Optimal Control Problem with State-control Constraints

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Abstract: Two-sector economy (raw material sector and production one) and external debt described by ordinary differential equation are considered. The problem is to minimized the external debt over fixed planning period by choice of an investment plan (control variables) in each sector. The control and state variables are subjected to state constraints of a complex nature leads to a complicating the formulation and properties of the MP. New objects appear: measure and functional Lagrange multipliers. It became necessary to examine the properties of these objects and analyze the inter-connections of the various parts of the MP. Otherwise it would be impossible to use MP [1]. First sector produces raw materials and primary products and the second sector produces final products (consumer goods and asset-forming products for both sectors). The economy is assumed to be open, i.e., the system exchanges products with foreign countries. The export volume of the second sector is assumed to be negligible compared with the export volume of the first sector and is ignored in the model. We assume that the production function of each sector has a constant yield per unit of assets from the economy scale and that the connection between the assets per employee \( x_i = x_i(t), i = 1, 2 \) and labor productivity \( y_i = y_i(t) \), is determined in the \( i \)-th sector by the function

\[
y_i = f_i(x_i), \quad i = 1, 2.
\]

where \( f_i(x_i) \) satisfies the neoclassical conditions \( f'_i(x_i) > 0 \) and \( f''_i(x_i) < 0 \) [2].

The dynamics of the assets in the sectors is governed by the equations

\[
\begin{align*}
\dot{x}_1 &= -\mu_1 x_1 + u_1 + u_2, \\
\dot{x}_2 &= -\mu_2 x_2 + u_3 + u_4,
\end{align*}
\]

where \( \mu_i, i = 1, 2 \), are the rates of depreciations, \( u_1 = u_1(t) \) is the flow of foreign investment into the first sector, \( u_2 = u_2(t) \) is the flow of asset-forming products from the second sector into the first sector, \( u_3 = u_3(t) \) is the flow of foreign investment into the second sector and \( u_4 = u_4(t) \) is the flow of asset-forming products from the second sector into the second sector itself.

The production balance equation of the first sector has the form

\[
f_1(x_1) = e_1 + \alpha_1(t)f_1(x_1) + \alpha_2(t)f_2(x_2),
\]

where \( e_1 = e_1(t) \) is the export flow of the first sector, and \( \alpha_1 f_1(x_1), i = 1, 2 \), is the consumption of the production of the first sector in the corresponding sectors.

Let us write out the balance equation for the second sector

\[
f_2(x_2) = u_2 + u_4 + c_2(t),
\]

where \( c_2 = c_2(t) \) is the production flow to non-investment consumption.

In the considered model we assume that the export and import flows determine the foreign debt dynamics. Let \( x_3 = x_3(t) \) be the external debt at the time \( t \). Then

\[
\dot{x}_3 = \mu_3 x_3 + u_1 + u_3 + u_5 - \nu e_1,
\]

where \( \mu_3 \) and \( \nu \) are some constants, and \( u_5 = u_5(t) \) is the flow of import of consumer goods. It is known that the MP reduces the initial control problem to the two-point (multi-point) boundary value problem.

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1 Introduction

An optimal control problem can be solved completely or at least qualitatively only with the use of the maximum principle (MP). Availability of state and control-state constraints of a complex nature leads to a complication of the formulation and properties of the MP. New objects appear: measure and functional Lagrange multipliers. It became necessary to examine the properties of these objects and analyze the interconnections of the various parts of the MP. Otherwise it would be impossible to use MP [1]. First sector produces raw materials and primary products and the second sector produces final products (consumer goods and asset-forming products for both sectors). The economy is assumed to be open, i.e., the system exchanges products with foreign countries. The export volume of the second sector is assumed to be negligible compared with the export volume of the first sector and is ignored in the model. We assume that the production function of each sector has a constant yield per unit of assets from the economy scale and that the connection between the assets per employee \( x_i = x_i(t), i = 1, 2 \) and labor productivity \( y_i = y_i(t) \), is determined in the \( i \)-th sector by the function

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The number of investigators have tried and rejected shooting methods for certain classes of boundary value problems which are particularly sensitive to
the initial adjoint values and are thereby troublesome numerically. On the other hand there are successful examples of shooting methods.

Pesch J., Bulirsch R. [3] used the multiple shooting method as well as homotopy techniques to overcome the obstacle of finding an appropriate initial guess for starting the multiple shooting iteration.

