Report about the Solutionsof Beams and Frames on Elastic Foundation Using FEM

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Abstract: - The paper contains numerical methods for the solution of plane beams and frames on an elastic foundation. In the first case, the solution uses beam element BEAM54 in the program ANSYS and the derivative of the stiffness matrix of the element is presented. The second approach uses a beam element with a contact element with the description of the derivative of the stiffness matrix for the frame on elastic foundation combination. Both solutions are compared with theoretical solution. The influence of the number of divisions the beam element on the accuracy of the solution is shown. There are presented some application of structures on elastic foundation (biomechanics&traumatology – external fixators for treatment of complicated bone fractures, mining industry - pressure distributions in the contact between mining supports and foot-wall as a problem of 3d body on elastic foundation).

Key-Words: - elastic foundation, Finite Element Method, beam element, contact element, theory, applications, biomechanics, traumatology, external fixators, mining, mining supports

1 Introduction
Solution of frames and beams on elastic foundation often occur in many practical cases for example, solution of building frames and constructions, buried gas pipeline systems and in design of railway tracks for railway transport, etc., see Fig.1.

Fig.1 Beam on elastic foundation is loaded by force F, couple M and distributed loading q.

Solution of beam on elastic foundation is a statically indeterminate problem of mechanics. In this case, we have the beam with elastic foundation along the whole length and width or only over some part of the length or width. Detailed explanation of theoretical solution can be found in [2], [3] and [4].Not all problems can be solved by theoretical approach (i.e. sometimes, the theoretical solution is very complicated). In solution of these complicated problems, the Finite Elements Method(FEM) can be applied. In this paper,FEM is applied for the solution of 2D and 3D beams, frames/structures on elastic Winkler's foundation.

2 Theoretical background for 2D beam on elastic foundation
The Winkler's foundation model is easy to formulate using energy concepts. The analysis of bending of beams on an elastic foundation (Winkler's model) is developed on the assumption that:
The surrounding foundation is utterly unaffected, see Fig.2a.

The strains are small.

The resisting pressure \( p_\| = K \nu / \text{Nm}^2 / \) in the foundation are proportional at every point to the deflection \( \nu = \nu(x) / \text{m} / \) normal to its surface at that point, where \( K / \text{Nm}^3 / \) is the modulus of the foundation.

The surrounding foundation is utterly unaffected, see Fig.2a.

Fig.2 Deflection of structure on elastic foundation under pressure \( p \) or distributed loading \( q \) (a) Winkler foundation, (b) elastic solid foundation

An area \( dA \, / \text{m}^2 / \) of the foundation surface acts like a linear spring of stiffness \( k / \text{Nm}^2 / \). Hence, \( k = p_\| \, / \nu = K \nu dA / \nu = K \, dA \). Strain energy \( U_\| / \text{N} / \) in a linear spring is \( U_\| = k \nu^2 / 2 \).

Now considering a structural element, perhaps a plate bending element or one face of a 3D solid element, which has an area \( A \) in contact with the foundation. Lateral deflection of area \( A \) normal to the foundation, is \( \nu = [N_i] \{d_i\} / \text{m} / \) normal to its surface at that point, where \( \{d_i\} / \text{m} / \) contains D.O.F. of element nodes in contact with foundation. Strain energy \( U / \text{N} / \) in foundation over area is

\[
U = \frac{1}{2} \int K \nu^2 \, dA = \frac{1}{2} \{d_i\}^T [K_i] \{d_i\},
\]

in which the Winkler's foundation stiffness matrix for the element is

\[
[K_i] = \int K [N_i]^T [N_i] \, dA.
\]

For example, if the problem deals with a beam on Winkler's foundation, \( [N_i] \) is identical to the shape function matrix \( [N] \) of the beam, where individual \( N_i \) are

\[
N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}, \quad N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2},
\]

\[
N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}, \quad N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}
\]

where \( dA = b \, dx / \text{m}^2 /, \) \( b / \text{m} / \) is the width of the beam face contact with the foundation and \( L / \text{m} / \) is length of the beam. We input eqn. (3) to (2), and get

\[
[K_i] = \begin{bmatrix}
13bL^2 & 11bL^2 & 9bL & 13bL^2 \\
35 & 210 & 70 & 420 \\
11bL^2 & bL^2 & 13bL^2 & bL^2 \\
210 & 105 & 420 & 140 \\
9bL & 13bL^2 & 13bL & 11bL^2 \\
70 & 420 & 35 & 210 \\
13bL^2 & bL^2 & 11bL^2 & bL^2 \\
420 & 140 & 210 & 105
\end{bmatrix}
\]

The stiffness matrix of beam without shear deformation can obtain the formal approach using equation

\[
[K_b] = \int [B^T]\frac{EI}{L} \, B \, dx,
\]

where \( B \) is the strain-displacement matrix, which is defined for beam by \( B = \frac{d^2N}{dx^2} \).

After mathematical solution of equation (5) using eqn. (3), we obtain the stiffness matrix for beam considering only bending moment and transversal load at the nodes

\[
[K_b] = \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix},
\]

where \( E / \text{Nm}^2 / \) is modulus of elasticity.

The second method of solving beams and frames is by using beam and contact element. Contact will be simulated by spring element between rigid ground and beam in the Fig.3. Stiffness matrix for spring element is as follows

\[
[K_{spring}] = \begin{bmatrix}
C & -C \\
-C & C
\end{bmatrix}
\]

where \( C / \text{Nm}^{-1} / \) is stiffness of spring.

Fig. 3 Beam and contact element

Global stiffness matrix for beam and spring element is given by combining eqn. (6) and eqn. (7), which is
\[
\begin{bmatrix}
12 + C_M & 6L & -12 & 6L & -C_M & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\
-12 & -6L & 12 + C_M & -6L & 0 & -C_M \\
6L & 2L^2 & -6L & 4L^2 & 0 & 0 \\
-C_M & 0 & 0 & 0 & C_M & 0 \\
0 & 0 & -C_M & 0 & 0 & C_M \\
\end{bmatrix} = \frac{EI}{L^2}, \quad (8)
\]

where \( C_M \) is the modified stiffness of spring solved by the following equation

\[
C_M = \frac{L^4}{EI}. \quad (9)
\]

3 Verification of numerical solution

According to chapter 2, let us consider the beam on elastic foundation shown in Fig.4, where the length of one-half of beam is \( L = 1.8 \) m. Beam is made of steel, which has the Young’s modulus \( E = 2 \times 10^8 \) MPa with rectangular cross-section area by parameters \( b = 200 \) mm and \( h = 400 \) mm. Foundation modulus \( K = 10^8 \) Nm\(^3\).

There are two approaches for the numerical solution of this beam. The first approach is using the BEAM54 element in ANSYS sw, see reference [5]. This approach can be used when the foundation is without compression resistance.

Fig.4 Beam on elastic foundation

If we consider compression resistance, we have to consider the approach using the contact element, for example CONTACT52, where compression resistance is prescribed by a gap. In our example, the gap is equal to zero. Because the program ANSYS contains beam elements with shear deformation, only BEAM54 is without shear deformation. For verification of mechanical contact, the element BEAM54 was considered and stiffness of elastic foundation is equal to zero.

Of course the accuracy of the result for FEM is influence by the number of elements over the length of beam. The verification examples used only one element over the length of beam \( L \). Influences of the number of divisions in both approaches are illustrated in the Fig.5 (deflection) and Fig.6 (bending moment).

4 Applications

Fig.7 Fracture of pelvis (a) anteroposterior radiograph - transverse with posterior wall acetabular fracture, (b) application of external fixator for treatment - two designs Option 1 and Option 2

There are a lot of applications of the structures on elastic foundation, for example see references [2], [3], [4], [6] and [7]:

- Applications in biomechanics & traumatology (i.e. FE solutions and design of new external fixators for treatment of complicated fractures of pelvis and its acetabulum), see Fig.7, 8 and references [6], [7].
- Applications in mining (i.e. FE solutions for pressure distributions in the mechanical contact between mining supports and foot-wall as a
problem of 3d body on elastic foundation), see Fig.9 and 10 and reference [3].

- Other applications, see references [2], [3], [4], [8] and [9].

Fig.8 External fixator for treatment of pelvis and its acetabulum - FE model, boundary conditions (A, B is elastic foundation, Ansys sw)

Fig.9 Mechanical contact between mining supports and foot-wall approximated via elastic foundation

Fig.10 Mechanical contact between mining supports and foot-wall approximated via elastic foundation (total displacement, MSC.MARC/MENTAT sw)

5 Conclusion

General solutions of FEM applications for the structures on elastic (Winkler's) foundations were derived, tested and discussed (two ways). Other examples, such as applications in mining and biomechanics, and references are mentioned.

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References:


