ for } -\infty < \rho \le c \\ \widetilde{y}_R\,(c,\rho), \text{ for } c \le \rho < \infty \end{cases} \tag{30}$$

where

$$\widetilde{y}_L(c,\rho) = \frac{A(c)}{\sqrt{k(c,\rho)}} \sin[\eta(c,\rho) + \alpha(c)] \tag{31a}$$

and

$$\begin{aligned} y_R(c,\rho) &= \frac{A(c)}{2\sqrt{\kappa(c,\rho)}} \cos[\alpha(c) - \pi/4]\, e^{[-\zeta(c,\rho)]} \\ &+ \frac{A(c)}{\sqrt{\kappa(c,\rho)}} \sin[\alpha(c) - \pi/4]\, e^{[\zeta(c,\rho)]} \end{aligned} \tag{31b}$$

But the constituents (functions: $\eta$ and $\zeta$) in (31a)-(31b) here rather reads:

$$\begin{aligned} \eta(c,\rho) &= \textstyle\int_\rho^c k(c,\rho)d\rho = \sqrt{e^{c-\rho} - 1} \\ &+ i[\tfrac{\rho - c}{2} + \ln(\sqrt{1 - e^{c-\rho}}) + 1], \rho < c \\ \zeta(c,\rho) &= \textstyle\int_c^\rho \kappa(c,\rho)d\rho = -\tfrac{c}{2} - \sqrt{1 - e^{c-\rho}} \\ &+ \ln[e^{\rho/2} + \sqrt{\ e^{\rho} - e^{c}}], c < \rho \end{aligned} \tag{32a}$$

(where $k$ and $\kappa$ are in usual meanings: $\kappa^2 = -k^2$) and the $c$ dependent coefficients can be found from the applications of the initial values given in (21) as follows:

$$\left\{ \begin{array}{l} \widetilde{y}(c,d_2) = y_L(c,d_2) = \alpha_2(c) \\ \partial_\rho \widetilde{y}(c,\rho)\Big|_{\rho=d_2} = \partial_\rho y_L(c,\rho)\Big|_{\rho=d_2} = \beta_2(c) \end{array} \right\} \Bigg|_{\left\{ \begin{array}{c} d_2 = 0 \\ \alpha_2(c) = 0 \\ \beta_2(c) = 1 \end{array} \right\}} \tag{32b}$$

$$\Rightarrow \begin{cases} A(c) = \frac{-\sqrt{2}}{(e^c - 1)^{1/4}} \text{ (for } 0 < c\infty) \\ \alpha(c) = \frac{ic}{2} - \sqrt{e^c - 1} - i\ln(1 + \sqrt{1 - e^c}), 0 < c < \infty \end{cases} \tag{32c}$$

Note that $\widetilde{y}(c,d_2)$ here in (32b) corresponds to $y_L(c,d_2)$ according to (30) since the initial values are chosen at $\rho = d_2$ which is in the CAR: $-\infty < d_2 = 0 < c$. General $(JWKB)_1$ solution in the other form given in (20), which is very important in our analysis, can similarly be written as follows [1,2]:

$$\begin{aligned} \widetilde{y}_1(c,\rho) &= \begin{cases} \frac{2}{\sqrt{k(c,\rho)}} \sin[\eta(c,\rho) + \pi/4], -\infty < \rho \le c \\ \frac{1}{\sqrt{\kappa(c,\rho)}} \exp[-\zeta(c,\rho)]\,, c \le \rho < \infty \end{cases} \\ \widetilde{y}_2(c,\rho) &= \begin{cases} \frac{1}{\sqrt{k(c,\rho)}} \sin[-\eta(c,\rho) + \pi/4]\,, -\infty < \rho \le c \\ \frac{1}{\sqrt{\kappa(c,\rho)}} \exp[\zeta(c,\rho)]\,, c \le \rho < \infty \end{cases} \end{aligned} \tag{33}$$

where $\eta(c,\rho)$ and $\zeta(c,\rho)$ are given in (32a), and applications of the initial values in (18) give:

$$\begin{cases} \widetilde{y}(c,d_2) = \widetilde{c}_1(c)\widetilde{y}_1(c,d_2) + \widetilde{c}_2(c)\widetilde{y}_2(c,d_2) = \alpha_2(c) \\ [\partial_\rho \widetilde{y}(c,\rho)]_{\rho=d_2} = \begin{bmatrix} \widetilde{c}_1(c)\partial_\rho \widetilde{y}_1(c,\rho) \\ + \widetilde{c}_2(c)\partial_\rho \widetilde{y}_2(c,\rho) \end{bmatrix}_{\rho=d_2} = \beta_2(c) \end{cases} \tag{34}$$

whose solutions for the $c$ dependent coefficients via the applications of the initial values in (21) give:

$$\begin{aligned} &\widetilde{c}_1(c) \qquad (35\text{a}) \\ &= \frac{\alpha_2(c)\partial_\rho \widetilde{y}_2(c,\rho)\,|_{\rho=d_2} - \beta_2(c)\widetilde{y}_2(c,d_2)}{\widetilde{\triangle}(c,d_2)} \Bigg|_{\left\{ \begin{array}{c} d_2 = 0 \\ \alpha_2(c) = 0 \\ \beta_2(c) = 1 \end{array} \right\}} \\ &= -\frac{\cos\left[\pm i\frac{c}{2} + \sqrt{e^c - 1} \mp i\ln(1 + \sqrt{1 - e^c}) + \frac{\pi}{4}\right]}{\sqrt{2}(e^c - 1)^{1/4}} \end{aligned}$$

and

$$\begin{aligned} &\widetilde{c}_2(c) \qquad (35\text{b}) \\ &= \frac{-\alpha_2(c)\partial_\rho \widetilde{y}_1(c,\rho)\,|_{\rho=d_2} - \beta_2(c)\widetilde{y}_1(c,d_2)}{\widetilde{\triangle}(c,d_2)} \Bigg|_{\left\{ \begin{array}{c} d_2 = 0 \\ \alpha_2(c) = 0 \\ \beta_2(c) = 1 \end{array} \right\}} \\ &= \frac{\sqrt{2}\sin\left[\pm i\frac{c}{2} + \sqrt{e^c - 1} \mp i\ln(1 + \sqrt{1 - e^c}) + \frac{\pi}{4}\right]}{(e^c - 1)^{1/4}} \end{aligned}$$

where the discriminant $\widetilde{\triangle}(c,d)$ (the Wronskian determinant) is similarly defined by

$$\widetilde{\triangle}(c,d_2) = \left| \begin{array}{cc} \widetilde{y}_1(c,d_2) & \widetilde{y}_2(c,d_2) \\ \frac{\partial}{\partial\rho}\widetilde{y}_1(c,\rho)\,|_{\rho=d_2} & \frac{\partial}{\partial\rho}\widetilde{y}_2(c,\rho)\,|_{\rho=d_2} \end{array} \right| \tag{36}$$

The graphs of the general exact and $(JWKB)_1$ solutions shown on the same graph (left column) and their difference (right column) in the transformed normal form (in $\rho$) for some $c$ values are given in Fig. 2.

**Remark 4** *Note that choosing either $\alpha_2$ or $\beta_2$ zero as we did here in* (21) *obviously simplifies the calculations of the coefficients in the general $(JWKB)_1$ solutions (see Eq.s* (35a)-(35b)*).*

### 3.2 Asymptotic Matching of the $(BDE)_1$ in the Normal Form

We can see from Fig. 2 that, $(JWKB)_1$ solutions are consistent with the exact solutions in the CAR where $-\infty < \rho < c$, but asymptotically are not (as $\rho$ increases in the CIR where $c < \rho < \infty$) as expected. Now, the question is whether we can see this without either interfering the exact results or consulting the present asymptotic matching rules given in (3)-(4) (and hence, (28b) here), and under what conditions

