Heat Transfer with Partial Boiling in System with Double Wall and Double Fins

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Abstract: - This paper considers steady state heat conduction in 2D geometry consisting of a double wall and double fins. We analyse a problem when boiling process is present, thereby nonlinear conditions are examined. An approximate analytical solution is constructed using conservative averaging method for L-type domains.

Key-Words: - double wall with double fins, L-type domain, temperature fields, stationary heat-conduction problem, boiling boundary condition, conservative averaging method, finite difference scheme

1 Introduction

Numerous studies have been conducted on systems with extended surfaces where the entire finned element is made from the same material. But in many research areas more complex structures have to be developed. Here we consider an element (let's call it *a double wall with double fins*, see Fig.1) that consists of a rectangle with silicon nanowire arrays covered with fluorine carbon coating. By controlling and modifying structure geometries, such fabricated assemblies are usually designed for boiling heat transfer enhancement (see, e.g., [1] - [3]).

This article is preceded by a publication [4], where we constructed a mathematical model for stationary heat conduction in 2D double wall with double fins when no boiling process is present. But here we consider a situation when there is boiling occurring on the fins' surface. Just like in the publications [9] - [11], [13] conservative averaging method is exploited to construct an approximate solution for the given problem.

Our mathematical models are new and quite a bit different than those where relatively simple fin assemblies are considered (see, e.g., [5] - [12]).

2 Formulation for 2D Problem

In this paper we focus on the simplest case when the process is stationary. Here the structure is 2D and it has constant properties.

Because of the geometrical and thermal symmetry of the model, we can divide it into several symmetrical L-shaped parts. It is sufficient to describe and analyse the problem for only one of those parts (see Fig.1).

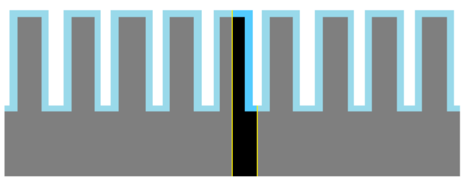

Fig.1: L-type domain

Such a domain can be reduced to the union of several nonintersecting rectangular subdomains with appropriate conjugation conditions along the lines connecting two neighbour domains. So, in this case we divide this sample into five rectangles (see Fig.2.).

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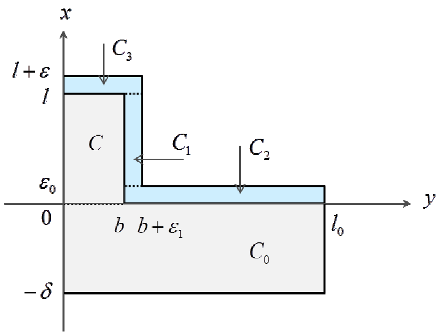

Fig.2: Definition of geometrical parameters for the sample

The temperature in the domain C_i is denoted by $V_i(x, y)$. And k_i and h_i represent thermal conductivity and heat transfer coefficient. For simplicity, let's assume that $k = k_0$ and $k_2 = k_3 = k_1$.

As the process is stationary, the temperature fields are described by Laplace equations:

$$\frac{\partial^2 V_i}{\partial x^2} + \frac{\partial^2 V_i}{\partial y^2} = 0, \; x, y \in C_i .$$

Let us impose the following boundary conditions.

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0(y)$$

specifies a heat flux at the wall surface $x = -\delta$. For simplicity reasons we henceforth assume the function $Q_0(y)$ to be constant, that is, $Q_0(y) = Q_0$.

Along the lines of symmetry $y = 0$ and $y = l_0$ symmetry boundary conditions

$$\frac{\partial V_i}{\partial n} = 0$$

are applied. Here n denotes the exterior normal to the boundary of the domains C_i.

But at the boundary $y = b + \varepsilon_1$ there is boiling occurring:

$$\left(\frac{\partial V_1}{\partial y} + \beta_1^1 V_1^m\right)_{y=b+\varepsilon_1} = 0,$$

where $\beta_1^1 = \dfrac{h_1}{k_1}$. As it was mentioned in the publications [13], the value of the index m should be taken equal to 3 or $3\frac{1}{3}$.

At the other sides of the sample we have boundary conditions for heat transfer between the body and the surrounding medium:

$$\frac{\partial V_i}{\partial n} + \beta_1^1 V_i = 0.$$

Assuming that there is no contact resistance between the connected parts, we also add conjugation conditions:

$$\left. V_i \right|_{x=\dots} = \left. V_j \right|_{x=\dots},$$

$$\left.\frac{\partial V_i}{\partial x}\right|_{x=\dots} = \frac{k_j}{k_i}\left.\frac{\partial V_j}{\partial x}\right|_{x=\dots},$$

$$\left. V_i \right|_{y=\dots} = \left. V_j \right|_{y=\dots},$$

$$\left.\frac{\partial V_i}{\partial y}\right|_{y=\dots} = \frac{k_j}{k_i}\left.\frac{\partial V_j}{\partial y}\right|_{y=\dots}.$$

Because of the boiling condition the model is nonlinear. Therefore, unlike it was done in our previous publications, we are going to take a slightly different approach to solve this problem.

3 Approximate Solution of Problem

Compared to the substrate, the upper layer is quite thin and the changes in temperature are considerably smaller here. It is therefore assumed that the temperature does not vary across the layer thickness. Under appropriate conjugation conditions we derive approximate expressions for the temperatures in the upper layer:

$$V_2(x, y) = v_2(y) = V_0(0, y), \tag{1}$$

$$V_1(x, y) = v_1(x) = V(x, b), \tag{2}$$

$$V_3(x, y) = v_3(y) = V(l, y). \tag{3}$$

Therefore only those steady-state conduction problems that are defined for the basic layer need to be solved. The Laplace equations

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{4}$$

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} = 0 \tag{5}$$

require boundary conditions on all sides of the new domain in which the solution is to be obtained.

From Section 2 we have

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0, \tag{6}$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = 0, \; \left.\frac{\partial V_0}{\partial y}\right|_{y=l_0} = 0, \tag{7}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = 0. \tag{8}$$

We derive expressions for the boundary conditions at $x = 0$, $x = l$, and $y = b$ by using appropriate

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conjugation conditions and formulae (1) – (3). For example, at $x=0$ we have

$$\left(\frac{\partial V_2}{\partial x}+\beta_1^1 V_2\right)\Bigg|_{x=\varepsilon_0}=\frac{1}{k_1}\left(k_0\frac{\partial V_0}{\partial x}+h_1 V_0\right)\Bigg|_{x=0}=0$$

or

$$\left(\frac{\partial V_0}{\partial x}+\beta_0^1 V_0\right)\Bigg|_{x=0}=0,\ y\in\left(b,l_0\right). \qquad (9)$$

And

$$\left(\frac{\partial V}{\partial x}+\beta_0^1 V\right)\Bigg|_{x=l}=0,\ y\in\left(0,b\right) \qquad (10)$$

$$\left(\frac{\partial V}{\partial y}+\beta_0^1 V^m\right)\Bigg|_{y=b}=0,\ x\in\left(0,l\right) \qquad (11)$$

at $x=l$ and $y=b$.

The continuity of temperature and heat flux at the interface between the wall and the fin is ensured by choosing these conjugation conditions:

$$V_0\big|_{x=-0}=V\big|_{x=+0}, \qquad (12)$$

$$\frac{\partial V_0}{\partial x}\bigg|_{x=-0}=\frac{\partial V}{\partial x}\bigg|_{x=+0}. \qquad (13)$$

3.1 Solution for Boiling on Fin

Using conservative averaging method (see [4], etc.), we are going to transform the given mathematical model into a more usable form.

Let's introduce the following average value of the function $V(x,y)$:

$$v(x)=\rho\int_0^b V(x,y)dy,\ \rho=b^{-1}. \qquad (14)$$

When integrating equation (4), we get:

$$\frac{d^2v}{dx^2}+\rho\frac{\partial V}{\partial y}\bigg|_{y=0}^{y=b}=0,\ x\in\left(0,l\right). \qquad (15)$$

For the temperature in the fin we will use the simplest approximation assuming that it is constant with respect to the argument y. So

$$v(x)=v_1(x). \qquad (16)$$

It may seem to be a first rough approximation pending a more accurate estimate.

Using boundary conditions (11) and (8), the differential equation (15) can be written in the form

$$\frac{d^2v}{dx^2}-\lambda^2 v^m(x)=0,\ x\in\left(0,l\right), \qquad (17)$$

where

$$\lambda^2=\rho\beta_0^1.$$

Applying operator (14) to (10) we get a boundary condition:

$$v'(l)+\beta_0^1 v(l)=0. \qquad (18)$$

3.1.1 Numerical Method for Problem in Fin

Now that we have an equation that depends on a single variable x, we can solve it using a finite difference scheme. Let's rewrite equation (17) as follows:

$$\frac{d^2v}{dx^2}-\lambda^2 v v^{m-1}=0. \qquad (19)$$

We are going to approximate this ordinary differential equation by a nonlinear combination

$$\frac{v_{i+1}^n-2v_i^n+v_{i-1}^n}{h_x^2}-\lambda^2\left(v_i^{n-1}\right)^{m-1}v_i^n=0, \qquad (20)$$

where v_i is a difference approximation of $v(x_i)$, and the grid points are:

$$x_i=l\left(1-\frac{i}{N}\right),\ i=\overline{0,N},\ \text{with}\ h_x=\frac{l}{N}.$$

But the superscript n denotes the number of iteration (as the differential equation is nonlinear we have to use iterative method to solve it).

Let's rewrite (20) like this (see [14]):

$$A_i v_{i-1}^n-C_i v_i^n+B_i v_{i+1}^n=-F_i,$$

$$F_i=0,\ A_i=B_i=\frac{1}{h_x^2}, \qquad (21)$$

$$C_i=A_i+B_i+\lambda^2\left(v_i^{n-1}\right)^{m-1}.$$

For the derivative in the boundary condition (18) we obtain a second order approximation using the differential equation (19) and its approximation (20) (see [14]):

$$\frac{dv}{dx}\bigg|_{x=l}=\frac{v_1^n-v_0^n}{h_x}+\frac{h_x}{2}\lambda^2\left(v_0^{n-1}\right)^{m-1}v_0^n+O\left(h_x^2\right).$$

So, we get an approximation of the condition (18) as

$$\frac{v_1^n-v_0^n}{h_x}+\frac{h_x}{2}\lambda^2\left(v_0^{n-1}\right)^{m-1}v_0^n+\beta_0^1 v_0^n=0$$

or

$$v_0^n=\chi_1 v_1^n$$

with coefficient χ_1 as

$$\chi_1=\left[h_x\left(\frac{1}{h_x}-\frac{h_x}{2}\lambda^2\left(v_0^{n-1}\right)^{m-1}-\beta_0^1\right)\right]^{-1}.$$

For other boundary condition we use conjugation condition (12) at a point $\left(0,b\right)$:

$$v_N^n=V_0\left(0,b\right). \qquad (22)$$

So, for the first iteration we have formulae:

$$v_i = \alpha_{i+1} v_{i+1} + \beta_{i+1}, \qquad (23)$$

with

$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i}, \; \beta_{i+1} = \frac{A_i \beta_i + F_i}{C_i - \alpha_i A_i}, \; i = \overline{1, N-1}$$

and

$$\alpha_1 = \chi_1, \; \beta_1 = 0.$$

From (21) and $\beta_1 = 0$ it can be seen that $\beta_i = 0$ for all $i = \overline{1, N}$.

Before we continue with the second iteration, we must seek a solution for the right part of the wall.

Let's note that for the zero-th iteration we can choose a linear function that satisfies the boundary conditions (6), (10):

$$v(x) = Q_0 \left(l - x + \frac{1}{\beta_0^1} \right), \; x \in [0, l].$$

3.2 Solution for Wall

Solution for the wall can be obtained by using the method of conservative averaging (see [4], [10] etc.).

Let us approximate the temperature field $V_0(x, y)$ using exponential approximation in the x direction. Therefore we use the following representation

$$V_0(x, y) = \\ = g_0(y) + \left(e^{-dx} - 1\right) g_1(y) + \left(1 - e^{dx}\right) g_2(y), \quad (24)$$

where the unknown functions g_i, $i = 0,1,2$, are found in such a way that the function (24) satisfies appropriate boundary conditions and integral equality (for more detailed information, see [4]):

$$v_0(y) = d \int_{-\delta}^{0} V_0(x, y) dx, \; d = \delta^{-1}. \qquad (25)$$

When integrating (24) over the segment $(-\delta, 0)$, we get

$$v_0(y) = g_0(y) + (e - 2) g_1(y) + e^{-1} g_2(y). \; (26)$$

Before we continue, we are going to divide the wall into two parts, where the right part of the wall occupies the domain $x \in (-\delta, 0)$, $y \in (b, l_0)$ but the left one is for $x \in (-\delta, 0)$, $y \in (0, b)$.

3.2.1 Solution for Right Part of Wall

Upon applying the boundary condition (9) to (24), we get

$$g_0(y)\left(-\frac{1}{2} d + \beta_0^1 + \frac{1}{2} d e^2 \right) \\ + v_0(y)\left(\frac{1}{2} d - \frac{1}{2} d e^2 \right) \\ + Q_0 \left(-\frac{1}{2} - e + \frac{1}{2} e^2 \right) = 0.$$

We can rewrite this expression as

$$g_0(y) = v_0(y) a_0 + Q_0 b_0, \qquad (27)$$

where

$$a_0 = \frac{d - de^2}{d - 2\beta_0^1 - de^2}, \; b_0 = -\frac{1 + 2e - e^2}{d - 2\beta_0^1 - de^2}.$$

Substituting (27) into the expression (24), reduces it to

$$V_0(x, y) = \\ = v_0(y) \begin{pmatrix} a_0 + \frac{1}{2}(a_0 - 1)\left(e^{-dx} - 1\right) + \\ + e^2 \frac{1}{2}(1 - a_0)\left(1 - e^{dx}\right) \end{pmatrix} \\ + Q_0 \begin{pmatrix} b_0 + \frac{1}{2}(b_0 + \delta)\left(e^{-dx} - 1\right) + \\ + \left(\delta e - e^2 \frac{1}{2}(\delta + b_0) \right)\left(1 - e^{dx}\right) \end{pmatrix}. \quad (28)$$

You can see that the function depends only on one unknown - the function $v_0(y)$. To find that the main differential equation (5) is integrated in the x direction

$$d \frac{\partial V_0}{\partial x} \Big|_{x=-\delta}^{x=0} + \frac{d^2 v_0}{dy^2} = 0. \qquad (29)$$

The boundary condition (6) and the definition (28) allows us to rewrite this identity as an ordinary differential equation

$$\frac{d^2 v_0}{dy^2} - \kappa^2 v_0(y) = \gamma Q_0, \quad y \in (b, l_0) \qquad (30)$$

where

$$\kappa^2 = -\frac{1}{2} d^2 \left(1 - e^2\right)\left(1 - a_0\right) = d\beta_0^1 a_0,$$

$$\gamma = -\frac{1}{2} d \left(b_0 d \left(e^2 - 1\right) + (1 - e)^2 \right).$$

But integration of the boundary condition (7_2) gives

$$v_0'(l_0) = 0. \qquad (31)$$

The solution of the problem (30), (31) is

$$v_0(y) = c_2 \left(e^{\kappa y} + \mu_0 e^{-\kappa y} \right) - \frac{\gamma Q_0}{\kappa^2} \qquad (32)$$

with

$$\mu_0 = e^{2\kappa l_0}$$

and c_2 as an unknown constant.

To ensure that the solution of the given problem is continuous in all the domain, it is necessary for the temperatures $V_0(x,y)$, $V(x,y)$ to coincide at the contact point $x=0$, $y=b$ between the fin and the right part of the wall. So, from the condition (22) and definition (28) we get:

$$v_N^n = V_0(0,b) = a_0 v_0(b) + b_0 Q_0(b). \quad (33)$$

Previously we assumed that the solution in the fin is constant in the y direction. That means (33) is true for all $y \in [0,b]$.

3.2.2 Solution for Left Part of Wall

When deriving equation for the left part of the wall, it is important to remember that for $x=0$ the functions $V_0(x,y)$, $V(x,y)$ must fulfil the conjugation conditions (12), (13). So, from (13) and (23) we get that

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = d\frac{dv}{dx}\bigg|_{x=+0} =$$

$$= d\left[\frac{v_{N-1}^n - v_N^n}{h_x} + \frac{h_x}{2}\lambda^2\left(v_N^{n-1}\right)^{m-1} v_N^n\right] =$$

$$= d\left[\frac{\alpha_N - 1}{h_x} + \frac{h_x}{2}\lambda^2\left(v_N^{n-1}\right)^{m-1}\right] v_N^n. \quad (34)$$

Now, using (33) formula (34) becomes

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} =$$

$$= d\left[\frac{\alpha_N - 1}{h_x} + \frac{h_x}{2}\lambda^2\left(v_N^{n-1}\right)^{m-1}\right]\left[a_0 v_0(b) + b_0 Q_0(b)\right].$$

From this expression we see that we should use iteration process to solve the equation for the left part of the wall as well. Before we find this equation, let's rewrite the last expression as

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = k_0^{n-1} v_0^n + \gamma_0^{n-1} Q_0. \quad (35)$$

Here

$$k_0^{n-1} = a_0 d\left[\frac{\alpha_N - 1}{h_x} + \frac{h_x}{2}\lambda^2\left(v_N^{n-1}\right)^{m-1}\right],$$

$$\gamma_0^{n-1} = b_0 d\left[\frac{\alpha_N - 1}{h_x} + \frac{h_x}{2}\lambda^2\left(v_N^{n-1}\right)^{m-1}\right].$$

If we now use (29) and conditions (35) and (6), then integrated equation becomes

$$\frac{d^2 v_0}{dy^2} + k_0^{n-1} v_0^n = -\gamma_0^{n-1} Q_0 - d Q_0. \quad (36)$$

From (7_1) we get a boundary condition:

$$v_0'(0) = 0. \quad (37)$$

So, the solution of the problem (36), (37) is:

for $k_0^{n-1} < 0$

$$v_0(y) = c_1\left(e^{\sqrt{-k_0^{n-1}}y} + e^{-\sqrt{-k_0^{n-1}}y}\right) - \frac{\left(\gamma_0^{n-1} + d\right)}{k_0^{n-1}} Q_0; \quad (38)$$

for $k_0^{n-1} > 0$ then

$$v_0(y) = c_1 \cos\left(\sqrt{k_0^{n-1}}\, y\right) - \frac{\left(\gamma_0^{n-1} + d\right)}{k_0^{n-1}} Q_0. \quad (39)$$

Another possibility when constructing a 1D equation for the left part of the wall is to use expression (28) and find the derivative in the direction of x. In that case

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = -\frac{1}{2} d^2\left[A_0 v_0(y) + B_0 Q_0\right],$$

$$A_0 = (a_0 - 1)\left(1 - e^2\right),$$

$$B_0 = (b_0 + \delta) + 2\left(\delta e - e^2 \frac{1}{2}(\delta + b_0)\right).$$

In this case we get a linear equation that is not dependent of iterations:

$$\frac{d^2 v_0}{dy^2} - \frac{1}{2} d^2 A_0 v_0(y) = \frac{1}{2} d^2 B_0 Q_0 - d Q_0. \quad (40)$$

3.2.3 Conjugation of Solutions

We have just found solution to the given problem. But we are still left with finding the unknown constants c_1 and c_2 in formulae (32) and (38)/(39). Those can be found from conditions:

$$v_0(b-0) = v_0(b+0),$$

$$\frac{d}{dx} v_0(b-0) = \frac{d}{dx} v_0(b+0).$$

Both constants are iteration-dependent. The iteration algorithm can be organised in this way: at first we perform the first iteration in the fin, finding α_i and constants c_1, c_2. After that we use (33) and (23) to get a value for $V_0(0,b)$ and calculate temperature in the fin. Afterwards we proceed with the iteration process.

For that other case when 1D equation for the left part of the wall, equation (40), can be solved without iterations, it is quite easy to find the unknown constants and the mean temperature in the wall - $v_0(y)$. Knowing this we can use representation (28) to calculate $V_0(0,y)$. According to (12) the latter gives v_N^n. So, the assumption that the temperature in the fin is constant in the y direction breaks down.

4 Conclusion

We have given a formulation of a problem for stationary heat conduction in 2D double wall with double fins when boiling is present. To solve this problem and construct an approximate analytical solution we have used conservative averaging method and finite difference scheme.

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