Mean square stability of discrete-time stochastic hybrid systems with interval time-varying delays

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Abstract—This paper is concerned with robust mean square stability of uncertain stochastic switched discrete time-delay systems. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the robust mean square stability for the uncertain stochastic discrete time-delay system is designed via linear matrix inequalities. Finally, some examples are exploited to illustrate the effectiveness of the proposed schemes.

Key–Words: Robust mean square stability, Discrete-time stochastic hybrid systems, Interval time-varying delays, Lyapunov functional, Linear matrix inequalities.

I. INTRODUCTION

Stochastic modelling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models. One of the most useful stochastic models which appear frequently in applications is the stochastic differential delay equations. In practice, we need estimate the parameters of systems. If the parameters are estimated using point estimations, the systems are described precisely and hence the study of the systems become relatively easier. On the other hand, if the parameters are estimated using confidence intervals, the systems become stochastic interval equations and the study of such systems are much more complicated.

Switched systems constitute an important class of hybrid systems. Such systems can be described by a family of continuous-time subsystems (or discrete-time subsystems) and a rule that orchestrates the switching between them. It is well known that a wide class of physical systems in power systems, chemical process control systems, navigation systems, auto-mobile speed change system, and so forth may be appropriately described by the switched model [1-8]. In the study of switched systems, most works have been centralized on the problem of stability. In the last two decades, there has been increasing interest in the stability analysis for such switched systems; see, for example, [9–16] and the references cited therein. Two important methods are used to construct the switching law for the stability analysis of the switched systems. One is the state-driven switching strategy [17–22]; the other is the time-driven switching strategy [23–25]. A switched system is a hybrid dynamical system consisting of a finite number of subsystems and a logical rule that manages switching between these subsystems (see, e.g., [22–26] and the references therein).

The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functional and linear matrix inequality (LMI) approach for constructing a common Lyapunov function [21–30]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays. In [20–29], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme.

This paper studies robust mean square stability problem for uncertain stochastic switched linear discrete-time delay with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to robustly mean square stable the uncertain stochastic linear discrete-time delay systems. By using improved Lyapunov-Krasovskii functional combined with LMIs technique, we propose new criteria for the robust mean square stability of the uncertain stochastic linear discrete-time delay system. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for robust mean square stability in terms of LMIs. Finally, some examples are exploited to illustrate the effectiveness of the proposed schemes.

The paper is organized as follows: Section II presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the robust mean square stability is presented in Section III. Numerical examples are provided to illustrate the theoretical results in Section IV, and the conclusions are drawn in Section V.

II. PRELIMINARIES

The following notations will be used throughout this paper. $R^+$ denotes the set of all real non-negative numbers; $R^n$ denotes the $n$-dimensional space with the scalar product of...
two vectors \( (x, y) \) or \( x^T y \); \( R^{n \times r} \) denotes the space of all matrices of \( (n \times r) \)-dimension. \( N^+ \) denotes the set of all non-negative integers; \( A^T \) denotes the transpose of \( A \); a matrix \( A \) is symmetric if \( A = A^T \).

Matrix \( A \) is semi-positive definite \( (A \geq 0) \) if \( \langle Ax, x \rangle \geq 0 \), for all \( x \in R^n \); \( A \) is positive definite \( (A > 0) \) if \( \langle Ax, x \rangle > 0 \) for all \( x \neq 0 \); \( A \geq B \) means \( A - B \geq 0 \). \( \lambda(A) \) denotes the set of all eigenvalues of \( A \); \( \lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\} \).

Consider an uncertain stochastic discrete systems with interval time-varying delay of the form

\[
x(k + 1) = (A_{\gamma} + \Delta A_{\gamma}(k))x(k) + (B_{\gamma} + \Delta B_{\gamma}(k))(x(k) - d(k)) + \sigma_{\gamma}(x(k), x(k) - d(k)), k \in N^+, \quad x(k) = u_k, \quad k = -d_2, -d_2 + 1, \ldots, 0, \tag{1}
\]

where \( x(k) \in R^n \) is the state, \( \gamma(.) : R^n \rightarrow N := \{1, 2, \ldots, N\} \) is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, \( \gamma(x(k)) = i \) implies that the system realization is chosen as the \( i^{th} \) system, \( i = 1, 2, \ldots, N \). It is seen that the system (1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state \( x(k) \) hits predefined boundaries. \( A_i, B_i, i = 1, 2, \ldots, N \) are given constant matrices and the time-varying uncertain matrices \( \Delta A_i(k) \) and \( \Delta B_i(k) \) are defined by:

\[
\Delta A_i(k) = E_{i\alpha}F_{i\alpha}(k)H_{i\alpha}, \quad \Delta B_i(k) = E_{i\beta}F_{i\beta}(k)H_{i\beta},
\]

where \( E_{i\alpha}, E_{i\beta}, H_{i\alpha}, H_{i\beta} \) are known constant real matrices with appropriate dimensions. \( F_{i\alpha}(k), F_{i\beta}(k) \) are unknown uncertain matrices satisfying

\[
F_{i\alpha}(k)F_{i\alpha}^T(k) \leq I, \quad F_{i\beta}(k)F_{i\beta}^T(k) \leq I, \quad k = 0, 1, 2, \ldots, \tag{2}
\]

where \( I \) is the identity matrix of appropriate dimension, \( \omega(k) \) is a scalar Wiener process (Brownian Motion) on \( (\Omega, \mathcal{F}, \mathcal{P}) \) with

\[
E[\omega(k)] = 0, \quad E[\omega^2(k)] = 1, \quad E[\omega(i)\omega(j)] = 0(i \neq j), \tag{3}
\]

and \( \sigma_i ; R^n \times R^n \rightarrow R^n, i = 1, 2, \ldots, N \) is the continuous function, and is assumed to satisfy that

\[
\sigma_i^T(x) = x(k) - d(k), \quad \sigma_i(x(k), x(k) - d(k)) \leq \rho_{11}x^T(k) + \rho_{12}x^T(k) - d(k) = \rho_1d(k), \tag{4}
\]

where \( \rho_{11} > 0 \) and \( \rho_{12} > 0 \), \( i = 1, 2, \ldots, N \) are known constant scalars. The time-varying function \( d(k) : N^+ \rightarrow N^+ \) satisfies the following condition:

\[
0 < d_1 \leq d(k) \leq d_2, \quad \forall k \in N^+
\]

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. The uncertain stochastic switched system (1) is robustly stable if there exists a switching function \( \gamma(.) \) such that the zero solution of the uncertain stochastic switched system is robustly stable.

Definition 2.2. The system of matrices \( \{J_i\}, i = 1, 2, \ldots, N \), is said to be strictly complete if for every \( x \in R^n \setminus \{0\} \) there is \( i \in \{1, 2, \ldots, N\} \) such that \( x^T J_i x < 0 \).

It is easy to see that the system \( \{J_i\} \) is strictly complete if and only if

\[
\bigcup_{i=1}^N \alpha_i = R^n \setminus \{0\},
\]

where

\[
\alpha_i = \{x \in R^n : x^T J_i x < 0\}, i = 1, 2, \ldots, N.
\]

Definition 2.3. The discrete-time system (1) is robustly stable in the mean square if there exists a positive definite scalar function \( V(k, x(k)) : R^n \times R^n \rightarrow R \) such that

\[
E[\Delta V(k, x(k))] = E[V(k + 1, x(k + 1)) - V(k, x(k))] < 0,
\]

along any trajectory of solution of the system (1).

Proposition 2.1. [31] The system \( \{J_i\}, i = 1, 2, \ldots, N \), is strictly complete if there exist \( \delta_i \geq 0, i = 1, 2, \ldots, N \), \( \sum_{i=1}^N \delta_i > 0 \) such that

\[
\sum_{i=1}^N \delta_i J_i < 0.
\]

If \( N = 2 \) then the above condition is also necessary for the strict completeness.

Proposition 2.2. (Cauchy inequality) For any symmetric positive definite matrix \( N \in M^{n \times n} \) and \( a, b \in R^n \) we have

\[
\pm a^T b \leq a^T N a + b^T N^{-1} b.
\]

Proposition 2.3. [31] Let \( E, H \) and \( F \) be any constant matrices of appropriate dimensions and \( F^T F \leq I \). For any \( \epsilon > 0 \), we have

\[
E F + H^T F^T E^T \leq \epsilon E E^T + \epsilon^{-1} H^T H.
\]

III. MAIN RESULTS

Let us set

\[
W_i = \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} \\ * & W_{i22} & W_{i23} \\ * & * & W_{i33} \end{bmatrix},
\]
where
\begin{align*}
W_{111} &= Q - P, \\
W_{112} &= S_1 - S_1A_i, \\
W_{113} &= -S_1B_i, \\
W_{122} &= P + S_1 + S_1^T + H_{ia}^T H_{ia} + S_1E_{ib}E_{ib}^T S_1^T, \\
W_{123} &= -S_1B_i, \\
W_{133} &= -Q + 2H_{ib}^T H_{ib} + 2\rho_1I, \\
J_i &= (d_2 - d_1)Q - S_1A_i - A_i^T S_1^T + 2S_1E_{ib}E_{ib}^T S_1^T + S_1E_{ib}E_{ib}^T S_1^T + H_{ia}^T H_{ia} + 2\rho_1I, \\
\alpha_i &= \{x \in \mathbb{R}^n : x^T J_i x < 0\}, i = 1, 2, \ldots, N, \\
\bar{\alpha}_1 &= \alpha_1, \quad \bar{\alpha}_2 = \alpha_i \setminus \bigcup_{j=1}^{i-1} \bar{\alpha}_j, \quad i = 2, 3, \ldots, N.
\end{align*}

The main result of this paper is summarized in the following theorem.

**Theorem 3.1.** The uncertain stochastic switched system (1) is robustly stable in the mean square if there exist symmetric positive definite matrices $P > 0, Q > 0$ and matrix $S_1$ satisfying the following conditions

(i) $\exists \delta_i \geq 0, i = 1, 2, \ldots, N, \sum_{i=1}^{N}\delta_i > 0 : \sum_{i=1}^{N}\delta_i J_i < 0.$

(ii) $W_i < 0, \quad i = 1, 2, \ldots, N.$

The switching rule is chosen as $\gamma(x(k)) = i$, whenever $x(k) \in \bar{\alpha}_i$.

**Proof.** Consider the following Lyapunov-Krasovskii functional for any ith system (1)

\[ V(k) = V_1(k) + V_2(k) + V_3(k), \]

where
\begin{align*}
V_1(k) &= x^T(k)Px(k), \\
V_2(k) &= \sum_{i=1}^{k-1} x^T(i)Qx(i), \\
V_3(k) &= \sum_{j=-d_2+1}^{-d_1+1} \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l),
\end{align*}

We can verify that
\[ \lambda_1 \|x(k)\|^2 \leq V(k). \]

Let us set \[ \xi(k) = [x(k) x(k+1) x(k - d(k)) \omega(k)]^T, \]

\[ H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}, \quad G = \begin{pmatrix} P & 0 & 0 & 0 \\ I & I & 0 & 0 \\ I & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \]

Then, the difference of $V_i(k)$ along the solution of the system (1) and taking the mathematical expectation, we obtained

\[ E[\Delta V_i(k)] = E[x^T(k+1)Px(k+1) - x^T(k)Px(k)] = E[\xi^T(k)H\xi(k) - 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \end{pmatrix}]. \]

because of

\[ \xi^T(k)H\xi(k) = x(k+1)Px(k+1), \]

\[ 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \end{pmatrix} = x^T(k)Px(k). \]

Using the expression of system (1)

\[ 0 = -S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_1(B_i + E_{ib}F_{ib}(k)H_{ib})x(k - d(k)) + S_1\sigma_i\omega(k), \]

we have

\[ E[-2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \end{pmatrix} = x^T(k)Px(k). \]

Therefore, from (7) it follows that

\[ E[\Delta V_1(k)] = E[x^T(k)[-P - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia} - A_i^T S_1^T - H_{ia}^T F_{ia}^T H_{ia}^T E_{ib}^T S_1^T]x(k) + 2x^T(k)[S_1A_i + S_1E_{ib}F_{ib}(k)H_{ib}]x(k - d(k)) + 2x^T(k)[-S_1\sigma_i - \sigma_i^T A_i - \sigma_i^T E_{ia}F_{ia}(k)H_{ia}]\omega(k) + x(k+1)[S_1H_{ia}^T x(k + 1) + 2x(k+1)[-S_1B_i - S_1E_{ib}F_{ib}(k)H_{ib}]x(k - d(k)) + 2x(k+1)[\sigma_i^T S_1^T \sigma_i \omega(k) + \omega^T(k)[-\sigma_i^T B_i - \sigma_i^T E_{ib}F_{ib}(k)H_{ib}]\omega(k) + \omega^T(k)[-2\sigma_i^T \sigma_i \omega(k)]. \]
By assumption (3), we have

\[
E[\Delta V_1(k)] = E[x^T(k)[-P - S_1 A_i - S_1 E_{ia} F_{ia}(k) H_{ia} \\
- A_i^T S_{11}^T - H_{ia}^T F_{ia}^T(k) E_{ia} S_{T1}^T] x(k) \\
+ 2x^T(k)[S_1 - S_1 A_i - S_1 E_{ia} F_{ia}(k) H_{ia}] x(k + 1) \\
+ 2x^T(k)[-S_1 B_i - S_1 E_{ib} F_{ib}(k) H_{ib}] x(k - d(k)) \\
+ x(k + 1)[S_1 + S_1^T] x(k + 1) \\
+ 2x(k + 1)[-S_1 B_i - S_1 E_{ib} F_{ib}(k) H_{ib}] x(k - d(k)) \\
- 2\sigma_i^T \sigma_i],
\]

Applying Proposition 2.2, Proposition 2.3, condition (2) and assumption (4), the following estimations hold

\[
- S_1 E_{ia} F_{ia}(k) H_{ia} - H_{ia}^T F_{ia}^T(k) E_{ia}^T S_{T1}^T \leq S_1 E_{ia} E_{ia} S_{T1}^T + H_{ia}^T H_{ia},
\]

\[
-2x^T(k)S_1 E_{ia} F_{ia}(k) H_{ia} x(k + 1) \leq \\
x^T(k)S_1 E_{ia} E_{ia} S_{T1}^T x(k) + x(k + 1) H_{ia}^T H_{ia} x(k + 1),
\]

\[
-2x^T(k)S_1 E_{ib} F_{ib}(k) H_{ib} x(k - d(k)) \leq \\
x^T(k)S_1 E_{ib} E_{ib}^T S_{T1}^T x(k) + x(k - d(k)) H_{ib}^T H_{ib} x(k - d(k)),
\]

\[
x^T(k + 1)S_1 E_{ib} E_{ib}^T S_{T1}^T x(k + 1) + x(k - d(k)) H_{ib}^T H_{ib} x(k - d(k)),
\]

\[
- \sigma_i^T (x(k), x(k - d(k)), k) \sigma_i (x(k), x(k - d(k)), k) \leq \\
\rho_{11} x^T(k) x(k) + \rho_{22} x^T(k - d(k)) x(k - d(k)).
\]

Therefore, we have

\[
E[\Delta V_1(k)] = E[x^T(k)[-P - S_1 A_i - A_i^T S_{11}^T \\
+ 2S_1 E_{ia} E_{ia}^T S_{T1}^T \\
+ S_1 E_{ib} E_{ib}^T S_{T1}^T + S_2 E_{ia} E_{ia}^T S_{21}^T \\
+ H_{ia}^T H_{ia} + 2\rho_{11} x(k) \\
+ 2x^T(k)[S_1 - S_1 A_i] x(k + 1) \\
+ 2x^T(k)[-S_1 B_i - S_2 A_i] x(k - d(k)) \\
+ x(k + 1)[S_1 + S_1^T] x(k + 1) \\
+ 2x(k + 1)[-S_1 B_i - S_1 E_{ib} F_{ib}(k) H_{ib}] x(k - d(k)) \\
+ x^T(k - d(k)) [2H_{ib}^T H_{ib} \\
+ 2\rho_{22} x(k - d(k))],
\]

The difference of \( V_2(k) \) is given by

\[
E[\Delta V_2(k)] = E[ \sum_{i=k+1-d(k+1)}^{k} x^T(i) Q x(i) \\
- \sum_{i=k-d(k)}^{k-1} x^T(i) Q x(i) ]
\]

Since \( d(k) \geq d_1 \) we have

\[
\sum_{i=k+1-d_1}^{k-1} x^T(i) Q x(i) \leq 0, \\
\]

and hence from (9) we have

\[
E[\Delta V_2(k)] \leq E[ \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i) Q x(i) \\
+ x^T(k) Q x(k) - x^T(k - d(k)) Q x(k - d(k))].
\]