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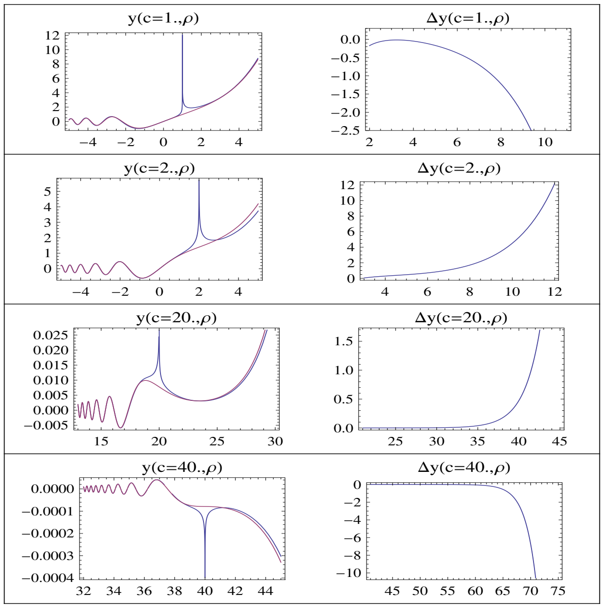


Figure 2: Graphs of the general exact and $(JWKB)_1$ solutions of the $(BDE)_1$ around the turning points (left column) and their difference (=errors) in the CIR (right column) ***in*** $(c,\rho)$ for some $c$ values.

they can be asymptotically-matched (in other words, what the asymptotic matching rule is).

From the JWKB theories we know that the formal expression of the $(JWKB)_N$ approximation written in (5) takes the form:

$$\widetilde{y}_N(c,\rho) = \exp\left[\frac{1}{\delta}\sum_{n=0}^{N}\delta^n S_n(c,\rho)\right], (\delta \to 0) \quad (37)$$

where the first three of them can be written in $(c,\rho)$ as follows [1,2]:

$$S_0(c,\rho) = \pm\int\sqrt{\kappa^2(c,\rho)}d\rho =: \pm A_0(c,\rho) = \begin{cases} S_{01}(c,\rho) = -A_0(c,\rho) \\ S_{02}(c,\rho) = A_0(c,\rho) \end{cases} \quad (38a)$$

$$S_1(c,\rho) = -\frac{1}{4}\ln\kappa^2(c,\rho) =: A_1(c,\rho) = \begin{cases} S_{11}(c,\rho) = A_1(c,\rho) \\ S_{12}(c,\rho) = A_1(c,\rho) \end{cases} \quad (38b)$$

$$S_2(c,\rho) = \pm\int\left\{\frac{\partial^2\left[\kappa^2(c,\rho)\right]/\partial\rho^2}{8\kappa^3(c,\rho)} - \frac{5\partial\left[\kappa^2(c,\rho)\right]/\partial\rho}{32\kappa^5(c,\rho)}\right\}d\rho \quad (38c)$$

$$=: \pm A_2(c,\rho) = \begin{cases} S_{21}(c,\rho) = -A_2(c,\rho) \\ S_{22}(c,\rho) = A_2(c,\rho) \end{cases}$$

where $\kappa^2 = -k^2$ as in the usual meaning. Here in $S_{ij}$, the first indice $i = 0,1,2$ represents the first three JWKB expansion terms, and the second indice $j(=1,2)$ represents two different sets due to the two-valuedness of these expansion terms as in [2]. For the $(JWKB)_1$ approximation, only the first two terms ($n = 0$ and up to $N = 1$ only) in (37) is used to give the well-known $(JWKB)_1$ formula in $(c,\rho)$ as:

$$\left.\begin{array}{l} \widetilde{y}_1(c,\rho) = \exp[\frac{S_{01}(c,\rho)}{\delta} + S_{11}(c,\rho)] \\ \widetilde{y}_2(c,\rho) = \exp[\frac{S_{02}(c,\rho)}{\delta} + S_{12}(c,\rho)] \end{array}\right\}, \delta \to 0 \quad (39)$$

which is equivalent with (33). As discussed above, according to (28b), the $(JWKB)_1$ solution in the CIR requires a cancellation of either $\widetilde{y}_1$ or $\widetilde{y}_2$ in Eq. (20) for the subdomain where the $(JWKB)_1$ applicability criterion holds. Note that both exact solution in (22) and $(JWKB)_1$ solution in (33) involves asymptotically diverging exponential terms in the CIR. As a result, this modification can fulfill the physical requirement, according to which the corresponding quantum mechanical system does not allow any asymptotically diverging term in the CIR (in both exact and $(JWKB)_1$ solution) [1–3].

Semiclassically, a successful (accurate w.r. to the exact solution) $(JWKB)_1$ solution of the TISE should have asymptotically descending $(JWKB)_1$ expansion terms with indices $n = 0$ and $n = 1$ and they should be bounded by the next term with indice $n = 2$ (which is not involved in the $(JWKB)_1$ solution, though) [1,2]. These requirements can be written by considering $\delta \to= 1$ to give [2]:

$$1 << \delta S_2 < S_1 < S_0/\delta\ ,\ \delta \to= 1 \quad (40)$$

Due to the two-valuedness, we can write the following proposition to determine which term ($\widetilde{y}_1$ or $\widetilde{y}_2$) exhibits the asymptotic requirements:

**Proposition 5** *In order to be an accurate $(JWKB)_1$ solution (and hence, in order to be a properly asymptotically matched $(JWKB)_1$ solution), the general JWKB expansion terms should satisfy the following inequalities:*

$$1 << \widetilde{S}_{21}(c,\rho\ or\ \rho(x)) < \widetilde{S}_{11}(c,\rho\ or\ \rho(x)) < \widetilde{S}_{01}(c,\rho\ or\ \rho(x)) \quad (41a)$$

$$1 << \widetilde{S}_{22}(c,\rho\ or\ \rho(x)) < \widetilde{S}_{12}(c,\rho\ or\ \rho(x)) < \widetilde{S}_{02}(c,\rho\ or\ \rho(x)) \quad (41b)$$

*where the definition of $\widetilde{S}_{ij}$ in [2] can be generalized (so as to involve also the transformed representation of our model $(BDE)_1$ under study) as follows:*

$$\widetilde{S}_{ij}(c,\rho\ or\ \rho(x)) = \begin{cases} |S_{ij}(c,\rho\ or\ \rho(x))|\ ,\ if\ S_{ij}(c,\rho\ or\ \rho(x)) \in \mathbb{C} \\ S_{ij}(c,\rho\ or\ \rho(x))\ ,\ if\ S_{ij}(c,\rho\ or\ \rho(x)) \in \mathbb{R} \end{cases} \quad (42)$$

**Proof:** Expansion terms in a specific problem (depending on the corresponding $f(c,\rho\ or\ \rho(x))$ in (17a)) may give real or complex $S_{ij}(c,\rho\ or\ \rho(x))$ elements in various (sub)regions within the domain. So,

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requirements given in (41a) and (41b), which are the natural consequence of (40) exhibiting a successful comparison according to their two-valuedness, makes the proof complete. □

**Corollary 6** *The expansion term(s) in the* $(JWKB)_1$ *solutions* ($S_{01}$, $S_{02}$, $S_{11}$, *and* $S_{12} = S_{11}$) *in* (39) *(building* (33) *for the* $(BDE)_1$*), whose associated* $\widetilde{S}_{ij}(c, \rho$ *or* $\rho(x))$ *elements determined from* (42) *do not obey* (41a)-(41b) *should be cancelled for successful asymptotic modification.*

Inequalities in (41a)-(41b) and the definition in (42) can freely be used in both representations independently and what we have for the transformed coordinates in $(c, \rho)$, which is also our main concern here, becomes:

$$\widetilde{S}_{ij}(c,\rho) = \begin{cases} |S_{ij}(c,\rho)|, \rho < c \because S_{ij}(c,\rho) \in \mathbb{C} \text{ in the CAR} \\ S_{ij}(c,\rho), c < \rho \because S_{ij}(c,\rho) \in \mathbb{R} \text{ in the CIR} \end{cases} \tag{43}$$

because from (38a)-(38b)-(38c) we have:

$$S_{01}(c,\rho) = -S_{02}(c,\rho) = -\frac{\sqrt{1-e^{c-\rho}}\,[-2\sqrt{e^{\rho}-e^{c}} + \rho e^{\rho/2} + 2e^{\rho/2} ln(1+e^{-\rho/2}\sqrt{e^{\rho}-e^{c}})]}{2\sqrt{e^{\rho}-e^{c}}} \tag{44a}$$

$$S_{11}(c,\rho) = S_{12}(c,\rho) = -\frac{1}{4} ln(\frac{1-e^{c-\rho}}{4}) \tag{44b}$$

$$S_{21}(c,\rho) = -S_{22}(c,\rho) = \frac{-6e^{c} - 4e^{\rho}}{48\sqrt{1-e^{c-\rho}}(e^{\rho}-e^{c})} \tag{44c}$$

from which we see that $S_{ij}(c,\rho) \in \mathbb{C}$ in the CAR (where $\rho < c$) and $S_{ij}(c,\rho) \in \mathbb{R}$ in the CIR (where $c < \rho$)). In order to make the comparison given in (41a)-(41b), we determine whether the elements are real or complex similar to [2]. So, the general expression (also including the SLP case in [2]) for the asymptotic matching can be written by (41a)-(41b)-(42) as the generalized asymptotic matching rule. If both (41a) and (41b) hold (we obviously see that this happens in the CAR), then the general $(JWKB)_1$ solution involves both $(JWKB)_1$ complementary functions ($\widetilde{y}_1$ and $\widetilde{y}_2$). But, if any of them does not hold (we again obviously see that this happens in the CIR), then the related complementary $(JWKB)_1$ function not obeying in that region (either $\widetilde{y}_1$ or $\widetilde{y}_2$) should cancel in the general $(JWKB)_1$ solution in that region (including also the SLP case in [2]). *Note that* (39) *with* (38a)-(38b)-(38c) *has an implication that* $S_{i1}$ *(and hence* $\widetilde{S}_{i1}$*) contributes to* $\widetilde{y}_1$ *and* $S_{i2}$ *(and hence* $\widetilde{S}_{i2}$*) contributes to* $\widetilde{y}_2$*. So, the non-obedient* $S_{i1}$ *terms require a cancellation of* $\widetilde{y}_1$ *and similarly,*

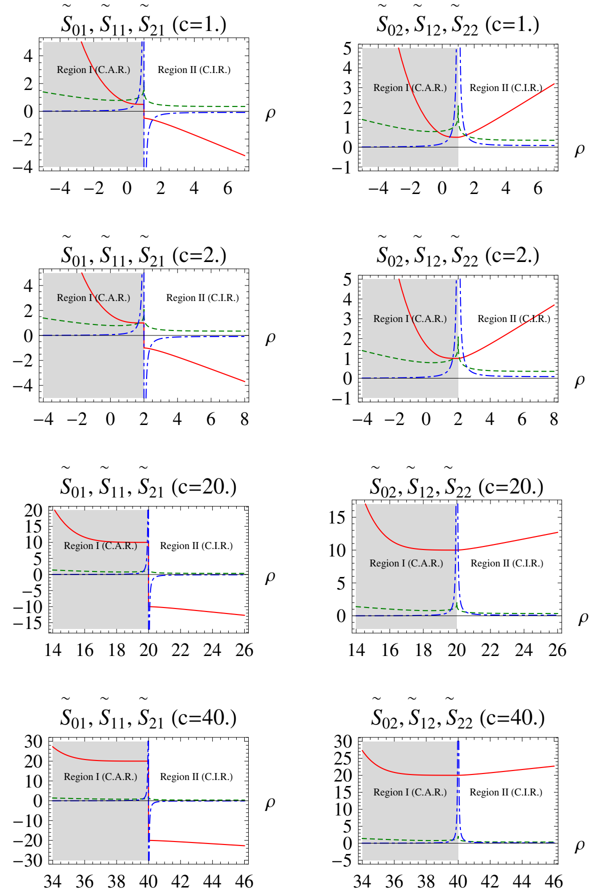


Figure 3: Graphs of $\widetilde{S}_{i,j=1}$ (left column) and $\widetilde{S}_{i,j=2}$ (right column) (where $i = 0, 1, 2$) ***in*** $\overline{(c, \rho)}$ for some specific $c$ values (solid curves: $for\ i = 0$, dotted curves: $for\ i = 1$ , and dashed curves: $for\ i = 2$).

*non-obedient* $S_{i2}$ *terms require a cancellation of* $\widetilde{y}_2$ *in the related subdomains.*

Graphs of the $\widetilde{S}_{i1}(c,\rho)$ and $\widetilde{S}_{i2}(c,\rho)$ for some $c$ values are given in Fig. 3. Comparing with the graph of $g(c,\rho)$ given in Fig. 1, we can see that $(JWKB)_1$ solutions should be in consistence with the exact solutions except for the non-obedient narrow regions given by (29a). We can also see that both (41a) and (41b) holds for the CAR whereas the CIR does not hold for both and hence needs a cancellation in the non-obedient $\widetilde{S}_{i1}$ term (left column graphs) in this region ($c < \rho < \infty$). This is the most remarkable consequence of our pure $(JWKB)_1$ analysis via Fig. 3 without interfering either the exact solutions or physical nature of the corresponding quantum mechanical system.

As a result, the necessary modifications according

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to our pure semiclassical analyses for the asymptotic matching of the $(BDE)_1$ in the transformed normal form representation should be as follows:

*(i)* Modification of the general exact solution in (22) (which will be used to compare with the modified $(JWKB)_1$ solutions):

$$y^{(m.)}(c,\rho) = \begin{cases} y, -\infty < \rho \leq c\ (CAR) \\ c_1(c)J_1(e^{\frac{c-\rho}{2}}), c \leq \rho < \infty\ (CIR) \end{cases} \tag{45}$$

Note that both exact and $(JWKB)_1$ solutions in (22) and (31b) of (30) (or (33) of (20)) have common form of exponentially increasing terms in the CIR: $c < \rho < \infty$, where a cancellation according to (41a)-(41b)-(42) is required for both exact and JWKB solutions (*See also* (47a) *below for comparison*).

*(ii)* Modification of the general $(JWKB)_1$ solution in (30):

$$\widetilde{y}^{(m.)}(c,\rho) = \begin{cases} \widetilde{y}_L\ ,\ \text{for}\ -\infty < \rho \leq c\ (CAR) \\ \widetilde{y}_R^{(m.)}\ ,\ \text{for}\ c \leq \rho < \infty\ (CIR) \end{cases} \tag{46a}$$

where

$$\widetilde{y}_R^{(m.)}(c,\rho) = \frac{A(c)}{2\sqrt{\kappa(c,\rho)}} \cos\left[\alpha(c) - \pi/4\right] Exp\left[-\zeta(c,\rho)\right] \tag{46b}$$

*(iii)* Modification of the general $(JWKB)_1$ solution in the other form (see Eq. (20) via (33)):

$$\widetilde{y}^{(m.)}(c,\rho) = \widetilde{c}_1(c)\widetilde{y}_1(c,\rho) + \widetilde{c}_2(c)\widetilde{y}_2^{(m.)}(c,\rho) \tag{47a}$$

where

$$\widetilde{y}_2^{(m.)}(c,\rho) = \begin{cases} \frac{1}{\sqrt{k(c,\rho)}} \sin[-\eta(c,\rho) + \pi/4]\ , -\infty < \rho \leq c\ (CAR) \\ 0\ \ , c \leq \rho < \infty\ (CIR) \end{cases} \tag{47b}$$

The superscript "(m.)" shows successfully asymptotically matched solutions here. In Fig. 3 we have non-obedient $\widetilde{S}_{i1}$ functions, requiring a cancellation of $S_{i1}$ which contributes to $\widetilde{y}_1$ (exhibiting exponentially increasing behavior) in the CIR. So, these asymptotic modifications are the results of our generalized asymptotic matching rules suggested here in (41a)-(41b)-(42), which is a semiclassical establishment of the existing asymptotic matching rules in (28a)-(28b) (or (3)-(4)). So, they can be used as an alternative, more general, and pure semiclassical (without interfering either exact solutions or physical nature of the system) asymptotic modification rule. Graphs of the anomalies for the unmodified and modified system for some $c$ values are given in Fig. 4 for comparison. It is clear that after the modification, the anomalies in the CIR have been removed successfully for the subdomain where $(JWKB)_1$ applicability criterion holds according to (29b) as has been aimed by the asymptotic modification process.

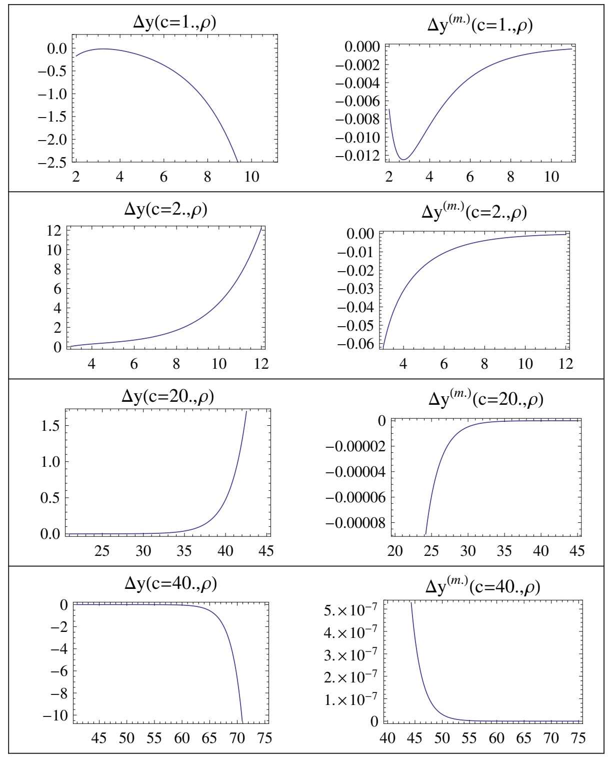


Figure 4: Errors in the unmodified (left column) and modified system (right column) ***in*** $(c,\rho)$ for some $c$ values.

## 4 Conclusion

In this work the $(BDE)_1$ has been chosen as a mathematical model to be studied by a pure semiclassical analysis (without interfering the physical nature of the system) since there exists subregions that $(JWKB)_1$ applicability criteria both holds and fails in some subregions of both CAR&CIR. Hereby presented generalized asymptotic matching rules regarding the $\widetilde{S}_{ij}$ matrix elements obtained from the $(JWKB)_N$ expansion terms show that, the general $(JWKB)_1$ solution of the $(BDE)_1$ for carefully chosen initial values needs asymptotic matching in the transformed (normal form) representation in the CIR where the $(JWKB)_1$ applicability criterion holds. Moreover, there is no need for asymptotic matching in the CAR where the $(JWKB)_1$ applicability criteria holds. These results, obtained by our pure semiclassical analysis, are consistent with the present conventional asymptotic matching rules given in (3)-(4) via [1, 2], which is a natural consequence of the physical nature of the related quantum mechanical system (normalizability requirement of quantum mechanical wave functions). The generalized asymptotic matching rules suggested in (41a)-(41b) with the generalized definition of $\widetilde{S}_{ij}$ in (42) are results of our pure semiclassical analyses where physical nature of the system is not interfered.

The main idea here is the semiclassical requirement that $(JWKB)_N$ expansion terms should be asymptotically decreasing as the term indice increases and bounded by the $(N+1)$th indiced term as stated in [1, 2]. Two valuedness of the $(JWKB)_N$ expansion terms as given in (38a)-(38b)-(38c) enables definitions of two different sets (corresponding to two complementary functions as in (39)) according to (42) for them so that the semiclassical requirements for the asymptotic matching can be tested for these two sets accordingly. Eqn. (39) shows that $S_{i1}$ (hence $\widetilde{S}_{i1}$ in our comparison function) contributes to $\widetilde{y}_1$ and, similarly, $S_{i2}$ (hence $\widetilde{S}_{i2}$ in our comparison function) contributes to $\widetilde{y}_2$. So, any violation of $\widetilde{S}_{i1}$ in (41a) requires a cancellation of $\widetilde{y}_1$, and any violation of $\widetilde{S}_{i2}$ in (41b) requires a cancellation of $\widetilde{y}_2$ in the related non-obedient (sub)domain provided that the $(JWKB)_1$ applicability criterion holds. Moreover, our alternative analyses enable a correct determination of which complementary function (solution) and where in the semiclassically solvable domain to cancel. So, the asymptotic matching rules in (41a)-(41b)-(42) obtained by our pure semiclassical analyses here without interfering physical (quantum mechanical) nature of the system seem to be an alternative (and also more general) equivalent asymptotic matching rules besides the present conventional rules given in (3)-(4) via [1, 2].

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