Valuation of Discretely-sampled Variance Swaps under Correlated Stochastic Volatility and Stochastic Interest Rates

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Abstract: In this paper, we evaluate the price of discretely-sampled variance swaps using a equity-interest rate hybrid model. Our modeling framework extends the Heston stochastic volatility model by including the Cox-Ingersoll-Ross stochastic interest rates and imposes correlation between the stochastic interest rate and volatility. It is known that one limitation of the hybrid models is that the analytical pricing formula is often unavailable due to the non-affinity property of hybrid models. An efficient semi-closed form pricing formula is derived for an approximation of the fully correlated hybrid model. Our pricing formula which involves solving two phases of three-dimensional partial differential equations is evaluated through numerical implementations to confirm its accuracy.

Key–Words: variance swaps, Heston-CIR hybrid model, stochastic volatility, stochastic interest rates, realized variance.

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1 Introduction

One of the contemporary developments in the financial world is the emergence of hybrid models, which link products from different asset classes such as stock, interest rate and commodities. Hybrid models can be generally categorized into two different types, namely hybrid models with full correlations or hybrid models with partial correlations between the engaged underlyings. The hybrid models are often solved using analytical or numerical approaches according to the complexities involved.

There has been an active research field concerning hybrid models with partial correlations between asset classes due to less complexity involved. Majority of the researchers focused on including either correlation between the stock and interest rate, or correlation between the stock and the volatility of the stock [10, 11, 20, 4, 13]. The study of hybrid models with full correlations between all state variables also attracted attention for improved model capability. The relevant works can be referred from [18, 12, 5, 14, 8].

Despite the relevance of imposing correlations as described above, the attention should be drawn on the ability of the hybrid models to keep their analytical and computational tractability. One possible approach to keep tractability is to implement some modifications to the models' correlation structure so that the property of affine diffusion models could hold. This framework which was adopted from [7] guarantees that the state vector would have closed or semi closed-form expressions. This is applicable with the aid of the characteristic functions obtained from Fourier transform techniques. Other advantages of affine diffusion models include the ability to replicate numerous shapes of the term structure, and also provide adequate fitting either to the whole or the initial term structure [16].

We present an equity-interest rate hybrid model where the interest rate is driven by the CIR interest rate model whereas the equity price follows the dynamics of the Heston stochastic volatility model. Our focus is on the pricing of discretely-sampled variance swaps with full correlations between the stock price, interest rate as well as the volatility of the stock. Note that an analytical variance swaps pricing formula of partially correlated Heston-CIR hybrid model is derived in [3]. However, since the fully correlated Heston-CIR hybrid model is analytically untractable, we modify the model such that the affinity property holds for the approximation of the hybrid model. The characteristic functions of the approximated hybrid model are then derived and it leads to a semi-closed form formula for variance swaps.

2 Solution Techniques for Pricing Variance Swaps

In this section we describe details about the Heston-CIR hybrid model and discuss our solution techniques for pricing discretely-sampled variance swaps with full correlations between the asset classes. In particular, we apply the deterministic approximation by [9] to obtain a semi closed-form solution for the pricing formula.

2.1 The Heston-CIR hybrid model

Under the real world probability measure $\mathbb{P}$, the hybridization of the Heston-CIR model can be described as:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{\nu(t)}S(t)dB_1(t), \\ d\nu(t) &= \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dB_2(t), \\ dr(t) &= \alpha(\beta - r(t))dt + \eta\sqrt{r(t)}dB_3(t), \end{aligned} \tag{1}$$

where $r(t)$ is the stochastic instantaneous interest rate in which α determines the speed of mean reversion for the interest rate process, β represents the interest rate term structure and η measures the volatility of the interest rate. In the stochastic instantaneous variance process $\nu(t)$, κ is its mean-reverting speed parameter, θ is its long-term mean and σ is its volatility. In order to ensure that the square root processes in $\nu(t)$ and $r(t)$ are always positive, it is required that the Feller conditions ($2\kappa\theta \geq \sigma^2$ and $2\alpha\beta \geq \eta^2$) are satisfied, refer to [6, 15]. The correlations involved in the model are given by $(dB_1(t), dB_2(t)) = \rho_{12}dt = \rho_{21}dt$, $(dB_1(t), dB_3(t)) = \rho_{13}dt = \rho_{31}dt$ and $(dB_2(t), dB_3(t)) = \rho_{23}dt = \rho_{32}dt$, where $-1 \leq \rho_{ij} \leq 1$ for all $i, j = 1, 2, 3$.

For any $0 \leq t \leq T$, let

$$\begin{aligned} Z(t) \;=\; \exp\Bigg[& -\frac{1}{2}\int_0^t (\gamma_1(s))^2 ds - \int_0^t \gamma_1(s)dB_1(s) \\ & -\frac{1}{2}\int_0^t (\gamma_2(s))^2 ds - \int_0^t \gamma_2(s)dB_2(s) \\ & -\frac{1}{2}\int_0^t (\gamma_3(s))^2 ds - \int_0^t \gamma_3(s)dB_3(s) \Bigg], \end{aligned}$$

where $\gamma_1(t) = \frac{\mu - r(t)}{\sqrt{\nu(t)}}$, $\gamma_2(t) = \frac{\lambda_1\sqrt{\nu(t)}}{\sigma}$ and $\gamma_3(t) = \frac{\lambda_2\sqrt{r(t)}}{\eta}$ are the market prices of risk (risk premium) of Brownian processes $B_1(t)$, $B_2(t)$ and $B_3(t)$, respectively. Here, λ_j ($j = 1, 2$) is the premium of volatility risk as illustrated in [15], where Breeden's consumption-based model is applied to yield a volatility risk premium of the form $\lambda(t, S(t), \nu(t), r(t)) = \lambda\nu$ for the CIR square-root process [1].

Define three processes $\widetilde{B}_1(t)$, $\widetilde{B}_2(t)$ and $\widetilde{B}_3(t)$ such that

$$\begin{aligned} d\widetilde{B}_1(t) &= dB_1(t) + \gamma_1(t)dt, \\ d\widetilde{B}_2(t) &= dB_2(t) + \gamma_2(t)dt, \\ d\widetilde{B}_3(t) &= dB_3(t) + \gamma_3(t)dt. \end{aligned}$$

According to Girsanov's theorem, $\mathbb{E}^{\mathbb{P}}[Z(T)] = 1$ and there exists a risk-neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $Z(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathscr{F}_t}$ and $\widetilde{B}_1(t)$, $\widetilde{B}_2(t)$ and $\widetilde{B}_3(t)$ are Brownian motions under $\mathbb{Q}$. In what follows, the conditional expectation at time t is denoted by $\mathbb{E}_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\cdot \mid \mathscr{F}_t]$, where $\mathscr{F}_t$ is the filtration up to time t. Under $\mathbb{Q}$, equations (1) are transformed into the following form

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sqrt{\nu(t)}S(t)d\widetilde{B}_1(t), \\ d\nu(t) &= \kappa^*(\theta^* - \nu(t))dt + \sigma\sqrt{\nu(t)}d\widetilde{B}_2(t), \\ dr(t) &= \alpha^*(\beta^* - r(t))dt + \eta\sqrt{r(t)}d\widetilde{B}_3(t), \end{aligned} \tag{2}$$

where $\kappa^* = \kappa + \lambda_1$, $\theta^* = \frac{\kappa\theta}{\kappa + \lambda_1}$, $\alpha^* = \alpha + \lambda_2$ and $\beta^* = \frac{\alpha\beta}{\alpha + \lambda_2}$ are the risk-neutral parameters.

Under $\mathbb{Q}$, (2) can be re-written as

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^* - \nu(t)) \\ \alpha^*(\beta^* - r(t)) \end{bmatrix} dt + \Sigma \times L \times \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}, \tag{3}$$

where

$$\Sigma = \begin{bmatrix} \sqrt{\nu(t)} & 0 & 0 \\ 0 & \sigma\sqrt{\nu(t)} & 0 \\ 0 & 0 & \eta\sqrt{r(t)} \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1 - \rho_{12}^2}} & \sqrt{1 - \rho_{13}^2 - \left(\frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1 - \rho_{12}^2}}\right)^2} \end{bmatrix}$$

such that

$$LL^\top = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ * & 1 & \rho_{23} \\ * & * & 1 \end{bmatrix}.$$

Here, $W_1(t)$, $W_2(t)$ and $W_3(t)$ are three Brownian motions under $\mathbb{Q}$ such that $dW_1(t)$, $dW_2(t)$ and $dW_3(t)$ are mutually independent and satisfy the following relation

$$\begin{bmatrix} d\widetilde{B}_1(t) \\ d\widetilde{B}_2(t) \\ d\widetilde{B}_3(t) \end{bmatrix} = L \times \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}.$$

2.2 Valuation of variance swaps

Variance swaps were first launched in the 1990s due to the breakthrough of volatility derivatives in the market. Since the payment of a variance swap is only made in a single fixed payment at maturity, it is one type of forward contract which is traded over the counter. A typical formula for the measure of realized variance (RV) is

$$RV = \frac{AF}{N} \sum_{j=1}^{N} \left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})} \right)^2 \times 100^2, \quad (4)$$

where $S(t_j)$ is the closing price of the underlying asset at the j-th observation time t_j, T is the lifetime of the contract and N is the number of observations. AF is the annualized factor which follows the sampling frequency to convert the above evaluation to annualized variance points. Assume there are 252 business days in a year, then AF is equal to 252 for daily sampling frequency. Similarly, if the sampling frequency is every month or every week, then AF will be 12 and 52, respectively. The measure of realized variance requires monitoring the path of underlying stock price discretely, usually at the end of each business day. For this purpose,we assume equally discrete observations to be compatible with the real market, which reduces to $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

At maturity time T, variance swaps rates can be evaluated as $V(T) = (RV - K) \times L$, where K is the annualized delivery price for the variance swap and L is the notional amount of the swap in dollars. The notional amount can be expressed in two terms which are variance notional and vega notional. Variance notional gives the dollar amount of profit or loss obtained from the difference of one point between the realized variance and the delivery price. Vega notional on the other hand, calculates the profit or loss from one point of change in volatility points. Since it is the market practice to define the variance notional in volatility terms, the notional amount is typically quoted in dollars per volatility point.

In the risk-neutral world, the value of a variance swap with stochastic interest rates at time t is the expected present value of its future payoff, that is, $V(t) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} (RV - K)L \right]$. This value should be zero at $t = 0$ since it is defined in the class of forward contracts. The above expectation calculation involves the joint distribution of the interest rates and the future payoff which is complicated to evaluate. Thus, it would be more convenient to use the bond price as the numeraire since the joint dynamics can be diminished by taking advantage of the property $P(T, T) = 1$.

Since the price of a T-maturity zero-coupon bond at $t = 0$ is given by $P(0, T) = \mathbb{E}_0^{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} \right]$, we can determine the value of K by changing $\mathbb{Q}$ to the T-forward measure $\mathbb{Q}^T$. It follows that

$$\mathbb{E}_0^{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} (RV - K)L \right] = P(0, T)\mathbb{E}_0^T (RV - K), \quad (5)$$

where $\mathbb{E}_0^T(\cdot)$ denotes the expectation with respect to the T-forward measure $\mathbb{Q}^T$ at $t = 0$. Thus, the fair delivery price of the variance swap is defined as $K = \mathbb{E}_0^T[RV]$.

2.3 Dynamics under the T-forward measure

Under the T-forward measure, the valuation of the fair delivery price for a variance swap is reduced to calculating the N expectations expressed in the form of

$$\mathbb{E}_0^T \left[\left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})} \right)^2 \right] \quad (6)$$

for $t_0 = 0$, some fixed equal time period Δt and N different tenors $t_j = j\Delta t$ $(j = 1, \cdots, N)$. It is important to note that we have to consider two cases $j = 1$ and $j > 1$ separately. For the case $j = 1$, $S(t_{j-1}) = S(0)$ is a known value at time $t_0 = 0$, instead of an unknown value of $S(t_{j-1})$ for any other cases with $j > 1$. In the process of calculating this expectation, the value j, unless otherwise stated, is regarded as a constant. Hence both t_j and t_{j-1} are regarded as known constants.

Based on the tower property of conditional expectations, the calculation of expectation (6) can be separated into two phases in the following form

$$\mathbb{E}_0^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \right] = \mathbb{E}_0^T \left[\mathbb{E}_{t_{j-1}}^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \right] \right]. \quad (7)$$

In the first phase, the computation involved is

$$\mathbb{E}_{t_{j-1}}^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \right] = E_{j-1}, \quad (8)$$

and in the second phase, we need to compute

$$\mathbb{E}_0^T \left[E_{j-1} \right]. \quad (9)$$

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Note that the numeraire under $\mathbb{Q}$ is $N_{1,t} = e^{\int_0^t r(s)ds}$, whereas the numeraire under $\mathbb{Q}^T$ is $N_{2,t} = P(t,T)$ (refer [2]). Implementation of the Radon-Nikodym derivative for these two numeraires gives the new dynamics for (3) under the forward measure, $\mathbb{Q}^T$ as follows:

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) - \rho_{13}B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)} \\ \kappa^*(\theta^* - \nu(t)) - \rho_{23}\sigma B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)} \\ \alpha^*\beta^* - [\alpha^* + B(t,T)\eta^2]r(t) \end{bmatrix} dt + \Sigma \times L \times \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}, \tag{10}$$

where

$$B(t,T) = \frac{2\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)}{2\sqrt{(\alpha^*)^2+2\eta^2}+\left(\alpha^*+\sqrt{(\alpha^*)^2+2\eta^2}\right)\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)}.$$

2.4 Price evaluation techniques

As illustrated in (8), we shall focus on a contingent claim denoted as $U_j(S(t),\nu(t),r(t),t)$, whose payoff at expiry t_j is $\left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2$. Applying standard techniques in the general asset valuation theory, the PDE for U_j over $[t_{j-1},t_j]$ can be described as

$$\begin{aligned} &\frac{\partial U_j}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 U_j}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 U_j}{\partial \nu^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 U_j}{\partial r^2} + \\ &\rho_{12}\sigma\nu S\frac{\partial^2 U_j}{\partial S\partial\nu} + \left[rS - \rho_{13}B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}S\right] \\ &\frac{\partial U_j}{\partial S} + \left[\kappa^*(\theta^*-\nu) - \rho_{23}\sigma B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}\right] \\ &\frac{\partial U_j}{\partial \nu} + \left[\alpha^*\beta^* - (\alpha^* + B(t,T)\eta^2)r\right]\frac{\partial U_j}{\partial r} \\ &+\rho_{23}\sigma\eta\sqrt{\nu(t)}\sqrt{r(t)}\frac{\partial^2 U_j}{\partial\nu\partial r} + \rho_{13}\eta\sqrt{\nu(t)}\sqrt{r(t)}S \\ &\frac{\partial^2 U_j}{\partial S\partial r} = 0 \end{aligned} \tag{11}$$

with the terminal condition

$$U_j(S,\nu,r,t_j) = \left(\frac{S}{S(t_{j-1})} - 1\right)^2. \tag{12}$$

It can be seen that the system (10) is not in the affine form due to the existence of non-affine terms $\rho_{13}B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}S$, $\rho_{23}\sigma B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}$, $\rho_{23}\sigma\eta\sqrt{\nu(t)}\sqrt{r(t)}$ and $\rho_{13}\eta\sqrt{\nu(t)}\sqrt{r(t)}S$. Therefore, it is impossible to obtain the characteristic function of equation (11) by standard techniques in [7], thus an approximation for the PDE (11) is needed.

The expectation $\mathbb{E}^T(\sqrt{\nu(t)})$ with stochastic processes $\nu(t)$ of the CIR-type process given by (10) can be approximated by (refer [9]):

$$\begin{aligned} &\mathbb{E}^T(\sqrt{\nu(t)}) \approx \\ &\sqrt{q_1(t)(\varphi_1(t)-1) + q_1(t)l_1 + \frac{q_1(t)l_1}{2(l_1+\varphi_1(t))}} \\ &=: \Lambda_1(t), \end{aligned} \tag{13}$$

with

$$\begin{aligned} q_1(t) &= \frac{\sigma^2(1-e^{-\kappa^* t})}{4\kappa^*}, \quad l_1 = \frac{4\kappa^*\theta^*}{\sigma^2}, \\ \varphi_1(t) &= \frac{4\kappa^*\nu(0)e^{-\kappa^* t}}{\sigma^2(1-e^{-\kappa^* t})}. \end{aligned} \tag{14}$$

In order to avoid further complications during derivation of the characteristic function and present a more efficient computation, the above approximation is further simplified as

$$\mathbb{E}^T(\sqrt{\nu(t)}) \approx m_1 + p_1 e^{-Q_1 t} =: \widetilde{\Lambda_1}(t), \tag{15}$$

where

$$\begin{aligned} m_1 &= \sqrt{\theta^* - \tfrac{\sigma^2}{8\kappa^*}}, \quad p_1 = \sqrt{\nu(0)} - m_1, \\ Q_1 &= -\ln[p_1^{-1}(\Lambda_1(1) - m_1)]. \end{aligned} \tag{16}$$

The same procedure can be applied to find the expectation of $\mathbb{E}^T(\sqrt{r(t)})$:

$$\begin{aligned} &\mathbb{E}^T(\sqrt{r(t)}) \approx \\ &\sqrt{q_2(t)(\varphi_2(t)-1) + q_2(t)l_2 + \frac{q_2(t)l_2}{2(l_2+\varphi_2(t))}} \\ &=: \Lambda_2(t), \end{aligned} \tag{17}$$

with

$$\begin{aligned} q_2(t) &= \frac{\eta^2(1-e^{-\alpha^* t})}{4\alpha^*}, \quad l_2 = \frac{4\alpha^*\beta^*}{\eta^2}, \\ \varphi_2(t) &= \frac{4\alpha^* r(0)e^{-\alpha^* t}}{\eta^2(1-e^{-\alpha^* t})}, \end{aligned} \tag{18}$$

$$\mathbb{E}^T(\sqrt{r(t)}) \approx m_2 + p_2 e^{-Q_2 t} =: \widetilde{\Lambda_2}(t), \tag{19}$$

and

$$\begin{aligned} &m_2 = \sqrt{\beta^* - \tfrac{\eta^2}{8\alpha^*}}, \quad p_2 = \sqrt{r(0)} - m_2, \\ &Q_2 = -\ln[p_2^{-1}(\Lambda_2(1) - m_2)]. \end{aligned} \tag{20}$$

Utilizing the above expectations of both stochastic processes, we are able to find $\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$ by employing properties of dependent random variables and instantaneous correlation.

Replace the non-affine terms $\rho_{13}B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}S$, $\rho_{23}\sigma B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}$, $\rho_{23}\sigma\eta\sqrt{\nu(t)}\sqrt{r(t)}$ and $\rho_{13}\eta\sqrt{\nu(t)}\sqrt{r(t)}S$ with their expectations $\rho_{13}B(t,T)\eta\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})S$, $\rho_{23}\sigma B(t,T)\eta\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$, $\rho_{23}\sigma\eta\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$ and $\rho_{13}\eta\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})S$, and the equation (11) is in the affine form. It leads us to the solution for the modified PDE with terminal condition (12).

Proposition 1. *If the underlying asset follows the dynamic process (10) and a European-style derivative written on this underlying asset has a payoff function $U(S,\nu,r,T) = H(S)$ at expiry T, then the solution of the associated PDE system of the derivative value*

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}\nu S^2\dfrac{\partial^2 U}{\partial S^2} + \dfrac{1}{2}\sigma^2\nu\dfrac{\partial^2 U}{\partial \nu^2} + \dfrac{1}{2}\eta^2 r\dfrac{\partial^2 U}{\partial r^2} + \\ \rho_{12}\sigma\nu S\dfrac{\partial^2 U}{\partial S\partial\nu} + \left[rS - \rho_{13}B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}S\right] \\ \dfrac{\partial U}{\partial S} + \left[\kappa^*(\theta^* - \nu) - \rho_{23}\sigma B(t,T)\eta\sqrt{r(t)}\sqrt{\nu(t)}\right] \\ \dfrac{\partial U}{\partial \nu} + \left[\alpha^*\beta^* - (\alpha^* + B(t,T)\eta^2)r\right]\dfrac{\partial U}{\partial r} \\ +\rho_{23}\sigma\eta\sqrt{\nu(t)}\sqrt{r(t)}\dfrac{\partial^2 U}{\partial\nu\partial r} + \rho_{13}\eta\sqrt{\nu(t)}\sqrt{r(t)}S \\ \dfrac{\partial^2 U}{\partial S\partial r} = 0 \\ U(S,\nu,r,T) = H(S), \end{cases} \tag{21}$$

can be expressed in closed form as:

$$\begin{aligned} &U(x,\nu,r,t) \\ &= \mathcal{F}^{-1}\left[e^{C(\omega,T-t)+D(\omega,T-t)\nu+E(\omega,T-t)r}\mathcal{F}[H(e^x)]\right], \end{aligned} \tag{22}$$

using generalized Fourier transform method (see [17]), where $x = \ln S$, $i = \sqrt{-1}$, $\tau = T - t$, and ω is the Fourier transform variable, and

$$\begin{cases} D(\omega,\tau) = \dfrac{a+b}{\sigma^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}}, \\ E(\omega,\tau) = \dfrac{(\alpha^* + B(T-\tau,T)\eta^2) + d}{\eta^2}\dfrac{1-e^{d\tau}}{1-he^{d\tau}}, \\ C(\omega,\tau) = \dfrac{\kappa^*\theta^*}{\sigma^2}\left[(a+b)\tau - 2\ln\left(\dfrac{1-ge^{b\tau}}{1-g}\right)\right] \\ +\dfrac{\alpha^*\beta^*}{\eta^2}\left[(\alpha^* + B(T-\tau,T)\eta^2) + d)\tau\right. \\ \left.-2\ln\left(\dfrac{1-he^{d\tau}}{1-h}\right)\right] \\ -2\rho_{13}\eta\omega i\int_0^\tau \mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)}) \\ B((T-s),T)ds \\ +\rho_{13}\eta\omega i\int_0^\tau \mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})E(\omega,s)ds \\ -\rho_{23}\sigma\eta\int_0^\tau \mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})D(\omega,s) \\ B((T-s),T)ds \\ +\rho_{23}\eta\sigma\int_0^\tau \mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})D(\omega,s) \\ E(\omega,s)ds, \\ a = \kappa^* - \rho_{12}\sigma\omega i, \quad b = \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \\ g = \dfrac{a+b}{a-b}, \\ d = \sqrt{(\alpha^* + B(T-\tau,T)\eta^2)^2 - 2\eta^2\omega i}, \\ h = \dfrac{(\alpha^* + B(T-\tau,T)\eta^2) + d}{(\alpha^* + B(T-\tau,T)\eta^2) - d}. \end{cases} \tag{23}$$

It can be seen in Proposition 1 that the solution for the PDE (11) is defined universally without fully specifying the terminal condition, which is (12) in our case. For convenience, let $I = S(t_{j-1})$ and $x = \ln S$. Note that the terminal condition involves the Dirac delta function, it is necessary to handle the solution of PDE (11) via the generalized Fourier transform. In particular, the corresponding Fourier transformation is defined as

$$\mathcal{F}[e^{i\xi x}] = 2\pi\delta_\xi(\omega), \tag{24}$$

where ξ is any complex number and $\delta_\xi(\omega)$ is the generalized delta function satisfying

$$\int_{-\infty}^{\infty}\delta_\xi(x)\Phi(x)dx = \Phi(\xi). \tag{25}$$

Conducting the generalized Fourier transform for the payoff function $H(S) = (\frac{S}{I} - 1)^2$ with respect to

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x results in

$$\mathcal{F}\left[\left(\frac{e^x}{I}-1\right)^2\right]=2\pi\left[\frac{\delta_{-2i}(\omega)}{I^2}-2\frac{\delta_{-i}(\omega)}{I}+\delta_0(\omega)\right]. \tag{26}$$

As a result, the solution of the PDE (11) is derived as follows:

$$\begin{aligned}&U_j(S,\nu,r,I,\tau)\\&=\mathcal{F}^{-1}\Big[e^{C(\omega,\tau)+D(\omega,\tau)\nu+E(\omega,\tau)r}\\&\quad 2\pi\left[\frac{\delta_{-2i}(\omega)}{I^2}-2\frac{\delta_{-i}(\omega)}{I}+\delta_0(\omega)\right]\Big]\\&=\frac{e^{2x}}{I^2}e^{\widetilde{C}(\tau)+\widetilde{D}(\tau)\nu+\widetilde{E}(\tau)r}-\frac{2e^x}{I}e^{\widehat{C}(\tau)+\widehat{E}(\tau)r}+1,\end{aligned} \tag{27}$$

where $t_{j-1}\le t\le t_j$ and $\tau=t_j-t$. We denote $\widetilde{C}(\tau)$, $\widetilde{D}(\tau)$ and $\widetilde{E}(\tau)$ as $C(-2i,\tau)$, $D(-2i,\tau)$ and $E(-2i,\tau)$ respectively. The specific forms of $\widetilde{C}(\tau)$, $\widetilde{D}(\tau)$ and $\widetilde{E}(\tau)$ are

$$\begin{cases}\widetilde{E}(\tau)=\dfrac{(\alpha^*+B(t_j-\tau,T)\eta^2)+\widetilde{d}}{\eta^2}\left(\dfrac{1-e^{\widetilde{d}\tau}}{1-\widetilde{h}e^{\widetilde{d}\tau}}\right),\\ \widetilde{D}(\tau)=\dfrac{\widetilde{a}+\widetilde{b}}{\sigma^2}\left(\dfrac{1-e^{\widetilde{b}\tau}}{1-\widetilde{g}e^{\widetilde{b}\tau}}\right),\\ \widetilde{C}(\tau)=\dfrac{\kappa^*\theta^*}{\sigma^2}\left[(\widetilde{a}+\widetilde{b})\tau-2\ln\left(\dfrac{1-\widetilde{g}e^{\widetilde{b}\tau}}{1-\widetilde{g}}\right)\right]\\ +\dfrac{\alpha^*\beta^*}{\eta^2}\Big[(\alpha^*+B(T-\tau,T)\eta^2)+\widetilde{d})\tau\\ -2\ln\left(\dfrac{1-\widetilde{h}e^{\widetilde{d}\tau}}{1-\widetilde{h}}\right)\Big]\\ -2\rho_{13}\eta\int_0^\tau\mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})\\ B((T-s),T)ds\\ +2\rho_{13}\eta\int_0^\tau\mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})E(s)ds\\ -\rho_{23}\sigma\eta\int_0^\tau\mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})D(s)\\ B((T-s),T)ds\\ +\rho_{23}\eta\sigma\int_0^\tau\mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})D(s)E(s)ds,\\ \widetilde{a}=\kappa^*-2\rho_{12}\sigma,\quad \widetilde{b}=\sqrt{\widetilde{a}^2-2\sigma^2},\quad \widetilde{g}=\dfrac{\widetilde{a}+\widetilde{b}}{\widetilde{a}-\widetilde{b}},\\ \widetilde{d}=\sqrt{(\alpha^*+B(t_j-\tau,T)\eta^2)^2-4\eta^2},\\ \widetilde{h}=\dfrac{(\alpha^*+B(t_j-\tau,T)\eta^2)+\widetilde{d}}{(\alpha^*+B(t_j-\tau,T)\eta^2)-\widetilde{d}}.\end{cases} \tag{28}$$

In addition, $\widehat{C}(\tau)$ and $\widehat{E}(\tau)$ are the notations for $C(-i,\tau)$ and $E(-i,\tau)$ respectively. The specific forms of $\widehat{C}(\tau)$ and $\widehat{E}(\tau)$ are

$$\begin{cases}\widehat{E}(\tau)=\dfrac{(\alpha^*+B(t_j-\tau,T)\eta^2)+\widehat{d}}{\eta^2}\left(\dfrac{1-e^{\widehat{d}\tau}}{1-\widehat{h}e^{\widehat{d}\tau}}\right),\\ \widehat{C}(\tau)=\dfrac{\alpha^*\beta^*}{\eta^2}\Big[(\alpha^*+B(t_j-\tau,T)\eta^2)+\widehat{d})\tau\\ -2\ln\left(\dfrac{1-\widehat{h}e^{\widehat{d}\tau}}{1-\widehat{h}}\right)\Big]\\ -\rho_{13}\eta\int_0^\tau\mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})\\ B((T-s),T)ds\\ +\rho_{13}\eta\int_0^\tau\mathbb{E}(\sqrt{\nu(T-s)}\sqrt{r(T-s)})E(s)ds,\\ \widehat{d}=\sqrt{(\alpha^*+B(t_j-\tau,T)\eta^2)^2-2\eta^2},\\ \widehat{h}=\dfrac{(\alpha^*+B(t_j-\tau,T)\eta^2)+\widehat{d}}{(\alpha^*+B(t_j-\tau,T)\eta^2)-\widehat{d}}.\end{cases} \tag{29}$$

We shall continue to carry out the second phase to find out the expectation (6) as described in Section 2.3. We aim to compute the expectation $\mathbb{E}_0^T[E_{j-1}]$, which will finally leads us to obtain the fair delivery price of a variance swap.

The inner expectation had been worked out by our first phase of the computation, and the solution is

$$\begin{aligned}[E_{j-1}]=&\,e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)\nu(t_{j-1})+\widetilde{E}(\Delta t)r(t_{j-1})}\\&-2e^{\widehat{C}(\Delta t)+\widehat{E}(\Delta t)r(t_{j-1})}+1.\end{aligned} \tag{30}$$

We use $G_j(\nu(t_{j-1}),r(t_{j-1}))$ to denote the above expression. To proceed with the outer expectation, $\mathbb{E}_0^T[G_j(\nu(t_{j-1}),r(t_{j-1}))]$, the corresponding expectation is represented by

$$\begin{aligned}&\mathbb{E}_0^T[G_j(\nu(t_{j-1}),r(t_{j-1}))]\\&=\mathbb{E}_0^T\Big[e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)\nu(t_{j-1})+\widetilde{E}(\Delta t)r(t_{j-1})}\\&\quad-2e^{\widehat{C}(\Delta t)+\widehat{E}(\Delta t)r(t_{j-1})}+1\Big].\end{aligned} \tag{31}$$

Based on the assumption that $\nu(t_{j-1})$ and $r(t_{j-1})$ are dependent with correlation ρ_{23}, the above expression can be approximated as

$$
\begin{aligned}
&G_j(\nu(0), r(0)) = \mathbb{E}_0^T[G_j(\nu(t_{j-1}), r(t_{j-1}))] \\
&= e^{\widetilde{C}(\Delta t)} \cdot \mathbb{E}_0^T\left[e^{\widetilde{D}(\Delta t)\nu(t_{j-1})+\widetilde{E}(\Delta t)r(t_{j-1})}\right] \\
&-2e^{\widehat{C}(\Delta t)} \cdot \mathbb{E}_0^T\left[e^{\widehat{E}(\Delta t)r(t_{j-1})}\right] + 1 \\
&\approx e^{\widetilde{C}(\Delta t)} \cdot \exp[\widetilde{D}(\Delta t)(q_1(t_{j-1})(l_1 + \varphi_1(t_{j-1}))) \\
&+\widetilde{E}(\Delta t)(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1}))) \\
&+\tfrac{\widetilde{D}(\Delta t)^2}{2}(q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1}))) \\
&+\tfrac{\widetilde{E}(\Delta t)^2}{2}(q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))) \\
&+\widetilde{D}(\Delta t)\widetilde{E}(\Delta t)\rho_{23}\sqrt{q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1}))} \\
&\sqrt{q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))}] \\
&-2e^{\widehat{C}(\Delta t)} \cdot \exp[\widehat{E}(\Delta t)(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1}))) \\
&+\tfrac{\widehat{E}(\Delta t)^2}{2}(q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1})))] + 1.
\end{aligned} \tag{32}
$$

The solution of the first phase becomes the terminal condition for the second phase, and the expectation (6) is obtained. However, as mentioned in Section 2.3, we have to consider two cases $j = 1$ and $j > 1$ separately. The case $j > 1$ follows directly the expression in (32). For the case of $j = 1$, the known value of the asset price at t_0 gives

$$
G(\nu(0), r(0)) = \mathbb{E}_0^T\left[\left(\frac{S(t_1)}{S(0)} - 1\right)^2\right], \tag{33}
$$

whose values can be evaluated from Proposition 1.

Combining these two cases, we are able to find the fair delivery price of a variance swap involving summation for the whole period from $j = 1$ to $j = N$ as outlined in (4)

$$
\begin{aligned}
K &= \mathbb{E}_0^T[RV] \\
&= \frac{100^2}{T}\left[G(\nu(0), r(0)) + \textstyle\sum_{j=2}^N G_j(\nu(0), r(0))\right],
\end{aligned} \tag{34}
$$

where N is a finite number denoting the total sampling times of the variance swap contract. Our solution technique involves derivation of the characteristic function using approximations in order to fulfill the affinity property for the fully correlated hybrid model.

2.5 Formula validation

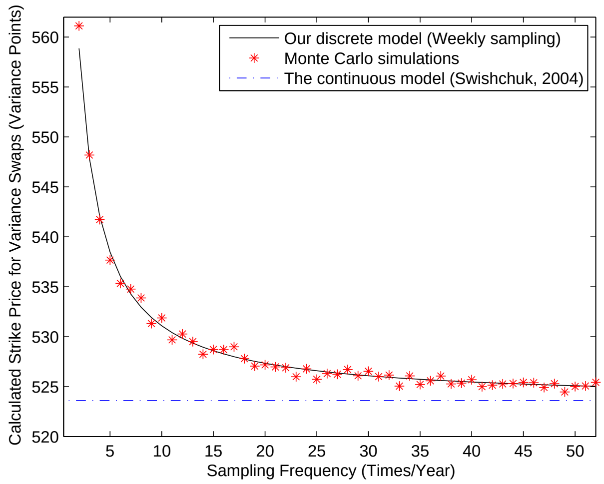

Figure 1: **Comparison of the delivery price of variance swaps produced between our model, continuous-sampling model and MC simulation.**

In order to analyze the performance of our approximation formula for evaluating prices of variance swaps as described in the previous section, we present some numerical results. Comparisons are made with the Monte Carlo (MC) simulation which resembles the real market, and the continuous-sampling variance swaps model in [19]. We perform the MC simulation using the Euler-Maruyama scheme with $200,000$ sample paths. In Figure 1, the results of all three compared models are plotted against each other, with the MC simulation results taken as the benchmark. It could be seen that our approximation formula results matches the MC simulation results quite well, whereas the continuous sampling model results do not provide a satisfactory match. To gain some insights of the relative difference between our formula and the MC simulation, we compare their relative percentage error. By taking $N = 52$ which is the weekly sampling frequency and $200,000$ path numbers, we find that the error produced is 0.07%, with further error reduction as path numbers reach $500,000$. Furthermore, even for small sampling frequency such as the quarterly sampling frequency when $N = 4$, our formula can be executed in 0.49 seconds compared to 27.7 seconds needed by the MC simulation.

3 Conclusion

This paper considers the Heston-CIR hybrid model for the pricing of discretely-sampled variance swaps with correlated stochastic volatility and stochastic interest rates. Since our model involves a full correlation structure between the state variables, we can only derive a semi-closed form approximation formula for the fair delivery price of variance swaps. We also consider the numerical implementation of our pricing formula which is validated to be fast and accurate via comparisons with the Monte Carlo simulation.

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