An Overview of the Spectral Estimation Methods

DIONYSIOS P. KALOGERAS
Telecommunications and Informatics
University of Peloponnesus
Kanigos St 14, Patras
GREECE
dkalogerias@uop.gr

Abstract: This paper presents an overview of the current work in Spectral Estimation methods focusing in the comparison of the Amplitude and Phase Estimation method and the Capon one, for the 1 and the 2 dimensional spectra. Unfortunately, their computational complexity tends to limit their usage in several cases, a problem that has previously been addressed by different authors. Finally this paper proposes implementations of faster algorithms in some cases.

Key-Words: Capon, APES, Spectrum estimation Methods, parametric, non parametric

1 Introduction

Many engineering problems, including high-resolution synthetic aperture radar (SAR) imaging and time-series analysis can be cast as spectral estimation problems (Erik Larsson, Jian LI, 2003). The classical approaches to spectral analysis include the discrete Fourier transform (DFT) and its variants. While the spectral resolution of these algorithms is rather poor (Erik Larsson, Jian LI, 2003), they do not make any a priori assumptions on the spectral shape and therefore they are very robust. An emerging class of methods that can overcome the resolution limit of the DFT and trade spectral resolution against robustness and statistical stability is the group of so-called adaptive filter-bank methods, which includes the classical Capon algorithm [1], and the more recent amplitude and phase estimation (APES) approach (Li, Stoica, 1996). Even though both APES and Capon are essentially equivalent to a Fourier transform as the amount of data goes to infinity [2] [3], their finite sample properties have shown to be promising in a number of experiments with both simulated and measured data.

This work examines the most known spectral estimation methods and presents the results of processing some 1D and 2D signals. To accomplish this aim, paper is structured in three parts, as follows:

a. In the first part, the basic theory of the spectral estimation methods is outlined
b. In the second part, some known signals are processed and finally some data collected by SAR systems is processed and the results are commented.

c. In the final and third part, conclusions and proposals for future research are presented.

2 The Spectral estimation Methods

Problem

According to the literature [4], the Spectral Estimation is defined as:

"from a finite record of a stationary data sequence, estimate how the total power is distributed over frequencies, or more practically, over narrow spectral bands (frequency bins)".

The methods that can lead the researcher to the Spectral Estimation as given above, are divided to:

a. Classical (Nonparametric) Methods: These methods parameterize \( \phi(\omega) \) by a finite-dimensional model. An example of this method cold be the passing of the data through a set of band-pass filters and the measuring the filter output powers.

b. Parametric (Modern) Approaches; They implicitly smooth the \( \phi(\omega) \) for \( \omega \rightarrow \pi \), by assuming that \( \phi(\omega) \) is nearly constant over bands where \( [\omega-\beta\pi, \omega+\beta\pi] \) and \( \beta<<1 \). An example model of this method is the following: the data is considered to be as a sum of a few damped sinusoids which parameters are asked to be estimated.

By comparing the 2 methods it is concluded [4] that the second method is more general than the first and requires \( \pi/\beta>1 \), in order to ensure that the numbers of estimated values \( (2\pi/2\pi\beta=1/\beta) \) are greater than \( N \). The trade-off between the above methods is the Robustness vs. Accuracy. While,
parametric methods may offer better estimates if data closely agrees with assumed model, the nonparametric may be better in every else case [4].

2.1 The theory
A simple way to estimate the spectrum in the case of missing data would be to compute the conventional DFT spectrum with the missing data set to zero [5] [2] [4]. As it is well known that this usually yields an estimated spectrum with serious artifacts, a number of methods for spectral estimation of gapped data have been developed:
The CLEAN algorithm [5] that estimates the spectrum by fitting and removing sinusoids to the data in an iterative fashion. Under certain circumstances this procedure can be interpreted as a deconvolution, and therefore carried out efficiently [5]. Many variants of CLEAN exist, but they are considered to be out of this paper field of interest.
The so-called multitaper [5] compute spectral estimates that are quadratic forms of the data. The coefficients in the corresponding quadratic functions are optimized according to certain criteria, and depend only on the gapping pattern. These methods are computationally rather burdensome, and it appears that they cannot overcome the resolution limit of the DFT [5].
A large body of work [5] for some early contributions) have addressed the problem of extrapolating incomplete Fourier data. There are also methods that attempt to fit an autoregressive (AR) model to the data and use this model to interpolate the gaps; however, doing so is often difficult: the AR model must be strictly stable, and hence the extrapolated samples decay to zero as the gap lengths grow.
In [6] the authors proposed an extension of the APES method to gapped data; this algorithm was later extended to two-dimensional data [7] This method was called gapped-data APES (GAPES) and based on an interpolation of the missing data under certain constraints. The GAPES method, can deal with quite general sampling patterns, but it is computationally rather complex. Furthermore, the ideas of [7] [6] [5] appear unfeasible for extending the Capon method to gapped data (which would be of some interest since the resolution capabilities of Capon can be higher than those of APES). In the following paragraphs these methods are going to be presented with analytical ways. But first of all, we must define the main terms used in spectral analysis.

The power spectrum: for the random signals, Power Spectral Density (PSD) of Random Signals is given by the following:
Let \( \{x(n)\} \) be wide-sense stationary (WSS) sequence with \( E[x(n)] = 0 \).
Then \( r(k) = E[x(n)x^*(n-k)] \).
With the properties of autocorrelation function \( r(k) \) to be:
\[
0 \leq r(k) \leq |r(k)|, \text{ for all } k
\]
That means that in the domain of \( \omega \), as also as the autocorrelation function are given by:
\[
P(\omega) = \sum_{k=-\infty}^{\infty} r(k) e^{-j\omega k}
\]
\[
r(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\omega) e^{j\omega k} d\omega
\]
Equation 1. The spectral estimation equation

Or for \( \omega = 2\pi f \)
\[
P(f) = \sum_{k=-\infty}^{\infty} r(k) e^{-j2\pi fk}
\]
\[
r(k) = \int_{-1/2}^{1/2} P(f) e^{j2\pi fk} df
\]
Equation 2. The spectral estimation equation

The following fig. 1 describes graphically the formulas noted above.

Fig 1. The graphically representation of the spectral estimation concept.

2.1.1 The nonparametric Methods
For the non parametric methods we have the following key-terms;

a. Periodogram: According to the literature there are possible Improvements to the Filter Bank Approach [4];

b. Correlogram: This term is defined as [1] [4] [8].

2.1.1.3 General Comments
By commenting the periodogram and correlogram functions we conclude that they provide very poor estimation [1] [2] [8] [4] of the real \( P(\omega) \), because
the variances of them are high. The reason is that they are come from a single realization of a random process.

2.1.2 The nonparametric Methods

The main remarks about parametric methods are the following \[4\] \[1\]:

\[ P(f) \text{ is described by 2 unknowns: } r(0) \text{ and } \sigma_f \text{. Once we know } r(0) \text{ and } \sigma_f \text{, we know } P(f) \text{ that means that we know the PSD } \[4\].

These methods assume no knowledge on \( P(f) \) so they have use too many unknowns. Also parametric Methods attempt to estimate \( r(0) \) and \( \sigma_f \).

\[ P(f) = \frac{r(0)}{\sqrt{2\pi\sigma_f}} e^{-\frac{1}{2}(f/\sigma_f)^2}, |f| < \frac{1}{2} \]

Fig 2. The PSD scheme

Researcher can obtain better estimates by using an appropriate data model with fewer unknowns. If data model wrong, \( P(f) \) will always be biased.

\[ \text{Estimate} \]

\[ \text{True PSD} \]

f.

So in order to use parametric methods, reasonably correct ‘a priori’ knowledge on data model is necessary. This is considered to be the main disadvantage of the methods.

The remarks here are:

We mostly consider real valued signals here.

The factors \( a_1, \ldots, a_p, b_1, \ldots, b_q \) are real coefficients.

Any continuous PSD can be approximated arbitrarily close by a rational PSD.

\[ H(\omega) = \frac{B(\omega)}{A(\omega)}, \]

Equation 3 The PSD rational filter

2.1.2.1 The Capon Method

The Capon Filter Design is focusing in the following problem:

Define the \( \min(h^*R_h) \) subject to \( h^*a(\omega)=1 \) This give as solution the \( h_o=R_h1/a(\omega)R^1a(\omega) \).

Then the power at the filter output is \( E\{|y(t)|^2\}=h_o^*R_hy_o=1/a(\omega)R^1a(\omega) \), which should be the power of \( y(t) \) in a passband centered on \( \omega \).

Consider the bandwidth as \( B=1/(m+1)=1/(\text{filter length}) \)

Capon uses one bandpass filter only, but it splits the N-data point sample into \( N/m \) subsequences of length \( m \) with maximum overlap.

2.1.2.2 The APES Method

Amplitude spectrum estimation (APES) is of great interest in a wide range of applications, including speech processing and analysis, time series analysis, geophysics, biomedical engineering, synthetic aperture radar imaging, etc \[9\]. The Amplitude and phase estimator (APES), the amplitude spectrum Capon (ASC) and the power spectrum Capon (PSC), are three popular high resolution spectral estimators, that can be interpreted as matched filter-band spectral estimators, \[10\]–[14]. However, direct implementation of these methods is prohibited due to the huge amount of computational power required.

2.1.2.2.1 The 1-D APES

First, the 1-D APES estimation is considered; the extension to the 2-D case is discussed later. Let , \( y(n), n=0,1,2,\ldots \) represent the data sequence under consideration. The matched filter-band spectral estimators aim in the design of a data dependent finite-impulse-response (FIR) filter of length \( M \), that passes the frequency \( \omega \) in \( y(n) \) without distortion, while at the same time attenuates all other frequencies \[1\] \[5\] \[9\] \[10\] \[4\]. Notice that in this paper forward type filter-band spectral estimators will be considered, for 1-D as well as for 2-D data sequences. however, the proposed method can be extended to handle forward-backward type estimators. Let \( h_{M} \in C^{M \times 1} \) be the FIR filter sought...
and $y_M(n)$ be the corresponding repressor vector, defined as

$$y_M(n) = [y(n), y(n-1), \ldots, y(n-M+1)]^T$$

Where $L = N - M + 2$

The above equation, which provides the 1-D APES spectrum, can be expressed in terms of three auxiliary polynomials:

$$\phi_{1hD}(\omega) = (L - \phi_{1g}(\omega))y_M(\omega) + \phi_{1g}(\omega)$$

Where

$$\phi_{1h}(\omega) = a_{1h}(\omega)R^{-1}_{M1,M2}a_M(\omega)$$

$$\phi_{1g}(\omega) = a_{1g}(\omega)G_{M2,L2}a_{L2}(\omega)$$

$$\phi_{1g}(\omega) = a_{1g}(\omega)H_{L1,L2}a_{L2}(\omega).$$

### 2.1.2.2.2 2-D APES

The 2-D APES as shown in [1] [9] [5] [4] [10] is defined as:

$$\Phi_{2D}(\omega_1, \omega_2) = \phi_{2h}(\omega_1, \omega_2)K^{-1}_{M1,M2}a_{M2}(\omega_1, \omega_2)$$

Where

$$\phi_{2h}(\omega_1, \omega_2) = a_{2h}(\omega_1, \omega_2)K^{-1}_{M1,M2}a_{M2}(\omega_1, \omega_2)$$

$$\phi_{2g}(\omega_1, \omega_2) = a_{2g}(\omega_1, \omega_2)G_{M2,L2}a_{L2}(\omega_1, \omega_2)$$

$$\phi_{2g}(\omega_1, \omega_2) = a_{2g}(\omega_1, \omega_2)H_{L1,L2}a_{L2}(\omega_1, \omega_2).$$

### 3. Case Studies

In the following paragraphs selected cases studies are presented.

#### 3.1 Case Study 2 - Sinusoids in Noise:

We consider a time-series $x(n)$ consisting of two complex sinusoids with amplitudes 1 and 0.75 and frequencies 0.75 and 0.8, respectively, embedded in white Gaussian noise with variance 0.01, by using the matlab code. In the following figure 5, the real part of the signal is shown.

![Fig 5 The result of the signal.](image)

#### 3.2 Case Study 4 APES vs CAPON

In this case study the data used for is shown in the following table 2. The graphs produced are shown in fig 9 to . By comparing the results the following issues have to be pointed out:

- a. APES seems to have better resolution of the amplitudes of the spectrum frequencies.
- b. CAPON has the ability to present more information.
- c. CAPON pulls its estimations to lower amplitude values, while APES seems to be more effective into this.
- d. Both methods are computationally precise.
- e. CAPON overestimates the existence of information in the spectrum.

The times needed to compute the graphs are given to the following table 1.

<table>
<thead>
<tr>
<th></th>
<th>1st Trial</th>
<th>2nd Trial</th>
</tr>
</thead>
</table>

Table 2. The data used for the study case.

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>PERIODOGRAM 2D</th>
<th>APES-2D</th>
<th>CAPON-2D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling Freq.</td>
<td>128</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st dimension of 2D Signal</td>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2nd dimension of 2D Signal</td>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SNR (db)</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Oversampling Factor in 1st dimension signal</td>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Oversampling Factor in 2nd dimension signal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vector Sample count 1st dimension signal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vector Sample count 1st dimension signal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of sign. components</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beta (estimated signal’s weight)</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In this case study data taken SAR systems are examined in order to figure out an MIG boundaries. By the results we can conclude that APES and CAPON present more precise results than FFT and PERIODOGRAM do. About the last one it can be said that it finds it difficult to extract the image boundaries. APES seems to present more precisely the signals’ amplitude. The methods of PERIODOGRAM and FFT2 cannot lower the noise effects, such they present dots near the airplane image. FFT is the fastest and consume less computation power in comparison with CAPON and APES. The figure’s results and the time are presented in the following figures (11h14) and table 3, respectively.

### Table 3. The time needed for every PSD estimation method

<table>
<thead>
<tr>
<th>Method</th>
<th>Time needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFT2</td>
<td>2 seconds</td>
</tr>
<tr>
<td>PERIODOGRAM</td>
<td>3 seconds</td>
</tr>
<tr>
<td>CAPON-2d</td>
<td>50 seconds</td>
</tr>
<tr>
<td>APES-2d</td>
<td>83 seconds</td>
</tr>
</tbody>
</table>

### 3.3 Case Study 5 APES vs CAPON

In this case study data taken SAR systems are examined in order to figure out an MIG boundaries. By the results we can conclude that APES and CAPON present more precise results than FFT and PERIODOGRAM do. About the last one it can be said that it finds it difficult to extract the image boundaries. APES seems to present more precisely the signals’ amplitude. The methods of PERIODOGRAM and FFT2 cannot lower the noise effects, such they present dots near the airplane image. FFT is the fastest and consume less computation power in comparison with CAPON and APES. The figure’s results and the time are presented in the following figures (11h14) and table 3, respectively.

```matlab
Y = sg_cissoid_2d(ones(2,1), [-0.2 0.2], [-0.1 0.3], [64 64], [64 64], 3);
```
11. The Periodogram estimation for the MIG data

Fig 12. The FFT2 results for the MIG data

Fig 13. The APES estimation for the MIG data

Fig 14. The Capon estimation for the MIG data

3.4 Case study VI

Another test run for simple noised signals (table 4) and the methods of FFT, Periodogram, Multitaper, Welch, Capon, APES and the results are presented in figures 15 to 18. By commenting this case study we can point that:

a. Welch and Blackman-Tukey are more efficient than Periodogram and FFT in noisy conditions.

b. Taylor lower the side lobes.

c. Capon has greater resolution than APES. Also CAPON can estimate better the frequencies of a complex signal than APES does.

d. On the other hand APES can estimate the amplitude better than CAPON.
4 Conclusion

Many useful conclusions come out through this work.

a. APES seems to have better resolution of the amplitudes of the spectrum frequencies.

b. CAPON has the ability to present more information.

c. CAPON pulls its estimations to lower amplitude values, while APES seems to be more effective into this.

d. Both methods are computationally precise.

e. CAPON overestimates the existence of information in the spectrum.

f. seems to have better resolution of the amplitudes of the spectrum frequencies.

g. CAPON has the ability to present more information.

h. CAPON pulls its estimations to lower amplitude values, while APES seems to be more effective into this.

i. Both methods are computationally precise.

j. CAPON overestimates the existence of information in the spectrum.

k. FFT seems to have better resolution of the amplitudes of the spectrum frequencies.
l. CAPON has the ability to present more information.
m. CAPON pulls its estimations to lower amplitude values, while APES seems to be more effective into this.
n. Both methods are computationally precise.
o. CAPON overestimates the existence of information in the spectrum.

A summary of the examined method is presented in Table 5.

Table 5. Summary of the main characteristics of the examined estimations methods.

<table>
<thead>
<tr>
<th>METHOD</th>
<th>Characteristics</th>
<th>Pros</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFT</td>
<td></td>
<td>Simple and easy to be developed</td>
<td>Great sidelobes</td>
</tr>
<tr>
<td>Periodogram</td>
<td></td>
<td>Fast method, cut sidelobes</td>
<td>Low resolution</td>
</tr>
<tr>
<td>Multitaper</td>
<td></td>
<td>Good performance in noise</td>
<td>Low resolutions</td>
</tr>
<tr>
<td>Welch</td>
<td>1. It cuts the samples into smaller subsamples which pass through band pass filters. 2. The greater the number of the subsequences the lower the variance and worse the resolution</td>
<td>High resolution, needs high computations</td>
<td></td>
</tr>
<tr>
<td>Capon</td>
<td>1. It basic idea is based on the two following principles: α. The sinusoids signals are passing through filter with the less distortion. β. Every other frequency than the principle is compressed with filter.</td>
<td>High resolution, and better than this of periodogram</td>
<td>Medium computation needs</td>
</tr>
</tbody>
</table>

2. A pass band filter is used in order to computer its spectra values. 3. The sample is divided into subfrequencies and results is the computations lead to the final result.

| Apes    | 1. It based on the estimation of the amplitude and the phase of the sinusoids signal. 2. It has more computation needs. | greater resolutions | medium computation needs in relation to the final results |

References:


