One-dimensional Intensive Steel Quenching Models

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Abstract: - In this paper we develop mathematical models for 1-D hyperbolic heat equations (wave equation or telegraph equation) and construct their analytical solutions for the determination of the initial heat flux for rectangular sample. In some cases we give expression of wave energy. Some solutions of time inverse problems are obtained in the form of 1st kind Fredholm integral equation, but others has been obtained in closed analytical form.

Key-Words: - Hyperbolic Equation, Exact Solution, Inverse Problem, Fredholm integral equation, Time inverse problem.

1 Introduction

The conventional steel quenching is usually performed in environmentally unfriendly oil or water/polymer solutions. Contrary to traditional method the intensive quenching process (IQP) uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions [1]-[5]. Traditionally for the mathematical description of the intensive quenching process, classical heat conduction equation is used. We proposed to use hyperbolic heat equation [6]-[13], [19]-[21] for more realistic description of the intensive quenching (IQ) process (especially for the initial stage of the process).

The idea of the usage of hyperbolic heat equation can be easily transferred to completely different sector of application - to the generation of electricity in sea or ocean by usage of wave energy [14]. It is important to note, that Ekergard and his co-authors examine the development of the system in time, describing the equipment with ordinary differential equation. Here we describe the equipment in development of both - in time as well as in spatial arrangement of equipment using the three-dimensional hyperbolic heat equation. Wave power plant has to work for long time period in moving environment – waves, see Wikipedia [15]. Therefore it is important to examine not only the development of equipment in time, but also the movement of its different components. We consider one dimensional statement for non-homogeneous equation with non-homogeneous boundary conditions.

2 Mathematical Formulation of 3-D Problem for IQP or Wave Power

Already in the introduction we noted that Professor M. Leijon, see [14] examined the development of system in time. Here we offer to consider the description of system in time and space. For this purpose instead of the ordinary differential equation, we consider the following partial differential equation:

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) + \Phi(x,y,z,t),\ a^2 = \frac{k}{c\rho}, \qquad (1)$$

$$x \in (0,l),\ y \in (0,b),\ z \in (0,w),\ t \in (0,T].$$

Here c is specific heat capacity, k - heat conductivity coefficient, ρ - density, τ_r - relaxation time. The source term $\Phi(x,y,z,t)$ can be from different parts of the same device or outer source. As the first step we use well known substitution:

$$V(x,y,z,t) = \exp\left(-\frac{t}{2\tau_r}\right) U(x,y,z,t). \qquad (2)$$

After transformation (2) equation (1) can be written in the following form:

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$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)-CU \\ +F(x,y,z,t), C=-\frac{1}{\tau_\tau^2}, a_\tau^2=\frac{a^2}{\tau_r}. \quad (3)$$

It is natural to assumption that planes $x=0, y=0, z=0$ are symmetry surfaces of the sample for the case of intensive steel quenching. In the case of wave energy we can assume different non-homogeneous boundary conditions:

$$\left(\frac{\partial U}{\partial x}-\alpha_1 U\right)\Big|_{x=0} = g_1(y,z,t), \alpha_i=\frac{h_i}{k}, \quad (4)$$

$$\left(\frac{\partial U}{\partial y}-\alpha_2 U\right)\Big|_{y=0} = g_2(x,z,t), \quad (5)$$

$$\left(\frac{\partial U}{\partial z}-\alpha_3 U\right)\Big|_{z=0} = g_3(x,y,t),\ i=1,2,3. \quad (6)$$

Here h_i is heat exchange coefficient. On all the other sides of steel device we have heat exchange with environment. For generalizing we assume following non-homogeneous third type boundary conditions (Robin conditions) on all the three outer sides:

$$\left(\frac{\partial U}{\partial x}+\beta_1 U\right)\Big|_{x=l} = g_4(y,z,t), \beta_i=\frac{h_i}{k}, \quad (7)$$

$$\left(\frac{\partial U}{\partial y}+\beta_2 U\right)\Big|_{y=b} = g_5(x,z,t), \quad (8)$$

$$\left(\frac{\partial U}{\partial z}+\beta_3 U\right)\Big|_{z=w} = g_6(x,y,t), i=1,2,3. \quad (9)$$

In fact it is possible to look at other types of boundary conditions: first (Dirichlet) and second (Neumann) type. After the transformation (2) then the differential equation (1) transforms into partial differential equation (3) without first time derivative. The initial conditions take the form:

$$U\big|_{t=0} = U_0(x,y,z), \quad (10)$$

$$\frac{\partial U}{\partial t}\Big|_{t=0} = U_1(x,y,z). \quad (11)$$

From the practical point of view in the steel quenching the condition (11) can be unrealistic. The initial heat flux must be determined theoretically. As additional condition we assume that either the temperature distribution or the heat fluxes distribution at the end of process is given (known):

$$U\big|_{t=T} = U_T(x,y,z), \quad (12)$$

$$\frac{\partial U}{\partial t}\Big|_{t=T} = V_1(x,y,z). \quad (13)$$

3 Solution of One Dimensional Problem for IQP

We will start with formulation of the mathematical model for thin in $y,z-$directions of steel part (one-dimensional model):

$$w \ll l, b \ll l. \quad (14)$$

Then in accordance with conservative averaging method [16]-[18],[7] we introduce following integral averaged value (one space-dimensional function):

$$u(x,t)=(bw)^{-1}\int_0^b dy\int_0^w U(x,y,z,t)dz, \\ \widehat{f}(x,t)=(bw)^{-1}\int_0^b dy\int_0^w F(x,y,z,t)dz. \quad (15)$$

Remember two inequalities (14): they show, that rectangle is thin and narrow. Really, integrating the three dimensional equation (3) in directions y,z we obtain:

$$\frac{1}{bw}\int_0^b dy\int_0^w\left[a_\tau^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)-CU\right]dz = \\ a_\tau^2\frac{\partial^2 u}{\partial x^2}-Cu+\widehat{f}(x,t)+\frac{a_\tau^2}{bw}\int_0^b dy\int_0^w\left(\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dz.$$

The last term can be transformed in following form:

$$\frac{a_\tau^2}{bw}\int_0^b dy\int_0^w\left(\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dz=$$
$$\frac{a_\tau^2}{bw}\left(\int_0^w \frac{\partial U}{\partial y}\bigg|_{y=0}^{y=b} dz+\int_0^b \frac{\partial U}{\partial z}\bigg|_{z=0}^{z=w} dy\right).$$

Taking into account heat exchange with environment, i.e. boundary conditions (5), (6), (8), (9) we obtain:

$$\frac{1}{bw}\left(\int_0^w \frac{\partial U}{\partial y}\bigg|_{y=0}^{y=b} dz+\int_0^b \frac{\partial U}{\partial z}\bigg|_{z=0}^{z=w} dy\right)=$$
$$-\frac{1}{bw}\int_0^w\left(\beta_2 U\big|_{y=b}+\alpha_2 U\big|_{y=0}+g_5\big|_{y=b}-g_2\big|_{y=0}\right)dz$$
$$-\frac{1}{bw}\int_0^b\left(\beta_3 U\big|_{z=w}+\alpha_3 U\big|_{z=0}+g_6\big|_{z=w}-g_3\big|_{z=0}\right)dy.$$

This allows us to assume the simplest approximation by constant for the function $U(x,y,z,t)$ in the $y,z-$directions:

$$U(x,y,z,t)=u(x,t)$$

As result we obtain 1-D differential equation with outer source term:

$$\frac{\partial^2 u}{\partial t^2}=a_\tau^2\frac{\partial^2 u}{\partial x^2}-cu+f(x,t),\ x\in(0,l),$$
$$t\in(0,T], \tag{16}$$
$$c=-\frac{1}{\tau_\tau^2}+a_\tau^2\left(\frac{\alpha_2+\beta_2}{b}+\frac{\alpha_3+\beta_3}{w}\right).$$

Their expressions are as follows:

$$\hat{g}_i(x,t)=\frac{1}{w}\int_0^w g_i(x,z,t)dz,\ i=2,5;$$
$$\hat{g}_j(x,t)=\frac{1}{b}\int_0^b g_j(x,y,t)dy,\ j=3,6; \tag{17}$$

$$f(x,t)=\hat{f}(x,t)+a_\tau^2\times$$
$$\left[\frac{\hat{g}_5(x,t)-\hat{g}_2(x,t)}{b}+\frac{\hat{g}_6(x,t)-\hat{g}_3(x,t)}{w}\right].$$

Initial conditions (14, (15) for the differential equation (20) are as follows:

$$u\big|_{t=0}=u_0(x),$$
$$u_0(x)=(bw)^{-1}\int_0^b dy\int_0^w U_0(x,y,z)dz, \tag{18}$$

$$\frac{\partial u}{\partial t}\bigg|_{t=0}=v_0(x),$$
$$v_0(x)=(bw)^{-1}\int_0^b dy\int_0^w U_1(x,y,z)dz. \tag{19}$$

The boundary conditions (4), (7) remain in the same form:

$$\left.\left(\frac{\partial u}{\partial x}-\alpha_1 u\right)\right|_{x=0}=g_1(t), \tag{20}$$

$$\left.\left(\frac{\partial u}{\partial x}+\beta_1 u\right)\right|_{x=l}=g_4(t). \tag{21}$$

Here

$$g_i(t)=(bw)^{-1}\int_0^b dy\int_0^w g_i(y,z,t)dz,\ i=1,4.$$

It is important to mark that one-dimensional approach is exact only if solution in other two directions is approximated with constant. At the end of this paragraph we will return to different one-dimensional models. This one-dimensional statement is substantially more realistic as statement given in our paper [9] because of taking into account heat or elasticity losses from flank sides $y=0,b$ and $z=0,w$. Firstly we assume firstly that we have non-homogeneous Klein-Gordon equation-with source term: $c\geq 0$. Solution of this one-dimensional direct problem (36)-(40) is well known, see, e.g. [22]:

$$u(x,t)=\frac{\partial}{\partial t}\int_0^l u_0(\xi)G(x,\xi,t)d\xi$$
$$+\int_0^l v_0(\xi)G(x,\xi,t)d\xi+H(x,t), \tag{22}$$

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$$H(x,t) = -a_\tau^2 \int_0^t g_1(\tau) G(x,0,t-\tau) d\tau + \int_0^t d\tau \int_0^l f(\xi,\tau) G(x,\xi,t-\tau) d\xi + a_\tau^2 \int_0^t g_4(\tau) G(x,l,t-\tau) d\tau.$$

The Green function has representation [22]-[25]:

$$G(x,\xi,t) = \sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \sqrt{a_\tau^2\lambda_i^2+c}},$$
$$\varphi_i(x) = \cos(\lambda_i x) + \frac{\alpha_1}{\lambda_i}\sin(\lambda_i x), \tag{23}$$
$$\|\varphi_i\|^2 = \frac{\alpha_1}{2\lambda_i^2} + \frac{l}{2}\left(1+\frac{\alpha_1^2}{\lambda_i^2}\right) + \frac{\beta_1}{2\lambda_i^2}\frac{\lambda_i^2+\alpha_1^2}{\left(\lambda_i^2+\beta_1^2\right)}.$$

The eigenvalues λ_i are positive roots of the transcendental equation:

$$\lambda = \frac{\lambda^2 - \alpha_1\beta_1}{\alpha_1+\beta_1}\tan(\lambda l).$$

It is easy to write out the so cooled "wave energy" [26]:

$$I_0(t) = \sum_{i=1}^{\infty} \frac{\sin^2\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\sqrt{a_\tau^2\lambda_i^2+c}}. \tag{24}$$

Right now we look at case $c<0, a_\tau^2\lambda_i^2+c<0$. In this case the Green function has different form [22]:

$$G(x,\xi,t) = \sum_{i=1}^{m-1} \frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}\right)}{\|\varphi_i\|^2 \sqrt{\left|a_\tau^2\lambda_i^2+c\right|}} + \sum_{i=m}^{\infty} \frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \sqrt{a_\tau^2\lambda_i^2+c}}. \tag{25}$$

Here the natural number m in the both sums is given by inequalities:

$$a_\tau^2\lambda_i^2 + c < 0, i = \overline{1, m-1},$$
$$a_\tau^2\lambda_i^2 + c \ge 0, i = \overline{m, \infty}.$$

In this case "wave energy" [26] has equality:

$$I_0(t) = \sum_{i=1}^{m-1} \frac{\sinh^2\left(t\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}\right)}{\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}} + \sum_{i=m}^{\infty} \frac{\sin^2\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\sqrt{a_\tau^2\lambda_i^2+c}}.$$

As it was told earlier, from experimental point of view initial condition (11) is unrealizable and the $v_0(x)$ must be calculated theoretically. The differentiation of solution (22) gives:

$$\frac{\partial}{\partial t}u(x,t) = \int_0^l u_0(\xi)\frac{\partial^2}{\partial t^2}G(x,\xi,t)d\xi + \int_0^l v_0(\xi)\frac{\partial}{\partial t}G(x,\xi,t)d\xi + \frac{\partial}{\partial t}H(x,t). \tag{26}$$

The additional conditions (12) and (13) at the end of process regarding t function $u(x,t)$ are as follows:

$$u\big|_{t=T} = u_T(x),$$
$$u_T(x) = (bw)^{-1}\int_0^b dy \int_0^w U_T(x,y,z)dz, \tag{27}$$

respectively

$$\left.\frac{\partial u}{\partial t}\right|_{t=T} = v_T(x),$$
$$v_T(x) = (bw)^{-1}\int_0^b dy \int_0^w V_T(x,y,z)dz. \tag{28}$$

Solution (26) at final moment $t=T$ gives:

$$u_T(x) = \int_0^l u_0(\xi)\left.\frac{\partial}{\partial t}G(x,\xi,t)\right|_{t=T} d\xi + \int_0^l v_0(\xi)G(x,\xi,T)d\xi + H(x,T)$$

or:

$$\int_0^l K(x,\xi)v_0(\xi)d\xi = f_0(x). \qquad (29)$$

Here

$$f_0(x) = u_T(x) - \int_0^l u_0(\xi)\frac{\partial}{\partial t}G(x,\xi,t)\Big|_{t=T} d\xi$$
$$-H\left(x,T\right), K(x,\xi) = G(x,\xi,T).$$

So as in our paper [6] we have obtained 1[st] kind Fredholm integral equation for the determination of unknown initial heat flux. For this ill-posed problem we have to use regularization method. If second additional condition is given from formula (26) we obtain following first kind Fredholm integral equation:

$$\int_0^l \tilde{K}(x,\xi)v_0(\xi)d\xi = g_0(x), \qquad (30)$$

$$\tilde{K}(x,\xi) = \frac{\partial}{\partial t}G(x,\xi,t)\Big|_{t=T}, g_0(x) = v_T(x) -$$
$$\frac{\partial}{\partial t}H\left(x,t\right)\Big|_{t=T} - \int_0^l u_0(\xi)\frac{\partial^2}{\partial t^2}G(x,\xi,t)\Big|_{t=T} d\xi.$$

There is an interesting situation, if both additional conditions (12), (13) are known. In this case we introduce new time argument:

$$\tilde{t} = T - t. \qquad (31)$$

The main differential equation (16) remains in its form, changes only the source term:

$$\frac{\partial^2 u}{\partial \tilde{t}^2} = a_\tau^2\frac{\partial^2 u}{\partial x^2} - cu + f\left(x, T-\tilde{t}\right),$$
$$x \in (0,l), \tilde{t} \in (0,T]. \qquad (32)$$

The boundary conditions (20), (21) change similarly:

$$\left(\frac{\partial u}{\partial x} - \alpha_1 u\right)\Big|_{x=0} = g_1\left(T-\tilde{t}\right),$$
$$\left(\frac{\partial u}{\partial x} + \beta_1 u\right)\Big|_{x=l} = g_4\left(T-\tilde{t}\right). \qquad (33)$$

Both additional conditions transforms to initial conditions for the equation (32):

$$u\big|_{\tilde{t}=0} = u_T(x),$$
$$\frac{\partial u}{\partial \tilde{t}}\Big|_{\tilde{t}=0} = -v_T(x). \qquad (34)$$

The solution of direct problem (32)-(34) is similar with the solution (22):

$$u(x,\tilde{t}) = \int_0^l u_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})d\xi$$
$$-\int_0^l v_T(\xi)G(x,\xi,\tilde{t})d\xi + H\left(x,\tilde{t}\right). \qquad (35)$$

The last term can be written in following form:

$$H\left(x,\tilde{t}\right) = -a_\tau^2\int_{T-\tilde{t}}^T g_1\left(\tau\right)G(x,0,\tilde{t}-T+\tau)d\tau$$
$$+\int_{T-\tilde{t}}^T d\tau\int_0^l f\left(\xi,\tau\right)G(x,\xi,\tilde{t}-T+\tau)d\xi \qquad (36)$$
$$+a_\tau^2\int_{T-\tilde{t}}^T g_4\left(\tau\right)G(x,l,\tilde{t}-T+\tau)d\tau.$$

For the heat flux we have an expression similar to formula (26):

$$\frac{\partial}{\partial \tilde{t}}u(x,\tilde{t}) = \int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2}G(x,\xi,\tilde{t})d\xi$$
$$-\int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})d\xi + \frac{\partial}{\partial \tilde{t}}H\left(x,\tilde{t}\right).$$

From here immediately follows a nice explicit representation for necessary initial heat flux:

$$v_0(x) = -\int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})\big|_{\tilde{t}=T} d\xi +$$
$$\int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2}G(x,\xi,\tilde{t})\big|_{\tilde{t}=T} d\xi + \frac{\partial}{\partial \tilde{t}}H\left(x,\tilde{t}\right)\Big|_{\tilde{t}=T} \qquad (37)$$

As we mentioned before, one-dimensional statement is approximated model of three-dimensional statement. If the boundary conditions (5), (6), (7), (8) are second kind (Neumann's) conditions $(\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0)$, than one-dimensional problem is exact statement of three-dimensional approach. If conditions (4), (7) are

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second type conditions ($\alpha_1 = \beta_1 = 0$), than formula (22) is correct, only the Green function has such form [22]-[25]:

$$G(x,\xi,t) = \frac{1}{l\sqrt{c}}\sin\left(t\sqrt{c}\right) + \frac{2}{l}\times$$
$$\sum_{i=1}^{\infty}\frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\sqrt{a_\tau^2\lambda_i^2+c}}, \quad (38)$$
$$\varphi_i(x) = \cos\left(\lambda_i x\right), \lambda_i = \frac{n\pi}{l}.$$

Similarly with formula (42) this is for $c \geq 0$. Solution in this case ($c < 0$) are in form:

$$G(x,\xi,t) = \frac{1}{l\sqrt{|c|}}\sinh\left(t\sqrt{|c|}\right) + \frac{2}{l}\times$$
$$\sum_{i=1}^{m-1}\frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}\right)}{\sqrt{\left|a_\tau^2\lambda_i^2+c\right|}} + \quad (39)$$
$$\sum_{i=m}^{\infty}\frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\sqrt{a_\tau^2\lambda_i^2+c}}.$$

We have two inequalities:

$$a_\tau^2\left(\frac{n\pi}{l}\right)^2 + c < 0, i = \overline{1, m-1},$$
$$a_\tau^2\left(\frac{n\pi}{l}\right)^2 + c \geq 0, i = \overline{m, \infty}.$$

All results can be applied to partial differential equation with first derivative to argument x:

$$\frac{\partial^2 w}{\partial t^2} = a_\tau^2\frac{\partial^2 w}{\partial x^2} - \nu\frac{\partial w}{\partial x} - \overline{c}u + \overline{f}\left(x,t\right).$$

In this case we use such transform:

$$w\left(x,t\right) = \exp\left(\frac{\nu x}{2a_\tau^2}\right)u\left(x,t\right).$$

After this transformation we have equation (16), in which new coefficients are:

$$c = \overline{c} - \frac{\nu^2}{4}, \; f\left(x,t\right) = \exp\left(-\frac{\nu x}{2a_\tau^2}\right)\overline{f}\left(x,t\right).$$

We would like to finish the one dimensional solution with a comparison of obtained here solutions of time inverse problem and solution from our paper [6]. The main distinction is in the form of both Green functions. In the paper [6] we have used the Green function for classical (parabolic) heat equation, but here we used Green function for the wave (hyperbolic) equation.

4 Simplifications for Homogeneous Initial Conditions

We would like to finish the one dimensional solution with a simplification for constant initial conditions:

$$u\big|_{t=0} = u_0(x) = u_0 = const,$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = w_0(x) = w_0 = const. \quad (40)$$

Solutions of time direct problem is such, see (22):

$$u(x,t) = u_0\int_0^l \frac{\partial}{\partial t}G(x,\xi,t)d\xi +$$
$$v_0\int_0^l G(x,\xi,t)d\xi + H\left(x,t\right). \quad (41)$$

Intensive steel quenching process with initial conditions (36) is very natural [8]-[12]. We have homogeneous equation (32) and homogeneous boundary conditions:

$$\frac{\partial^2 u}{\partial \tilde{t}^2} = a_\tau^2\frac{\partial^2 u}{\partial x^2} - cu,$$
$$\left.\left(\frac{\partial u}{\partial x} - \alpha_1 u\right)\right|_{x=0} = 0,$$
$$\left.\left(\frac{\partial u}{\partial x} + \beta_1 u\right)\right|_{x=l} = 0.$$

It means that we have:

$$H\left(x,t\right) = 0.$$

Solution (37) can be simplified:

$$u(x,t) = v_0\int_0^l G(x,\xi,t)d\xi + u_0\int_0^l \frac{\partial}{\partial t}G(x,\xi,t)d\xi$$
$$= I_0 + I_1.$$

We can integrate both integrals. For I_0:

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$$I_0 = v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2\sqrt{a_\tau^2\lambda_i^2+c}} \int_0^l \varphi_i(\xi)d\xi$$

$$= v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i\sqrt{a_\tau^2\lambda_i^2+c}} \times \quad (42)$$

$$\left[\sin(\lambda_i\xi) - \frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l}.$$

Similarly we can integrate second integral I_1:

$$I_1 = u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\cos\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i} \times \quad (43)$$

$$\left[\sin(\lambda_i\xi) - \frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l}.$$

We can use representations (38), (39) for testing our proposed method. For IQP we have positive initial temperature (hot steel sample) and negative heat flux: $u_0 > 0, v_0 < 0$ .Temperature field is given by formula:

$$u(x,t) = v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i\sqrt{a_\tau^2\lambda_i^2+c}_i}$$

$$\times\left[\sin(\lambda_i\xi) - \frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l} + \quad (44)$$

$$u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\cos\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i} \times$$

$$\left[\sin(\lambda_i\xi) - \frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l}.$$

For the heat flux we have an expression:

$$\frac{\partial}{\partial t}u(x,t) = u_0\int_0^l \frac{\partial^2}{\partial t^2}G(x,\xi,t)d\xi$$

$$+v_0\int_0^l \frac{\partial}{\partial t}G(x,\xi,t)d\xi = I_2 + I_0,$$

$$I_2 = u_0\int_0^l \frac{\partial^2}{\partial t^2}G(x,\xi,t)d\xi.$$

$$I_2 = u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sqrt{a_\tau^2\lambda_i^2+c}\,\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i}$$

$$\times\left[\frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi) - \sin(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l}.$$

Finally we have:

$$\frac{\partial}{\partial t}u(x,t) = u_0 \sum_{i=1}^{\infty}\left[\frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi) - \sin(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l}$$

$$\times\frac{\varphi_i(x)\sqrt{a_\tau^2\lambda_i^2+c}\,\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i} + \quad (45)$$

$$v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x)\sin\left(t\sqrt{a_\tau^2\lambda_i^2+c}\right)}{\|\varphi_i\|^2 \lambda_i\sqrt{a_\tau^2\lambda_i^2+c}} \times$$

$$\left[\sin(\lambda_i\xi) - \frac{\alpha_1}{\lambda_i}\cos(\lambda_i\xi)\right]\Bigg|_{\xi=0}^{\xi=l}.$$

Select arbitrary $u_0 > 0, v_0 < 0$ we can calculate $u(x,T) = u_T(x), \frac{\partial}{\partial t}u(x,T) = v_T(x)$. By solving time reverse problem (35) we find $\bar{u}_0 > 0, \bar{v}_0 < 0$. To use formula (44), (45) we can solve two different time reverse problem. First of them is problem with initial conditions:

$$u\big|_{\tilde{t}=0} = \bar{u}_T, \bar{u}_T = \int_0^l u_T(\xi)d\xi$$

$$\frac{\partial u}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -\bar{v}_T, \bar{v}_T = \int_0^l v_T(\xi)d\xi.$$

Second problem is with "small" initial conditions:

$$u(x,T) = u_T(x) - \bar{u}_T,$$

$$\frac{\partial}{\partial t}u(x,T) = -v_T(x) + \bar{v}_T.$$

The difference between $u_0 > 0, v_0 < 0$ and $\bar{u}_0 > 0, \bar{v}_0 < 0$ will show the precisely of our method.

5 Conclusions

We have constructed some one-dimensional solutions for direct and time inverse problems for

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hyperbolic heat equation. The solutions for determination of initial heat flux are obtained either in the form of Fredholm integral equation of 1st kind with continuous kernel or in closed analytical form.

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