On a Theory of Fuzzy Numbers and Fuzzy Arithmetic

Yingxu Wang
International Institute of Cognitive Informatics and Cognitive Computing (ICIC)
Dept. of Electrical and Computer Engineering, Schulich School of Engineering
University of Calgary
2500 University Drive NW, Calgary, Alberta T2N 1N4
CANADA
Tel: (403) 220 6141, Fax: (403) 282 6855
yingxu@ucalgary.ca http://www.ucalgary.ca/icic/

Abstract

Fuzzy numbers in number theory are a foundation of fuzzy sets and fuzzy mathematics that extend the domain of numbers from those of real numbers to fuzzy numbers. Fuzzy arithmetic is a system of fuzzy operations on fuzzy numbers. A theory of fuzzy arithmetic is presented towards a fuzzy mathematical structure for fuzzy inference and cognitive computation. The mathematical models of fuzzy numbers and their algebraic properties enable rigorous modelling of fuzzy entities in fuzzy systems and efficient manipulation of fuzzy variables in fuzzy analysis, fuzzy inference, and fuzzy computing. The denotational mathematical structure of fuzzy arithmetic not only explains the fuzzy nature of human perceptions and language semantic representation, but also enables cognitive machines and fuzzy systems to mimic human fuzzy inference mechanisms in cognitive informatics, cognitive computing, soft computing, cognitive linguistics, measurement theory, and computational intelligence.

Keywords: Fuzzy number theory, fuzzy functions, fuzzy arithmetic, fuzzy algebra, fuzzy mathematics, fuzzy systems, fuzzy semantics, fuzzy inference, denotational mathematics, cognitive computing, computational intelligence.

1. Introduction


The domains of number theories in mathematics have been continuously expanding from binary numbers (\(\mathbb{B}\)), natural numbers (\(\mathbb{N}\)), integers (\(\mathbb{Z}\)), and real numbers (\(\mathbb{R}\)) to fuzzy numbers (\(\mathbb{F}\)) and hyperstructures (\(\mathbb{H}\)) [Kline, 1972; Zadeh, 1975a; Artin, 1991; Smith, 2001; Timothy, 2008; Sibley, 2009; Wang, 2002, 2008b, 2011a, 2013, 2014c]. It demonstrates an interesting course of advances in human ability of abstraction and quantification in order to deal with the real-world entities and their perceptive representations in the brain. The difference between \(\mathbb{N}\) and \(\mathbb{Z}\) is that the members of the former are divisible by one, while those of the latter may not. The characteristic of the domain of fuzzy numbers \(\mathbb{F}\) is a 2-dementional hyperstructure, \(\mathbb{F} = \mathbb{R} \times \mathbb{R} = \{(\mathbb{R}, \mathbb{R})\}\) , with a crisp set of member elements in \([-\infty, +\infty]\) and an associate crisp set of degrees of membership in \([0, 1]\) for each of the members.

Sets are the fundamental mathematical means for abstraction, which is the essence of mathematics. Therefore, set theory becomes the foundation of almost all branches of mathematics. Lotfi A. Zadeh extends the classic set theory to fuzzy sets [Zadeh, 1965] and fuzzy logic [Zadeh, 1975b, 2008]. It is found that, although logical inferences may be carried out on the basis of crisp sets and predicate logic, more inference mechanisms and rules such as those of intuitive, empirical, heuristic, perceptive, and semantic inferences, are fuzzy and uncertain [Zadeh, 1971, 1975b; Wang, 2012b, 2014a, 2014b].

It is recognized that fuzzy numbers as a foundation of fuzzy sets and fuzzy mathematics are yet to be rigorously modeled and intensively studied [Zadeh, 1975a; Yager, 1986; Wang & Klir, 1992; Ross, 1995; Wang, 2014b, 2014c]. Fuzzy numbers were used to be treated as a fuzzy set with a convex triangle membership function for arbitrary number of elements with a uniformed incremental among values of the elements. Arithmetical operations on fuzzy numbers, such as those of addition, subtraction, multiplication, and division, were initially formulated by Lotfi A. Zadeh known as the extension principle of fuzzy sets [Zadeh, 1975a].
Definition 1. Zadeh’s extension principle for fuzzy set composition is a fuzzy union of fuzzy intersections of two fuzzy sets \( \tilde{X} \) and \( \tilde{Y} \), which derives a composite fuzzy set \( \tilde{X} \ast \tilde{Y} \), i.e.:

\[
\mu_{\tilde{X} \ast \tilde{Y}}(x \ast y) = \bigvee_{x \in \tilde{X}, y \in \tilde{Y}} \min(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y))
\]

where * denotes a fuzzy arithmetic operation, \( \ast \in \{+, -, \times, \div\} \), and \( |\tilde{X}| \) the cardinal size of \( \tilde{X} \) or the number of elements in \( \tilde{X} \).

Example 1. Given two fuzzy variables described by the following convex triangle fuzzy sets:

\( \tilde{X} = \{(0.0,2),(1.1,0),(2.0,2)\} \) and \( \tilde{Y} = \{(2.0,3),(3.1,0),(4.0,3)\} \)

Applying the extension principle, a fuzzy addition between the fuzzy variables \( \tilde{X} + \tilde{Y} \) can be derived as follows:

\[
\tilde{X} + \tilde{Y} = \{(0.0,2),(1.1,0),(2.0,2)\} \cup \{(2.0,3),(3.1,0),(4.0,3)\} \\
= \{(0,0),(1.0,0),(2.0,2),(2.0,3),(3.1,0),(4.0,3)\}
\]

where \( \bigvee \) denotes the fuzzy union operator and \( \bigwedge \) denotes the fuzzy intersection operator.

It is noteworthy via Example 1 that the extension method for fuzzy arithmetic is extremely complicated for such a simple operation on two small fuzzy numbers. It would be much more difficult when the fuzzy numbers are distributed in a wide range of points in the fuzzy sets. Therefore, in general, it is almost impossible to manually or mentally manipulate arbitrary fuzzy arithmetical operations as for those of real numbers.

This problem leads to a number of proposed solutions and algorithms for fuzzy arithmetic such as fuzzy interval analysis and fuzzy vectors [Dubois & Prade, 1980; Ross, 1995; BISC, 2014]. Because of the intensive complexity involved in fuzzy number representations and operations, various approximate algorithms were proposed for fuzzy set extensions such as those of the vertex method [Dong & Shah, 1987] and the DSW algorithm [Dong et al., 1985]. However, they are not mathematically intuitive enough and are over complicated for arbitrary fuzzy variable operations.

Theoretically, both the mathematical models and physical meanings of conventional fuzzy numbers and their fuzzy arithmetical operations were not rigorously defined in literature. The uncertainty error of fuzzy numbers, \( \varepsilon \), is used to be modeled by integers as \( \varepsilon \geq 1 \) with a certain membership value in conventional fuzzy set representation. However, it would be otherwise in general engineering practice where a fuzzy number is always \( \tilde{n} \pm \varepsilon \) where \( \varepsilon < 1 \) as a decimal number.

In order to address the aforementioned challenging problems, an improved theory of fuzzy numbers and fuzzy arithmetic is presented for applications in fuzzy inference systems, cognitive robots, cognitive informatics, cognitive computing, and computational intelligence. In the remainder of this paper, a mathematical model of fuzzy numbers in the fuzzy variable discourse is introduced in Section 2, which extends the domain of number theories from real numbers \( \mathbb{R} \) to fuzzy numbers \( \mathbb{F} \). An arithmetic structure on fuzzy numbers is developed in Section 3 with the abstract model of fuzzy arithmetic and formal operators of fuzzy addition, subtraction, multiplication, and division. A set of algebraic properties of fuzzy numbers and fuzzy arithmetic are explored in Section 4. The denotational mathematical structure of fuzzy numbers and fuzzy arithmetic not only explains the fuzzy nature of human abstract perceptions and language semantic representation, but also enables cognitive machines and fuzzy systems to mimic the human fuzzy inference mechanisms in cognitive informatics, cognitive computing, fuzzy measurement theory, fuzzy engineering mathematics, and computational intelligence.

### 2. Mathematical Model of Fuzzy Numbers

It is recognized that the discourse of classic mathematics \( \mathbb{R} \) is built on real numbers \( n \in \mathbb{R} \), arithmetic operators \( \ast, \), and functions \( f(x) \), i.e., \( \mathbb{R} \cong (\mathbb{R}, \ast, f) \). As that of \( \mathbb{R} \), the discourse of fuzzy mathematics can be formally described as follows.
**Definition 2.** The discourse of fuzzy mathematics \( \mathfrak{H}_\mathbb{F} \) is a triple:
\[
\mathfrak{H}_\mathbb{F} \triangleq (\mathbb{F}, \bullet, f)
\]  
where \( \mathbb{F} \) is a universal set of fuzzy numbers \( \tilde{\mathbb{F}} \in \mathbb{F}, \) \( \bullet \) a set of fuzzy arithmetic operators, and \( f \) a set of fuzzy functions.

Details of each element of the discourse of fuzzy mathematics will be formally described in the following subsections.

### 2.1 The Domain of Fuzzy Numbers

The concepts of fuzzy numbers and fuzzy variables are a natural extension of classic numbers from the real number domain \( \mathbb{R} \) to that of fuzzy numbers \( \mathbb{F} \).

**Definition 3.** The domain of fuzzy numbers \( \mathbb{F} \) is a fuzzy set of the range of values of all fuzzy variables \( \tilde{\mathbb{F}} \) and associate degrees of memberships \( \mu_\mathbb{F}(\tilde{\mathbb{F}}) \) in the hyperstructure of the fuzzy mathematical discourse \( \mathfrak{H}_\mathbb{F} \), i.e.:
\[
\mathbb{F} \triangleq \mathbb{R} \times \mathbb{R} \sqsubseteq \mathfrak{H}_\mathbb{F} = \{(\mathbb{R}, 1) | \mathbb{R} = [-\infty, \infty] \land 1 = [0,1] \subseteq \mathbb{R}\}
\]  
where \( \sqsubseteq \) denotes the dimension inclusion or that set \( \mathbb{S}_i \) is an elemental dimension of a hyperstructure \( \mathfrak{G} \) with \( n \) multiple dimensions, i.e.:
\[
\mathfrak{G} \triangleq \prod_{i=1}^{n} \mathbb{S}_i, \quad \mathbb{S}_i \subseteq \mathfrak{G}
\]  
where \( \prod_{i=1}^{n} \mathbb{S}_i \) is the big-R notation [Wang, 2002] that denotes an iterative operation on or a recurring structure of \( \mathbb{S}_i \) [Wang, 2002].

The domain of fuzzy numbers \( \mathbb{F} \) enlarges the discourse of mathematics from the domains of binary, integer, natural, and real numbers to a two-dimensional domain of fuzzy numbers \( \mathbb{F} = \mathbb{R} \times \mathbb{R} \).

In the hyperstructural domain \( \mathbb{F} \), a fuzzy number carries two dimensions of uncertainties for its base, lower, and upper values in the accuracy dimension of quantities as well as the degrees of memberships of its values in the confidentiality dimension. This notion leads to the formal definition of fuzzy numbers as follows.

**Definition 4.** A fuzzy number, \( \tilde{n} \), in \( \mathbb{F} \subseteq \mathfrak{H}_\mathbb{F} \) is a set of pairs \( \tilde{n} = \{ (\tilde{n}v, \tilde{n}\mu) \} \) where its values \( \tilde{n}v \) is constrained by a fuzzy errors \( \varepsilon \) and the associate uncertainty, \( \tilde{n}\mu \) is determined by a membership function \( \mu_\mathbb{F}(\tilde{n}\varepsilon) \) with respect to the base value \( n \) and \( n \pm \varepsilon \), respectively, i.e.:
\[
\tilde{n} \triangleq \{ (\tilde{n}v, \tilde{n}\mu) | \tilde{n}v = n \varepsilon \land \tilde{n}\mu = \mu_\mathbb{F}(n \varepsilon) \} 
\]  
\[
= \{(n-\varepsilon, \mu_\mathbb{F}(n-\varepsilon)), (n, \mu_\mathbb{F}(n)), (n+\varepsilon, \mu_\mathbb{F}(n+\varepsilon)) \}
\]  
where \( n \setminus \varepsilon = n \) represents the most accurate value of \( \tilde{n} \) with \( \mu_\mathbb{F}(n) = 1.0 \), and \( \varepsilon \) the upper and lower limits of the fuzzy errors of \( \tilde{n} \), respectively.

According to Definition 4, a fuzzy number is a fuzzy quantification of a fuzzy variable on both its accuracy and confidentiality. A fuzzy variable can be illustrated by a three-point convex triangle fuzzy membership function in \( \mathbb{F} \).

**Example 2.** Given \( a=2, \varepsilon_a=0.2, \) and \( \mu_\mathbb{F}(a \varepsilon_a) = \{0.5,1,0.7\} \) corresponding to \( a \varepsilon_a \), a fuzzy number \( \tilde{a} \), \( \tilde{a} \in \mathbb{F} \), can be formally expressed according to Definition 4 as follows:
\[
\tilde{a} = \{(a, \varepsilon_a, \mu_\mathbb{F}(a \varepsilon_a)) | a=2 \wedge \varepsilon_a=0.2 \wedge \mu_\mathbb{F}(a \varepsilon_a) = \{0.5,1,0.7\} \}
\]  
\[
= \{(2-0.2, 2-0.2), (2, \mu_\mathbb{F}(2)), (2+0.2, 2+0.2)\}
\]  
\[
= \{(1.8,0.5), (2,1.0), (2.2,0.7)\}
\]  
where the degrees of membership \( \mu_\mathbb{F}(a \varepsilon_a) \) are case specific.

**Example 3.** A fuzzy number \( \tilde{b} \), \( \tilde{b} \in \mathbb{F} \), can be similarly expressed according to Definition 4 as follows:
\[
\tilde{b} = \{(b, \varepsilon_b, \mu_\mathbb{F}(b \varepsilon_b)) | b=5 \wedge \varepsilon_b=0.3 \wedge \mu_\mathbb{F}(b \varepsilon_b) = \{0.8,1,0.4\} \}
\]  
\[
= \{(4.7,0.8), (5,1.0), (5.3,0.4)\}
\]  

### 2.2 Properties of Fuzzy Numbers

**Corollary 1.** The domains of \( \mathbb{B}, \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{R} \) are subsets or special cases of \( \mathbb{F} \), i.e.:
\[
\mathbb{B} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}
\]  
\[
\subseteq \mathbb{F} = \mathbb{R} \times \mathbb{R} = \{(\mathbb{R}, \mathbb{R}) \} \subseteq \mathfrak{H}_\mathbb{F}
\]  

**Definition 5.** The fuzziness of a fuzzy number, \( \varphi(\tilde{n}) \), in \( \mathbb{F} \subseteq \mathfrak{H}_\mathbb{F} \) is the rate of changes of its degree of membership.
over unit distance of its base value \( n \) in both the left and right sides, respectively, i.e.:

\[
\varphi^l(\tilde{n}) = \frac{d \mu_\varepsilon(n)}{d \varepsilon} = \frac{1 - \mu_\varepsilon(n - \varepsilon)\varepsilon}{\varepsilon} = \frac{1 - \mu_\varepsilon(n + \varepsilon)\varepsilon}{\varepsilon}
\]

It is obvious that the shaper the triangle membership function for a given fuzzy number, the higher the degree of its fuzziness. When the triangle membership function of the fuzzy number is symmetric, the left and right fuzziness are the same:

\[
\varphi^l(\tilde{n}) = \varphi^r(\tilde{n}), \text{ iff } \mu_\varepsilon(n - \varepsilon) = \mu_\varepsilon(n + \varepsilon)
\]

Example 4. The fuzziness the fuzzy number \( \tilde{b} \) given in Example 3, \( \tilde{b} \in F \), can be determined according to Definition 5 as follows:

\[
\varphi(\tilde{b}) = \begin{cases} 
\varphi^l(\tilde{b}) = \frac{1 - \mu_\varepsilon(b - \varepsilon)}{\varepsilon} = \frac{1 - 0.8 - 0.3}{0.7} = 0.3 \\
\varphi^r(\tilde{b}) = \frac{1 - \mu_\varepsilon(b + \varepsilon)}{\varepsilon} = \frac{1 - 0.4 - 0.3}{2.0} = 0.3
\end{cases}
\]

Corollary 2. The fuzziness of an unfuzzy real number \( n \in \mathbb{R} \) is always zero, i.e.:

\[
\varphi(\tilde{n} | n \in \mathbb{R}) = \varphi^l(n | n \in \mathbb{R}) = 0
\]

Eq. 9 can be directly proven based on Definition 5. It indicates that a real number in \( \mathbb{R} \) is a special case of fuzzy numbers in \( F \) where its fuzziness is zero.

Corollary 3. A fuzzy number \( \tilde{n} \) as denoted by a fuzzy set degrades to a single real number \( n \) known as the base value of \( \tilde{n} \) when the fuzzy errors of \( \tilde{n} \) is zero, i.e.:

\[
\tilde{n} = n, \text{ iff } \varepsilon = 0
\]

Proof. Corollary 4 can be directly proven according to Definition 5 as follows:

\[
\tilde{n} = \{(n, \varepsilon), \mu_\varepsilon(n, \varepsilon) | \varepsilon = 0 \land \mu_\varepsilon(n, 1.0) = 1.0
\]

= \{(n, \mu_\varepsilon(n))\}

= \{(n, 1.0)\}

= n

Both Corollaries 4 and 5, as well as Definition 4, indicate the approach to real number fuzzification.

3. Arithmetic of Fuzzy Numbers

Classic fuzzy arithmetic is based on Zadeh’s extension principle [Zadeh, 1975a] as described in Definition 1. However, it is over complicated and does not work on disintersected fuzzy numbers (sets) in general. On the basis of the mathematical models of fuzzy numbers and variables as created in Section 2, a set of fuzzy arithmetic operations can be formally defined in this section. The typical fuzzy arithmetic operators are addition, subtraction, multiplication, and division [Zadeh, 1975a; Ross, 1995; BISC, 2014], which form an algebraic structure of fuzzy arithmetic.

3.1 The Abstract Algebraic Model of Fuzzy Arithmetical Operations

The generic operator of fuzzy arithmetic on fuzzy numbers and variables can be introduced as a unique abstract operation in the following theorem.

Theorem 1. The abstract arithmetic operator \( \tilde{\bullet} \) on fuzzy numbers \( \tilde{x} \) and \( \tilde{y} \) in \( F \subseteq \mathbb{R} \) is:

\[
\tilde{x} \tilde{\bullet} \tilde{y} = \{([x - \varepsilon_x], \min(\mu_x(x - \varepsilon_x), \mu_y(y - \varepsilon_y))), (x \bullet y, 1.0), ([x + \varepsilon_x], \min(\mu_x(x + \varepsilon_x), \mu_y(y + \varepsilon_y))) \}
\]

(13)

where \( \tilde{\bullet} = \{+, -, \times, \div\} \) represent the arithmetical operators of addition, subtraction, multiplicatons, and division on fuzzy numbers, respectively. Correspondingly, \( \bullet = \{+,-,\times,\div\} \) denote the classic arithmetical operators on real numbers or the base of the corresponding fuzzy numbers.
3.2 The Arithmetic Operators on Fuzzy Numbers

On the basis of the generic abstract model of arithmetical operators on fuzzy numbers, the instances of fuzzy addition, subtraction, multiplication, and division, i.e., \( \mathbb{F} = \{+, -, \times, \div\} \), will be formally derived based on Theorem 1 in this subsection with specific elaborations.

### 3.2.1 Fuzzy Addition of Fuzzy Numbers

#### Definition 6.
The arithmetic operator of fuzzy addition \( \sim + \), \( \sim \in \mathbb{F} \), on two fuzzy numbers \( \tilde{x} \) and \( \tilde{y} \) in \( \mathbb{F} \subseteq \mathcal{U}_{\tilde{x}} \) can be derived from Theorem 1 as follows:

\[
\tilde{x} + \tilde{y} \triangleq \{(x - e_x, \mu_{\tilde{x}}(x - e_x)), (x + e_x, \mu_{\tilde{x}}(x + e_x))\} \bullet \\
\{(y - e_y, \mu_{\tilde{y}}(y - e_y)), (y + e_y, \mu_{\tilde{y}}(y + e_y))\} \bullet \\
\{(x - e_x, \mu_{\tilde{x}}(x - e_x), \mu_{\tilde{y}}(y - e_y)), \\
(x + e_x, \mu_{\tilde{y}}(x + e_x), \mu_{\tilde{y}}(y + e_y))\} \\
\{(x - e_x, \mu_{\tilde{x}}(x - e_x), \mu_{\tilde{y}}(y + e_y)), \\
(x + e_x, \mu_{\tilde{x}}(x + e_x), \mu_{\tilde{y}}(y - e_y))\} \\
\{(x + e_x, \mu_{\tilde{x}}(x + e_x), \mu_{\tilde{y}}(y - e_y)), \\
(x - e_x, \mu_{\tilde{x}}(x - e_x), \mu_{\tilde{y}}(y - e_y))\}
\]

\[ (14) \]

#### Corollary 6.
The hybrid arithmetic operator \( \tilde{\sim} \) on a fuzzy number \( \tilde{n} \) and a real number \( k \) is:

\[
k \tilde{\sim} \tilde{n} \triangleq \{(k \bullet (n - e_n), \mu_{\tilde{n}}(n - e_n)), \\
(k \bullet n, 1.0), \\
(k \bullet (n + e_n), \mu_{\tilde{n}}(n + e_n))\}
\]

\[ (15) \]

#### Proof.
Corollary 6 can be proven based on Theorem 1 and Definition 4 given \( e_k = 0 \), and \( \mu(k) = 1 \) as follows:

\[
k \tilde{\sim} \tilde{n} \triangleq \{(k - e_k, \mu_{\tilde{k}}(k - e_k)), (k, \mu_{\tilde{k}}(k)), (k + e_k, \mu_{\tilde{k}}(k + e_k))\} \bullet \\
\{(n - e_n, \mu_{\tilde{n}}(n - e_n)), (n, \mu_{\tilde{n}}(n)), (n + e_n, \mu_{\tilde{n}}(n + e_n))\} \bullet \\
\{(k \bullet (n - e_n), \mu_{\tilde{n}}(n - e_n)), \\
(k \bullet n, 1.0), \\
(k \bullet (n + e_n), \mu_{\tilde{n}}(n + e_n))\} \\
\{e_k = 0, \text{ and } \mu_{\tilde{k}}(k \pm e_k) = 1\} \\
\{(k \bullet (n - e_n), \mu_{\tilde{n}}(n - e_n)), \\
(k \bullet n, 1.0), \\
(k \bullet (n + e_n), \mu_{\tilde{n}}(n + e_n))\}
\]

\[ (16) \]

Therefore, a hybrid fuzzy arithmetical operation on mixed fuzzy and real numbers is a special case as described in Theorem 1, which will be illustrated in Section 3.2 by specific instances.
3.2.2 Fuzzy Subtraction of Fuzzy Numbers

Definition 8. The arithmetic operator of fuzzy subtraction \( \sim \), \( \sim \in \mathcal{E} \), on two fuzzy numbers \( \tilde{x} \) and \( \tilde{y} \) in \( \mathbb{F} \subseteq \mathcal{U}_x \) can be derived from Theorem 1 as follows:

\[
\tilde{x} - \tilde{y} = \{ ([x - e_x] - [y - e_y], \min(\mu^-_X(x - e_x), \mu^-_Y(y - e_y))),
(x - y, 1.0),
([x + e_x] - [y + e_y], \min(\mu^-_X(x + e_x), \mu^-_Y(y + e_y))) \}
\]

(19)

Example 7. Applying the fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \) as given in Examples 2 and 3, the fuzzy subtraction \( \tilde{b} - \tilde{a} \) can be carried out according to Definition 8 as follows:

\[
\tilde{b} - \tilde{a} = \{ ([b - e_b] - [a - e_a], \min(\mu^-_A(a - e_a), \mu^-_B(b - e_b))),
(b - a, 1.0),
([b + e_b] - [a + e_a], \min(\mu^-_A(a + e_a), \mu^-_B(b + e_b))) \}
\]

\[
= \{ (4.7 - 1.8, \min(0.5, 0.8)),
(5.0 - 2.0, 1.0),
(5.3 - 2.2, \min(0.7, 0.4)) \}
\]

\[
= \{ (2.9, 0.5), (3.0, 1.0), (3.1, 0.4) \}
\]

However, \( \tilde{a} - \tilde{b} \) yields different result as follows:

\[
\tilde{a} - \tilde{b} = \{ ([a - e_a] - [b - e_b], \min(\mu^-_A(a - e_a), \mu^-_B(b - e_b))),
(a - b, 1.0),
([a + e_a] - [b + e_b], \min(\mu^-_A(a + e_a), \mu^-_B(b + e_b))) \}
\]

\[
= \{ (1.8 - 4.7, \min(0.5, 0.8)),
(2.0 - 5.0, 1.0),
(2.2 - 3.3, \min(0.7, 0.4)) \}
\]

\[
= \{ (-2.9, 0.5), (-3.0, 1.0), (-3.1, 0.4) \}
\]

The comparative results indicate that \( \tilde{b} - \tilde{a} = \tilde{b} + (-\tilde{a}) = -(\tilde{a} - \tilde{b}) \).

Definition 9. The hybrid fuzzy subtraction \( \sim \), \( \sim \in \mathcal{E} \), on a fuzzy number \( \tilde{x} \) in \( \mathbb{F} \subseteq \mathcal{U}_x \) and a real number \( y \) is a special case of Definition 8 as follows:

\[
\tilde{x} - y = \{ ((x - e_x) - y, \mu^-_X(x - e_x)),
(x - y, 1.0),
((x + e_x) - y, \mu^-_X(x + e_x)) \}
\]

(20)

3.2.3 Fuzzy Multiplication of Fuzzy Numbers

Definition 10. The arithmetic operator of fuzzy multiplication \( \times \), \( \times \in \mathcal{E} \), on two fuzzy numbers \( \tilde{x} \) and \( \tilde{y} \) in \( \mathbb{F} \subseteq \mathcal{U}_x \) can be derived from Theorem 1 as follows:

\[
\tilde{x} \times \tilde{y} = \{ ([x - e_x] \times [y - e_y], \min(\mu^-_X(x - e_x), \mu^-_Y(y - e_y))),
(x \times y, 1.0),
([x + e_x] \times [y + e_y], \min(\mu^-_X(x + e_x), \mu^-_Y(y + e_y))) \}
\]

(21)

Example 8. Applying the fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \) as given in Examples 2 and 3, the fuzzy addition \( \tilde{a} \times \tilde{b} \) can be carried out according to Definition 10 as follows:

\[
\tilde{a} \times \tilde{b} = \{ ([a - e_a] \times [b - e_b], \min(\mu^-_A(a - e_a), \mu^-_B(b - e_b))),
(a \times b, 1.0),
([a + e_a] \times [b + e_b], \min(\mu^-_A(a + e_a), \mu^-_B(b + e_b))) \}
\]

\[
= \{ (1.8 \times 4.7, \min(0.5, 0.8)),
(2.0 \times 5.0, 1.0),
(2.2 \times 3.3, \min(0.7, 0.4)) \}
\]

\[
= \{ (8.5, 0.5), (10.0, 1.0), (11.7, 0.4) \}
\]

Definition 11. The hybrid fuzzy multiplication \( \times \), \( \times \in \mathcal{E} \), on a fuzzy number \( \tilde{x} \) in \( \mathbb{F} \subseteq \mathcal{U}_x \) and a real number \( y \) is a special case of Definition 10 as follows:

\[
\tilde{x} \times y = \{ ((x - e_x) \times y, \mu^-_X(x - e_x)),
(x + y, 1.0),
((x + e_x) \times y, \mu^-_X(x + e_x)) \}
\]

(22)

3.2.4 Fuzzy Division of Fuzzy Numbers

Definition 12. The arithmetic operator of fuzzy division \( \div \), \( \div \in \mathcal{E} \), on two fuzzy numbers \( \tilde{x} \) and \( \tilde{y} \) in \( \mathbb{F} \subseteq \mathcal{U}_x \) can be derived from Theorem 1 as follows:

\[
\tilde{x} \div \tilde{y} = \{ ([x - e_x] \div [y - e_y], \min(\mu^-_X(x - e_x), \mu^-_Y(y - e_y))),
(x \div y, 1.0),
([x + e_x] \div [y + e_y], \min(\mu^-_X(x + e_x), \mu^-_Y(y + e_y))) \}
\]

(23)
Example 9. Applying the fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \) as given in Examples 2 and 3, the fuzzy division \( \tilde{a} \div \tilde{b} \) can be carried out according to Definition 12 as follows:

\[
\tilde{a} \div \tilde{b} = \{(a - \varepsilon_a) \div [b - \varepsilon_b], \min(\mu_A(a - \varepsilon_a), \mu_B(b - \varepsilon_b)), (a \div b, 1.0),
(a + \varepsilon_a) \div [b + \varepsilon_b], \min(\mu_A(a + \varepsilon_a), \mu_B(b + \varepsilon_b))\}
\]

\[
\{(1.8 \div 4.7, \min(0.5,0.8)), (2.0 \div 5.0, 1.0), (2.2 \div 5.3, \min(0.7,0.4))\}
\]

\[
\{(0.38,0.5),(0.4, 1.0),(0.42,0.4)\}
\]

Example 10. Similarly, the fuzzy division \( \tilde{b} \div \tilde{a} \) can be carried out according to Definition 12 as follows:

\[
\tilde{b} \div \tilde{a} = \{(5 - 0.3) \div (2 - 0.2), \min(0.5,0.8)), (5.0 \div 2.0, 1.0),
((5 + 0.3) \div (2 + 0.2), \min(0.7,0.4))\}
\]

\[
\{(2.6,0.5),(2.5,1.0),(2.4,0.8)\}
\]

The comparative results indicate that \( \tilde{a} \div \tilde{b} = \tilde{a} \times \tilde{b}^{-1} = (\tilde{b} \div \tilde{a})^{-1} \).

Definition 13. The hybrid fuzzy division \( \tilde{x} \div \tilde{y} \in \tilde{\mathbb{F}} \), on a fuzzy number \( \tilde{x} \) in \( \mathbb{F} \subseteq \tilde{\mathbb{F}} \) and a real number \( y \) is a special case of Definition 12 as follows:

\[
\tilde{x} \div y \triangleq \{(x - \varepsilon_x) \div y, \mu_{\tilde{x}}(x - \varepsilon_x)), (x \div y, 1.0),
(x + \varepsilon_x) \div y, \mu_{\tilde{x}}(x + \varepsilon_x))\}
\]

Examples 5 through 10 provided in this subsection demonstrates that the formal fuzzy arithmetical operations can be much efficiently manipulated by manual calculations as for those of real numbers.

### 4. Properties of Fuzzy Algebra on Fuzzy Numbers and Variables

The basic properties of fuzzy algebra on fuzzy numbers in \( \mathbb{F} \subseteq \tilde{\mathbb{F}} \) can be expressed by the commutative, associative, distributive, inversive, idempotent, and identity laws as summarized in Table 1. In Table 1, the first category of properties is applied to fuzzy arithmetic on all fuzzy numbers; while the second category is for hybrid fuzzy operations between mixed fuzzy and real numbers.

It is noteworthy that \( \tilde{\cdot} \) and \( \div \) do not obey the commutative and associative laws of fuzzy arithmetic. With regards to the identity law of fuzzy arithmetic, \( \tilde{a}^{-1} \) share the same base value of \( \tilde{a} \). However, its membership value is constrained by the minimum, i.e.:

\[
\tilde{a}^{-1} = \{(a, \min(\mu_a(a - \varepsilon_a),\mu_a(0 - \varepsilon_0)))\} \cup \{(a, \min(\mu_a(a + \varepsilon_a),\mu_a(0 + \varepsilon_1)))\}
\]

\[
\{(a, \min(\mu_a(a - \varepsilon_a),\mu_a(0 - \varepsilon_0)))\}
\]

<table>
<thead>
<tr>
<th>No.</th>
<th>Property</th>
<th>Fuzzy algebra</th>
<th>Hybrid fuzzy algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Commutative</td>
<td>( \tilde{a} + \tilde{b} = \tilde{b} + \tilde{a} )</td>
<td>( \tilde{a} + \tilde{b} = \tilde{b} + \tilde{a} )</td>
</tr>
<tr>
<td>2</td>
<td>Associative</td>
<td>( \tilde{a} \times \tilde{b} = \tilde{b} \times \tilde{a} )</td>
<td>( \tilde{a} \times \tilde{b} = \tilde{b} \times \tilde{a} )</td>
</tr>
<tr>
<td>3</td>
<td>Distributive</td>
<td>( \tilde{a} \times (\tilde{b} \div \tilde{a}) = \tilde{b} \div \tilde{a} )</td>
<td>( \tilde{a} \times (\tilde{b} \div \tilde{a}) = \tilde{b} \div \tilde{a} )</td>
</tr>
<tr>
<td>4</td>
<td>Inversive</td>
<td>( \tilde{b} - \tilde{a} = \tilde{b} - \tilde{a} )</td>
<td>( \tilde{b} - \tilde{a} = \tilde{b} - \tilde{a} )</td>
</tr>
<tr>
<td>5</td>
<td>Idempotent</td>
<td>( \tilde{a}^{-1} = (0, \min(\mu_{\tilde{a}}(a - \varepsilon_a), \mu_{\tilde{a}}(a + \varepsilon_a))) )</td>
<td>( \tilde{a}^{-1} = (0, \min(\mu_{\tilde{a}}(a - \varepsilon_a), \mu_{\tilde{a}}(a + \varepsilon_a))) )</td>
</tr>
<tr>
<td>6</td>
<td>Identity</td>
<td>( \tilde{a} \times \tilde{a} = \tilde{a} )</td>
<td>( \tilde{a} \times \tilde{a} = \tilde{a} )</td>
</tr>
</tbody>
</table>
So do

\[ \tilde{\theta} = \{0, \min(\mu_2(a - \varepsilon), \mu_3(0 - \varepsilon))\} \] (26)

Each algebraic property of fuzzy arithmetic on fuzzy numbers as listed in Table 1 can be proven by directly applying the related definition as given in Section 3. The theories of fuzzy numbers, fuzzy arithmetic, and their algebraic properties establish a rigorous framework for abstracting fuzzy entities as fuzzy numbers in fuzzy systems and for manipulating fuzzy variables in fuzzy analysis, inference, and computation in cognitive computing and soft computing.

5. Conclusions

A theory of fuzzy numbers, fuzzy arithmetic, and their algebraic properties has been presented for abstracting fuzzy entities as fuzzy numbers in fuzzy systems and for manipulating fuzzy variables in fuzzy analysis, inference, and computation. Classic fuzzy arithmetical methods have been explored. The mathematical models of abstract fuzzy numbers and the fuzzy discourse have been introduced that extends the domain of number theories from real numbers in \( \mathbb{R} \) to fuzzy numbers in \( \mathbb{F} \). The arithmetic structure on fuzzy numbers has been developed based on the formal model of fuzzy arithmetical operations. A set of algebraic properties of fuzzy numbers and fuzzy arithmetic has been formally elicited.

The denotational mathematical structure of fuzzy arithmetic has not only explained the fuzzy nature of human perceptions and linguistic semantics, but also enabled cognitive machines and fuzzy systems to mimic the human fuzzy inference mechanisms in cognitive informatics, cognitive computing, fuzzy engineering mathematics, and computational intelligence. A wide range of applications of fuzzy arithmetic have been identified in fuzzy systems, cognitive robots, soft computing, inference systems, fuzzy measurement theory, cognitive semantics, cognitive computing, and cognitive robotics.

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