The difference of \( V_3(k) \) is given by

\[
E[\Delta V_3(k)] = E[ \sum_{j=-d_2+1}^{-d_1+1} \sum_{i=k+j}^{k} x^T(l) Q x(l) \\
- \sum_{j=-d_2+1}^{-d_1+1} \sum_{l=k+j}^{k-1} x^T(l) Q x(l) ]
\]

Therefore, we have

\[
E[\Delta V_3(k)] = E[ \sum_{j=-d_2+1}^{-d_1+1} \sum_{l=k+j}^{k} x^T(l) Q x(l) \\
- \sum_{j=-d_2+1}^{-d_1+1} \sum_{l=k+j}^{k-1} x^T(l) Q x(l) \\
- x^T(k + j - 1) Q x(k + j - 1) \\
+ x^T(k + j) Q x(k + j) \\
- E[(d_2 - d_1)x^T(k) Q x(k) \\
- \sum_{j=k+1-d_2}^{k-d_1} x^T(j) Q x(j)].
\]
Since $d(k) \leq d_2$, and

$$\sum_{i=k+1-d(k)+1}^{k-d_1} x^T(i)Qx(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Qx(i) \leq 0,$$

we obtain from (10) and (11) that

$$E[\Delta V_2(k) + \Delta V_3(k)] \leq E[(d_2 - d_1 + 1)x^T(k)Qx(k) - x^T(k - d(k))Qx(k - d(k))].$$

(12)

Therefore, combining the inequalities (8), (12) gives

$$E[\Delta V(k)] \leq E[x^T(k)J_i(x(k)) + \psi^T(k)W_i\psi(k)],$$

(13)

where

$$\psi(k) = [x(k)x(k+1)x(k-d(k))]^T,$$

$$W_i = \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} \\ * & W_{i22} & W_{i23} \\ * & * & W_{i33} \end{bmatrix},$$

$$W_{i11} = Q - P,$$

$$W_{i12} = S_i - S_iA_i,$$

$$W_{i13} = -S_iB_i,$$

$$W_{i22} = P + S_i + S_i^T + H_{ia}^TH_{ia} + S_1E_{ib}E_{ib}^TS_1^T,$$

$$W_{i23} = -S_iB_i,$$

$$W_{i33} = -Q + 2H_{ia}^TH_{ib} + 2\rho_2I,$$

and

$$J_i = (d_2 - d_1)Q - S_iA_i - A_i^TS_i^T + 2S_1E_{ia}E_{ib}^TS_i^T + S_1E_{ib}E_{ib}^TS_1^T + H_{ia}^TH_{ia} + 2\rho_1I.$$
Moreover, the sum

$$\delta_1 J_1(R, Q) + \delta_2 J_2(R, Q) = \begin{bmatrix} -0.5761 & -0.0042 \\ -0.0042 & -3.9164 \end{bmatrix}$$

is negative definite; i.e. the first entry in the first row and the first column $-0.5761 < 0$ is negative and the determinant of the matrix is positive. The sets $\alpha_1$ and $\alpha_2$ are given as

$$\alpha_1 = \{(x_1, x_2) : -0.2170x_1^2 - 0.0052x_1x_2 - 1.8633x_2^2 < 0\},$$

$$\alpha_2 = \{(x_1, x_2) : 0.3591x_1^2 + 0.0032x_1x_2 + 2.0531x_2^2 > 0\}.$$

Obviously, the union of these sets is equal to $R^2 \setminus \{0\}$. The switching regions are defined as

$$\overline{\alpha}_1 = \{(x_1, x_2) : -0.2170x_1^2 - 0.0052x_1x_2 - 1.8633x_2^2 < 0\},$$

$$\overline{\alpha}_2 = \alpha_2 \setminus \overline{\alpha}_1.$$

By Theorem 3.1 the uncertain system is robustly stable and the switching rule is chosen as $\gamma(x(k)) = i$ whenever $x(k) \in \alpha_i$.

V. CONCLUSION

This paper has proposed a switching design for the robust stability of uncertain stochastic switched discrete time-delay systems with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the robust stability for the uncertain stochastic switched discrete time-delay system is designed via linear matrix inequalities. Numerical examples are provided to illustrate the theoretical results.

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