

Kalman Filter Design, Modeling, Simulation and Practical Applications

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Abstract: - In this paper the design of Kalman filter for linear single input-output systems, modeling and simulation results for practical applications are developed. Based on theoretical very good literature on this theme, but too few applications, the author presents the necessary mathematical equations for designing a comprehensive Kalman filter, and the methods to calculate the matrix, more precisely the transition matrix $\Phi(\Delta t)$. In the paper you find many practical examples of direct order systems together with models and good simulation results.

Key-Words: - Kalman filter, design model, matrix determination, practical applications, simulation results

1 Introduction

At any time the practical systems are strongly influenced by noise or perturbation coming from the process. To control this it's necessary to measure the outputs, or estimate the state of process with high fidelity by filtering the measures and reducing or eliminating the noise. A very good method is the Kalman filter. For each method we have a lot of theoretical papers and academic books, with good mathematical rigurocity but there are too few practical applications. We consider that comprehensive presentation of general practical algorithm and the specific examples which can easily be used as models for technical practitioners and others.

This is the main target of this paper which contains the presentation of the theory of Kalman filter in comprehensive mode oriented to design and applications.

Now we consider the "n" order system linear, observable and stationary described by the transfer function, and the state equations in discrete form:

$$G(s) = \frac{y}{u} = \frac{a_0 + a_1 \cdot s + \dots + a_{n-1} \cdot s^{n-1}}{b_0 + b_1 \cdot s + \dots + b_{n-1} \cdot s^{n-1} + s^n}$$

$$x_k = A \cdot x_{k-1} + b \cdot U_k \quad (1)$$

$$y_k = z_k = H \cdot x_k$$

The filter consists of two stages: first to predict the state \hat{x}_k^- and error P_k^- from present state \hat{x}_{k-1}^- , error P_{k-1}^- and noise co-variance matrix Q named "predictor" and second, to correct them as \hat{x}_k , P_k using \hat{x}_k^- , P_k^- , Kalman coefficient K_k and the

system matrix, A, B, H, named "corrector". The algorithm works in discrete sequences, time is discrete passing from k-1 to k in Δt time, with co-variance matrix state error R. The algorithm can be done in any programming language but very practical is to build in Simulink platform which is close to technical implementation. The equations for Kalman filter for predictor and corrector are on the following forms:

Predictor:

$$\hat{x}_k^- = A \cdot \hat{x}_{k-1}^- + B \cdot u_k$$

$$z_k = H \cdot \hat{x}_k^-$$

$$P_k^- = A \cdot P_{k-1}^- \cdot A^T + Q \quad (2)$$

Corrector:

$$K_k = P_k^- \cdot H^T - (H \cdot P_k^- \cdot H^T + R)^{-1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k \cdot (z_k - H \cdot \hat{x}_k^-) \quad (3)$$

$$P_k = (I - K_k \cdot H) \cdot P_k^-$$

In the above equations A is the transition matrix equivalent to $\Phi(t, t_0)$, B=b, $H = c^T \cdot \Phi$, $R = (0.1 \dots 0.001)$, I is the unity matrix. Q is the Gauss-Laplace covariance matrix error for outputs which can be evaluated by a probabilistic approach of the process or estimated.

2 Problem formulation

To show how this filter works we consider the case where we need to determine with good precision the value of a motor internal resistance by 20 measurements presented bellow.

The measurement system is very simple, the template is a step $\theta(t)$ and the output value of the resistance R_0 , $y = R \cdot u$, resulting $G(s)=R$.

This is an instantaneous system $x=0$, transition matrix $\Phi = 1$, $H=1$, $I=1$, $Q=0.001$, $R=0.1$, $v_k = 0.001$. The last three are choose.

The equations of the filter are:

Predictor:

$$\begin{aligned} \hat{x}_k^- &= \hat{x}_{k-1} \\ z_k &= \hat{x}_k + 0.001 \\ P_k^- &= P_{k-1} + 0.001 \end{aligned}$$

Corrector:

$$K_k = \frac{P_k^-}{P_k^- + 0.1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k \cdot (z_k - \hat{x}_k^-)$$

$$P_k = (I - K_k) \cdot P_k^-$$

The average of the measurements

$$R_m = \frac{\sum_{i=1}^{20} R_i}{20} = 78.45 \Omega$$

The model and simulation results are presented in fig.1 having the value $R=78.35\Omega$ better then average.

R_i	74	75	75	81	78	87	79	83	76	77
	78	74	76	80	82	78	77	76	82	74

a)

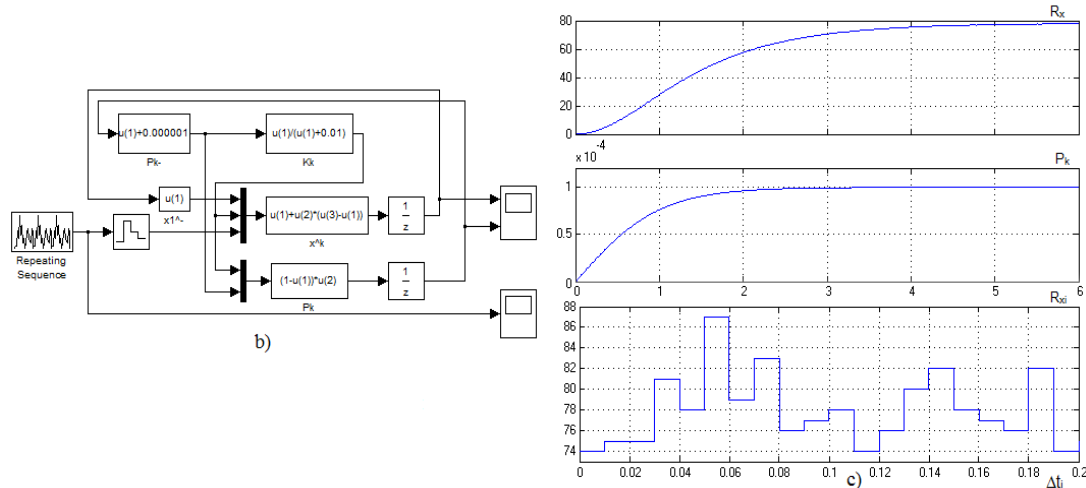


Fig.1. Kalman filter: a) measurements; b) model; c) simulation results.

The difficult problem which produces confusion or mistakes is getting the A matrix which is in fact $\Phi(t, t_0) = \Phi(t - t_0) = \Phi(\Delta t)$.

Notes that: it's necessary to do an observer realization state equations from G(s) matrix.

$$\begin{cases} \dot{x} = A \cdot x + b \cdot u \\ y = c^T \cdot x \end{cases} \quad (3)$$

$$\begin{cases} \dot{x} = \begin{bmatrix} -b_{n-1} & 1 & 0 & \dots & 0 \\ -b_{n-2} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -b_0 & 0 & 0 & \dots & 0 \end{bmatrix} \cdot x + \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \dots \\ a_0 \end{bmatrix} \cdot u \\ y = [1 \ 0 \ 0 \ \dots \ 0] \cdot x \end{cases}$$

The solutions of this equation, considering $u=0$ are:

$$\begin{cases} x = \Phi(t, t_0) \cdot x(t_0) = e^{A(t-t_0)} \cdot x(t_0) \\ y = H(t, t_0) \cdot x(t_0) = c^T \cdot e^{A(t-t_0)} \cdot x(t_0) \end{cases}$$

The transition matrix $e^{A(t-t_0)}$ can be determined by the inversion Laplace operator L^{-1} using the relation:

$$e^{A \cdot \Delta t} = L^{-1}[(s \cdot I - A)^{-1}]$$

where A is matrix from equations (3).

For example, a second order system has the following matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{aligned} \Phi(\Delta t) = e^{A \cdot \Delta t} &= L^{-1} \left\{ \begin{bmatrix} \frac{s}{s^2 + 1} & \frac{1}{s^2 + 1} \\ -\frac{1}{s^2 + 1} & \frac{s}{s^2 + 1} \end{bmatrix} \right\} = \\ &= \begin{bmatrix} \cos(\Delta t) & \sin(\Delta t) \\ -\sin(\Delta t) & \cos(\Delta t) \end{bmatrix} \end{aligned}$$

$$\Phi(\Delta t) \Big|_{\Delta t=0.01} = \begin{bmatrix} 0.999 & 0.0099 \\ -0.0099 & 0.999 \end{bmatrix}$$

$$H = c^T \cdot \Phi = [1 \ 0] \cdot \begin{bmatrix} 0.999 & 0.0099 \\ -0.0099 & 0.999 \end{bmatrix} = [0.999 \ 0.0099]$$

The final Kalman filter equations become:

Predictor:

$$\hat{x}_k^- = \Phi \cdot \hat{x}_{k-1}^-; \hat{x}_k^- \in R^{nx1}; \Phi \in R^{n \times n}$$

$$z_k = H \cdot \hat{x}_k^- + v_k; z_k \in R^{1 \times 1}; H \in R^{1 \times n}; v_k \in R^{1 \times 1}$$

$$P_k^- = \Phi \cdot P_{k-1}^- \cdot \Phi^T + Q; P_k^- \in R^{n \times n}; Q \in R^{n \times n}$$

Corrector:

$$K_k = \frac{P_k^- \cdot H^T}{(H \cdot P_k^- \cdot H^T + R)}$$

$$K_k \in R^{nx1}; R \in R^{1 \times 1}; R \in [0.01, 0.00001]$$

$$\hat{x}_k = \hat{x}_k^- + K_k \cdot (z_k - H \cdot \hat{x}_k^-)$$

$$P_k = (I - K_k \cdot H) \cdot P_k^-$$

Now, we will achieve the general equations for Kalman filter of first and second order processes.

First order system

$$G(s) = \frac{a_0}{b_0 + s}$$

The Observer matrixes are: $A = -b_0$, $b = a_0$, $c^T = 1 = H$.

$$\begin{cases} \dot{x} = -b_0 \cdot x + a_0 \cdot x \\ y = x \end{cases}$$

$$\Phi(t) = e^{A \cdot t} = e^{-b_0 \cdot t}; \Phi(\Delta t) = e^{-b_0 \cdot \Delta t} = \Phi$$

$$v_k = 0.001; Q = 0.001; R = 0.01$$

The Kalman filter equations are:

Predictor:

$$\hat{x}_k^- = \Phi \cdot \hat{x}_{k-1}^-$$

$$z_k = \hat{x}_k^- + 0.001$$

$$P_k^- = \Phi \cdot P_{k-1}^- \cdot \Phi^T$$

Corrector:

$$K_k = \frac{P_k^-}{P_k^- + 0.1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k \cdot (z_k - \hat{x}_k^-)$$

$$P_k = (I - K_k) \cdot P_k^-$$

Second order system

$$G(s) = \frac{a_0 + a_1 \cdot s}{b_0 + b_1 \cdot s + s^2}$$

The Observer matrixes are:

$$A = \begin{bmatrix} -b_1 & 1 \\ -b_0 & 0 \end{bmatrix}; b = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}; c^T = [1 \ 0] = H$$

$$\begin{cases} \dot{x} = A \cdot x + b \cdot u \\ y = c^T \cdot x \end{cases}$$

The $\Phi(t)$ matrix will be determined as follows:

• Eigen values:

$$|\lambda \cdot I - A| = 0; \lambda_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4 \cdot b_0}}{2}$$

• $\Phi(t) = e^{A \cdot t} = e^{\lambda_1 \cdot t} \cdot A_{10} + e^{\lambda_2 \cdot t} \cdot A_{20}$

• The A_{10} , A_{20} matrixes will be determined for:

$$f(A) = f(\lambda_1) \cdot A_{10} + f(\lambda_2) \cdot A_{20}$$

In the above equation the function f will be $\lambda - \lambda_1$ and $\lambda - \lambda_2$ which generate two matrix equations to calculate A_{10} and A_{20} .

• If the eigen value are different the matrix is:

$$e^{A \cdot t} = \frac{1}{\lambda_1 - \lambda_2} \cdot [e^{\lambda_1 \cdot t} \cdot (A - \lambda_2 \cdot t) - e^{\lambda_2 \cdot t} \cdot (A - \lambda_1 \cdot t)] = \Phi$$

Other used method is to invert the Laplace transformation L^{-1} :

$$e^{A \cdot t} = L^{-1} \left\{ [s \cdot I - A]^{-1} \right\}$$

If A matrix has a particular form and $A^k = 0$ then:

$$e^{A \cdot t} = I + \frac{A}{1!} \cdot t + \frac{A^2}{2!} \cdot t^2 + \dots + \frac{A^{k-1}}{(k-1)!} \cdot t^{k-1}$$

For example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; A^2 = 0; e^{A \cdot t} = I + A \cdot t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \Phi$$

The Kalman filter equations are:

Predictor:

$$\hat{x}_k^- = \Phi \cdot \hat{x}_{k-1}^-; \hat{x}_k^- \in R^{2 \times 1}; \Phi \in R^{2 \times 2}$$

$$z_k = H \cdot \hat{x}_k^- + v_k; z_k \in R^{1 \times 1}; H \in R^{1 \times 2}; v_k = 0.1$$

$$P_k^- = \Phi \cdot P_{k-1}^- \cdot \Phi^T + Q; P_k^- \in R^{2 \times 2}; Q \in R^{2 \times 2}$$

Corrector:

$$K_k = \frac{P_k^- \cdot H^T}{(H \cdot P_k^- \cdot H^T + R)}; K_k \in R^{2 \times 1}; Q \in R^{2 \times 2}$$

$$\hat{x}_k = \hat{x}_k^- + K_k \cdot (z_k - H \cdot \hat{x}_k^-); K_k \in R^{2 \times 1}; R = [0.01, 0.00001]$$

$$P_k = (I - K_k \cdot H) \cdot P_k^-$$

3 Problem solution

In this paragraph will design, model and simulate the Kalman filter for a real first and second order process.

First order hydro process

In fig.2 is presented a process of water accumulation with its parameters.

The hydro process equations:

$$\frac{dV}{dt} = q_1 - q_2; q_2 = f(h) = a \cdot h; h = \frac{q_2}{a}$$

$$V = S \cdot h; \frac{a}{S} = k; a_0 = b_0 = 0.05$$

$$G(s) = \frac{q_2}{q_1} = \frac{a_0}{s + b_0}; a_0 = b_0 = k$$

The Observer matrixes are:

$$A = -b_0; b = a_0; b = k; c^T = 1 = H$$

$$\begin{cases} \dot{x} = -b_0 \cdot x + a_0 \cdot u \\ y = x \end{cases}$$

The filter Kalman simulation results are presented in fig.3. The input is step mixed with a Gaussian noise. The filter pulls out the output from noise with good accuracy.

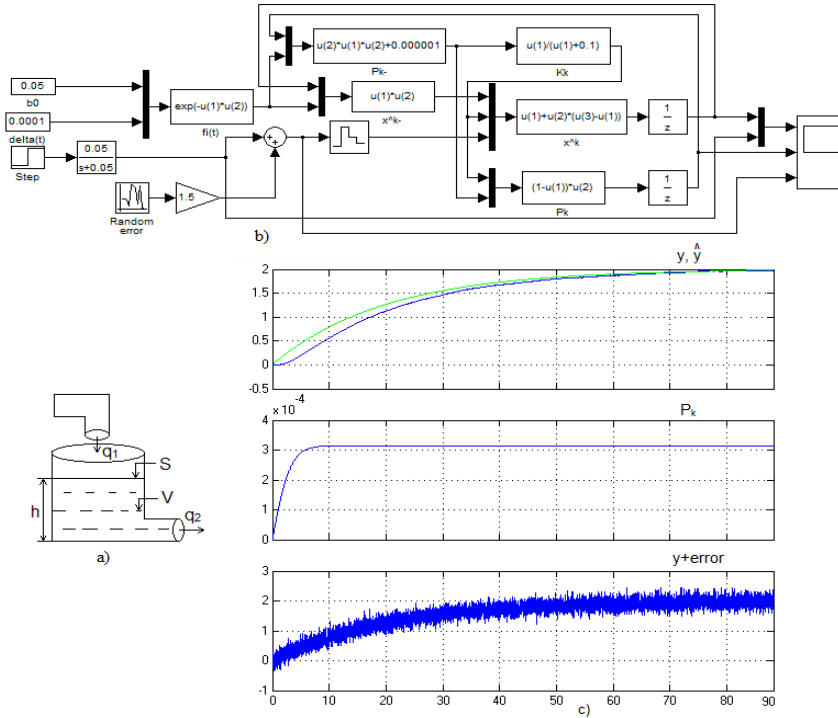


Fig.3. Kalman filter: a) hydro process; b) model; c) simulation results.

Second order DC servo-drive

A servo-drive with constant magnetic field Φ , resistance R, inductance L, rotor inertia J, load $M_s=0$, torque M, current i is controlled by DC voltage u and results the speed ω , as shown in fig.4.a. The drive equations are:

$$u = R \cdot i + L \frac{di}{dt} + k_e \cdot \Phi \cdot \omega; M = k_m \cdot \Phi \cdot i$$

$$J \frac{d\omega}{dt} = M - M_s = M$$

From the above results the input-output equation and the transfer function:

$$\frac{d^2\omega}{dt^2} + b_1 \frac{d\omega}{dt} + b_0 \cdot \omega = a_0 \cdot u$$

$$G(s) = \frac{\omega}{u} = \frac{a_0}{s^2 + b_1 \cdot s + b_0}$$

$$a_0 = \frac{1}{k_e \cdot \Phi}; b_0 = \frac{k_e \cdot k_m \cdot \Phi^2}{L \cdot J}; b_1 = \frac{R}{L}$$

For: $a_0 = 1786; b_0 = 178.6; b_1 = 6.25$ the matrixes are:

$$A = \begin{bmatrix} -6.25 & 1 \\ -178.6 & 0 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 1786 \end{bmatrix}; c^T = [1 \ 0] = H$$

$$e^{A \cdot t} = L^{-1} \left\{ [s \cdot I - A]^{-1} \right\} = \begin{bmatrix} \frac{s}{(s + 3.1)^2 + 13^2} & \frac{1}{(s + 3.1)^2 + 13^2} \\ \frac{178.6}{(s + 3.1)^2 + 13^2} & \frac{s + 6.2}{(s + 3.1)^2 + 13^2} \end{bmatrix}$$

$$\Phi = e^{A \cdot t} = \begin{bmatrix} e^{-3.1 \cdot t} [\cos(13.1 \cdot t) - 0.24 \cdot \sin(13 \cdot t)] & 0.0077 \cdot e^{-3.1 \cdot t} \cdot \sin(13 \cdot t) \\ -13 \cdot e^{-3.1 \cdot t} \cdot \sin(13 \cdot t) & e^{-3.1 \cdot t} [\cos(13 \cdot t) + 0.24 \cdot \sin(13 \cdot t)] \end{bmatrix}$$

$$e^{A \cdot \Delta t} \Big|_{\Delta t=0.01 \text{ s}} = \begin{bmatrix} 0.962 & 0.023 \\ -0.31 & 0.9766 \end{bmatrix}$$

The Simulink model and simulation results as presented in the fig.4.

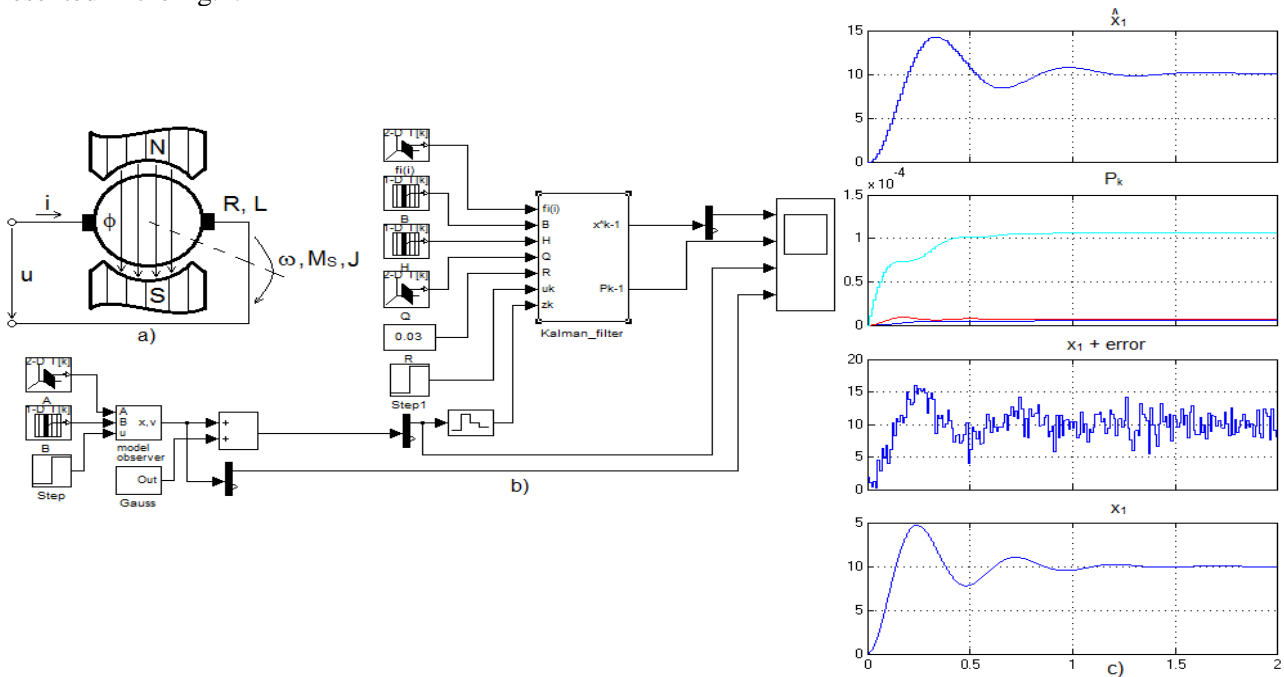


Fig.4. Kalman filter: a) servo-drive; b) model; c) simulation results.

4 Conclusion

In this paper we design in a comprehensive way the general Kalman filter for SISO linear systems, beginning from the transfer function. The main problem solved in this paper is the algorithm for the transition function $\Phi(\Delta t)$ as part of the Kalman filter equations.

In the last paragraph the practical examples are developed first and second order systems together with the Kalman filter equations, Simulink model and simulation results. The very accurate results of the simulation validate the methods used in the paper and represent a recommendation to use this experience for other applications.

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