Our purpose is to describe the procedures for numerical solving optimal control problems with local constraints. The list of examined questions covers a boundary value problem, numerical integration of stiff ordinary differential equations, preliminary assessment of the type of contact with a state constraint (geometry of the optimal trajectory), incorrectly posed nonlinear (linear) programming.

The basis of the proposed numerical methods is formed by introduction and perturbation of parameters and the processing the obtaining data in the course of continuation of the solution. In particular, we suggest the embedded problems depending on parameter. It gives us opportunity to get analytical solution in order to verify numerical solutions for special values of the parameter.

Note that application of Pontriagin’s maximum principle for dynamic optimization made appearance hamiltonian dynamic systems (HDS) in economics literature in the 1960’s. By 1970’s, the hamiltonian approach to dynamic economics became commonplace. Applications range from economic growth theory, fluctuations, capital theory, dynamic profit, intertemporal production and consumption plans, foreign investments, resource allocation, pollution, natural resources, portfolio allocation, optimal financing and advertising, to name only a few [4]. Not only local and global stability but also structural stability can be fruitfully investigated in the framework of HDS. The dynamics of these systems with constraints could be quite complex when some key parameter reaches some critical level, which causes the system to lose its stability. These items are covered under the heading of bifurcation theory, catastrophe theory and chaos [5].

2 Statement of the Optimal Control Problem

We consider a model of a two-sector open economic system on the time interval $[0, T]$; all variables of the model are nonnegative and are normed per one employee. The term $\mu_3 x_3$ takes into account foreign debt service payments, and the coefficient $\nu$ relates the internal and external prices.

We assume that the consumption flow can not be less than some minimum value, namely,

$$u_5(t) + c_2(t) \geq c_{\text{min}}, \quad t \in [0, T].$$

The asset level in the second sector can not drop below some critical level

$$x_2(t) \geq x_{2\text{min}}, \quad t \in [0, T].$$

The initial state of system (2) and the initial value of the external debt are given

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30}. \tag{8}$$

In the following, we restrict our consideration to the problem of minimization of the external debt, namely,

$$x_3(T) \rightarrow \min$$

under the boundary conditions

$$x_1(T) = x_{11}, \quad x_2(T) = x_{21}. \tag{10}$$

We assume that the control variables are subjected to constraint

$$0 \leq u_i(t) \leq u_{i1}, \quad i = 1, 3. \tag{11}$$

We obtain the following control problem.

Problem A. Find the minimum of the function (9) under constraints (1)-(8), (10), (11).

3 First Approximation

We introduce the following approximation

$$f_1(x_1) = \beta_1 x_1, \quad x_1(t) \leq a_1, \quad \alpha_1(t) = \alpha_{11},$$
$$f_2(x_2) = \beta_2 x_2, \quad x_2(t) \leq a_2, \quad \alpha_2(t) = \alpha_{21},$$
$$c_2(t) = c_{21},$$

where $\beta_1, \beta_2, a_1, a_2, \alpha_{11}, \alpha_{21}, c_{21}$ are constants. The equality control-state constraint (4) is substituted by inequality constraint

$$u_2 + u_4 \leq \beta_2 x_2 - c_{21}. \tag{12}$$

For $x_1(t) < a_1, x_2(t) < a_2, u_2 + u_4 \leq \beta_2 x_2 - c_{21}$ we have Pontriagin problem. In this case closed-form analytic solutions may be obtained by use of maximum principle.

Availability of state constraints leads to a complication of the formulation and properties of the maximum principle. The main problem is to define the set of active indices. We use factor analysis method for preliminary estimation of the optimal trajectory [6]. As a result we have the large scale linear programming problem. We solved this problem by method of extending solution by a parameter.
In the following we apply the obtained solution for forming the hypothesis about the structure of the optimal trajectory. Every hypothesis is checked in the frame of the maximum principle.

The control-state constraint (12) is treated the same way.

The solution of the first approximation problem can then be used as initial guess for the problem A. For this purpose we introduced the parameter $\alpha$ in ordinary differential equation (5) and control-state constraint (12)

$$
\dot{x}_3 = \mu_3 x_3 + u_1 + u_3 + u_5 - \nu \alpha [(1 - \alpha_1) f_1(x_1) - \alpha_2 f_2(x_2)] - \gamma (1 - \alpha) [(1 - \alpha_1) \beta_1 x_1 - \alpha_2 \beta_2 x_2],
$$

$$
u \alpha [(1 - \alpha_1) f_1(x_1) - \alpha_2 f_2(x_2)] - \gamma (1 - \alpha) [(1 - \alpha_1) \beta_1 x_1 - \alpha_2 \beta_2 x_2],
$$

$$
0 \leq \alpha \leq 1.
$$

The homotopy chain gives us the good initial guesses for all adjoint variables.

4 Change of Variables

The equations (2), (5) are stiff. Stiff differential equations are equations which are ill conditioned in a computational sense. In our case we can decrease stiffness by change of variables.

We put $y_1 = \ln x_1, y_2 = \ln x_2, y_3 = \ln x_3$. As the result we have

$$
\dot{y}_1 = -\mu_1 + (u_1 + u_2) e^{-y_1},
$$

$$
\dot{y}_2 = -\mu_2 + (u_3 + u_4) e^{-y_2},
$$

$$
\dot{y}_3 = \mu_3 + (u_1 + u_3 + u_5) e^{-y_3} - \gamma_1 e^{y_1 - y_3} + \gamma_2 e^{y_2 - y_3}.
$$

By the reverse change we can determine optimal control at every point $t$. It must be noted that direct system (2), (5) is stiff only. We use the equation (15) to compute $x_1, x_2, x_3$.

5 Existence and Uniqueness of the Optimal Control

Dikusar V. [2] has proved existence theorem for the canonical Dubovitsky-Milyutin problem in the form: find $\min J, J = F(x, u, t)$,

$$
\dot{x} = a_1(x, t) u + b_1(x, t),
$$

$$
g(x, u, t) = a_2(x, t) + b_2(x, t) = 0,
$$

$$
\Phi(x, u, t) \leq 0,
$$

$$
K(p) = 0, \varphi(p) \leq 0, p = (x(t_0), x(t_1)).
$$

The general optimal control problem with parameters can be reduced by standard methods to problem (16). Hence, the proof of existence of the homotopy chain is based on existence theorem for the canonical Dubovitsky-Milyutin problem. The assumptions are listed in [1].

It is clear that our problem satisfies the conditions of the existence theorem. We state the uniqueness of optimal control for the problem A.

Theorem. The optimal control for the problem A exists and is unique.

6 Example

We put

$$
\mu_1 = \mu_2 = \mu_3 = 5 \cdot 10^{-3}, \quad T = 100, \quad u_{11} = 0.1,
$$

$$
u_{21} = 0.3, \quad u_{31} = 0.1, \quad u_{41} = 0.3, \quad u_{51} = 0,
$$

$$
x_1(0) = 15, \quad x_2(0) = 25, \quad x_3(0) = 2400,
$$

$$
x_1(t) \geq 35, \quad x_2(t) \geq 20, \quad \beta_1 = \beta_2 = 1.
$$

Using factor analysis we produce the following solution:

$$
u_1(t) = u_{11}, \quad u_2(t) = u_{21}, \quad t \in [0, t_{11}],
$$

$$
u_2(t) = \mu_1 x_1 = \mu_1 a_1, \quad u_1(t) \equiv 0, \quad t \in [t_{11}, T],
$$

$$
u_3(t) \equiv 0, \quad t \in [0, T],
$$

$$
u_4(t) \equiv 0, \quad t \in [0, t_{21}],
$$

$$
u_4(t) = \mu_2 x_2, \quad t \in [t_{21}, T],
$$

$$
u_2(t) = x_2(t) e^{-\mu_2 t}, \quad t \in [0, t_{21}],
$$

$$
u_2(t) = x_2(t), \quad t \in [t_{21}, T],
$$

$$
t_{21} = \frac{1}{\mu_2} \ln \frac{x_2(t)}{x_2(t_{21})},
$$

$$
t_{11} = \frac{1}{\mu_1} \ln \frac{\mu_1 C_1}{a_1 \mu_1 - u_{11} - u_{21}},
$$

$$
C_1 = x_1(0) - \frac{u_{11} + u_{21}}{\mu_1}.
$$

The solution (17) satisfies the maximum principle.

7 The Non-regular Maximum Principle

We consider the control-state equality constraint (4). For the maximal flows $u_2 = u_{21}, u_4 = u_{41}$, we have

$$
\lambda_{21} - \lambda = 0, \quad \lambda_{41} - \lambda = 0,
$$

$$
\lambda_{21}(u_2 - u_{21}) = 0, \quad \lambda_{41}(u_4 - u_{41}) = 0.
$$

Relations (18) determine the non-regular maximum principle. The system (18) for the Lagrange multipliers has non-trivial solution. Now we have proved the existence of non-regular maximum principle.

References:


