A New Approach of the Induction Machine Mathematical Model

EMIL POP, PETRE VAMVU, CAMELIA BARBU
Department of Control Engineering, Computers, Electrical and Power Engineering
University of Petrosani
20 University Street, 332006, Petrosani
ROMANIA
emilpop2001@yahoo.com , petre_vamvu@yahoo.com , tabacarucamelia@yahoo.com, www.upet.ro

Abstract: - In this paper an original approach based on the asymptotic approximation and the system theory was used to solve the nonlinear mathematical model of the induction machine. The mathematical model of induction machine has several nonlinear equations that cannot be integrated to determine the input/output equations or transfer function. The main objective of this paper is to find a method to solve the equations of mathematical model. For this reason first contribution was to replace speed with an asymptotic function then decupling the feedback loop and replacing it with an equivalent feedback control voltage. The second contribution consist in a special arrangement of mathematical model matrix to allow splitting off the decomposition model, consisting of forth order transition matrix, in two diagonal second order matrices that permit to write a linear state model. This interesting model was simulated and has the same state and output variation very close to the consecrated models. This approach permits to determine the solutions of model and the transfer matrices on the channels input (voltages), state (fluxes) and output (currents). At the final of the paper the theoretically results are validated by simulation using a real induction machine with good results.

Key-Words: - induction machine, special approximated model, state equations, solutions of equations transfer matrix, modeling, simulation

1 Introduction

Consider the normal three phase induction machine in the normal conditions of electro mechanic and electromagnetic parameters. We work in a vector space of continuous and derivable functions having two orthogonal Clarke coordinates d, q with s, r indexes for stator and respectively rotor.

The notations Ψ_{ij}, i_{ij}, u_{ij}, represents the fluxes, currents and voltages, L_i, R_i, represent the inductances and resistances. The M, Ms, ω, J are the torque, load, speed and inertia of drive [3].

The well known 10 machine equations from which three are nonlinear are given below:

$$\dot{\Psi}_{sd} = -R_s \cdot i_{sd} + u_{sd}$$
$$\dot{\Psi}_{sq} = -R_s \cdot i_{sq} + u_{sq}$$
$$\dot{\Psi}_{rd} = -R_r \cdot i_{rd} + u_{rd} - \omega \cdot \Psi_{rq}$$
$$\dot{\Psi}_{rq} = -R_r \cdot i_{sd} + u_{sd} + \omega \cdot \Psi_{rd}$$
$$\Psi_{sd} = L_s \cdot i_{sd} + L_m \cdot i_{rd}$$
$$\Psi_{sq} = L_s \cdot i_{sq} + L_m \cdot i_{rq}$$
$$\Psi_{rd} = L_r \cdot i_{rd} + L_m \cdot i_{sd}$$
$$\Psi_{rq} = L_r \cdot i_{rq} + L_m \cdot i_{sq}$$
$$\dot{\omega} = \frac{M - M_s}{J}$$
$$M = \frac{3}{2} \cdot \frac{L_m}{L_s \cdot L_r - L_m^2} (\Psi_{rd} \cdot \Psi_{rq} - \Psi_{rq} \cdot \Psi_{sd})$$

This model cannot be integrated due to lack of nonlinear equations for the input/output transfer function. That is a difficult problem especially in control applications which require the identification of parameters, the estimation for state or the implementation of advanced algorithms [7]. Of course there are many solution which use approximations by linearization in steady state regime, but those are used in limited applications.

Now first we organize the two order vector space. Supply voltage vector on d-coordinate:

$$u_{da} = [u_{sd}, u_{rd}]^T = [U_{sm} \sin \omega_0 \cdot t, -U_{rm} \cdot \cos \omega_0 \cdot t]^T$$

Supply voltage vector on q-coordinate:

$$u_{qa} = [u_{sq}, u_{rq}]^T = [-U_{sm} \cos \omega_0 \cdot t, -U_{rm} \cdot \sin \omega_0 \cdot t]^T$$

Fluxes are two components state vector:

$$\Psi_d = [\Psi_{sd}, \Psi_{rd}]^T, \ \Psi_q = [\Psi_{sq}, \Psi_{rq}]^T$$

Currents are two components output vectors:

$$i_d = [i_{sd}, i_{rd}]^T, \ i_q = [i_{sq}, i_{rq}]^T$$

Feedback voltage vector on d coordinate:

$$u_{fq} = B \cdot \Psi_q \cdot \omega = [0, \omega \cdot \Psi_{rq}]^T$$

ISBN: 978-1-61804-171-5

Feedback voltage vector on q coordinate:

$$u_{fd} = B \cdot \Psi_d \cdot \omega = [0, \omega \cdot \Psi_{rd}]^T$$

The matrices used are:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix},\ D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Elements of matrix *A* are:

$$a = -\frac{R_s L_r}{L_s \cdot L_r - L_m^2},\ b = \frac{R_s L_m}{L_s \cdot L_r - L_m^2}$$

$$c = \frac{R_r L_m}{L_s \cdot L_r - L_m^2},\ d = -\frac{R_r L_s}{L_s \cdot L_r - L_m^2}$$

Elements of matrix *C* are:

$$c_1 = \frac{a}{R_s \cdot L_r},\ c_2 = -\frac{b}{R_s \cdot L_r}$$

$$c_3 = \frac{c}{R_s \cdot L_r},\ c_4 = -\frac{d}{R_s \cdot L_r}$$

Constant element's used are:

$$k = \frac{3}{2} \cdot \frac{L_m}{L_s \cdot L_r - L_m^2},\ \alpha = \frac{E_2}{U_1},\ a_f = \frac{1}{T}$$

We write an interesting, compact matrix model with two symmetric matrices:

$$\begin{bmatrix} \dot{\Psi}_d \\ \dot{\Psi}_q \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \cdot \begin{bmatrix} \Psi_d \\ \Psi_q \end{bmatrix} + \begin{bmatrix} u_{da} - u_{fq} \\ u_{qa} - u_{fd} \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \cdot \begin{bmatrix} \Psi_d \\ \Psi_q \end{bmatrix} \quad (2)$$

$$M = k \cdot \Psi_d^T \cdot Q \cdot \Psi_q \quad (3)$$

$$\dot{\omega} = \frac{M - M_s}{J} \quad (4)$$

Equations (1), (2) look like state model and (3) and (4) are scalar depending of the previously terms.

In Fig.1 is depicted the comprehensive model based on equations above.

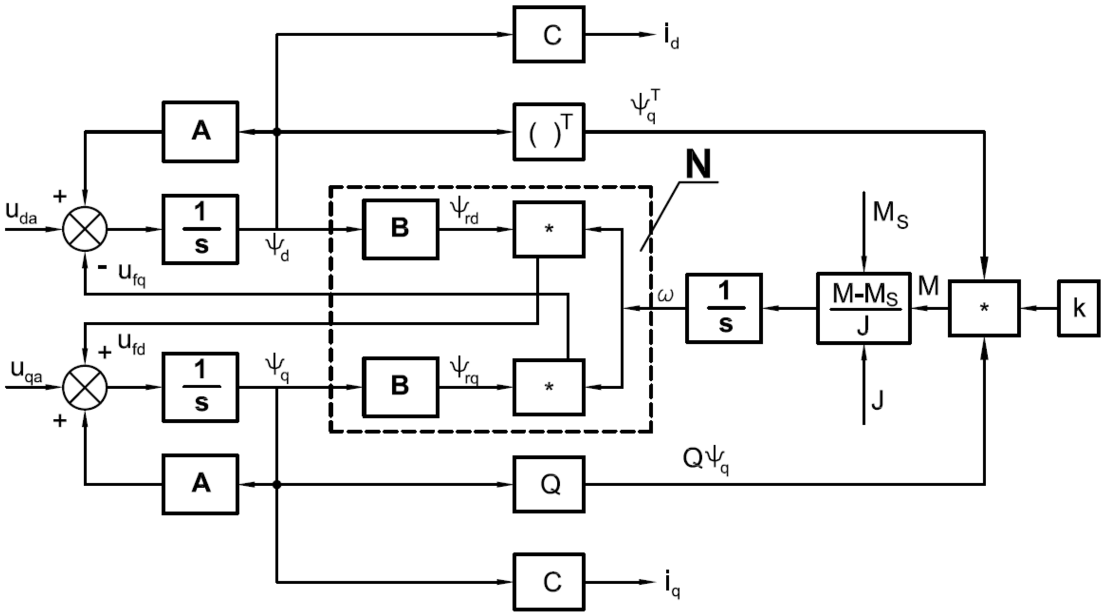

Fig. 1 The comprehensive induction motor mathematical model

Fig.2 presents the simulation results for speed, torque and feedback voltages.

Fluxes depend by sine and cosine functions. The speed can be approximated more accuratly using Berstain polynoms or by asymptotic relation.

2 Problem Formulation

We aproximate the feedback voltages as follows:

$$u_{fd} = -\alpha \cdot U_{sm} \cdot \frac{\omega_s}{\omega_0} \cdot (1 - e^{-a_f \cdot t}) \cdot \cos \omega_0 \cdot t \quad (5)$$

$$u_{fq} = -\alpha \cdot U_{sm} \cdot \frac{\omega_s}{\omega_0} \cdot (1 - e^{-a_f \cdot t}) \cdot \sin \omega_0 \cdot t \quad (6)$$

In the equations above U_{sm}, ω_0 are the voltages supply parameters, ω_s is the steady state speed, α and a_f are the calibration parameters

The nonlinear block N was replaced by Feedback Voltage Generator (FVG) and the mathematically model became almost linear, excepting torque equation.

The new simulation results depicted by Fig.2 are close to those of nonlinear model..

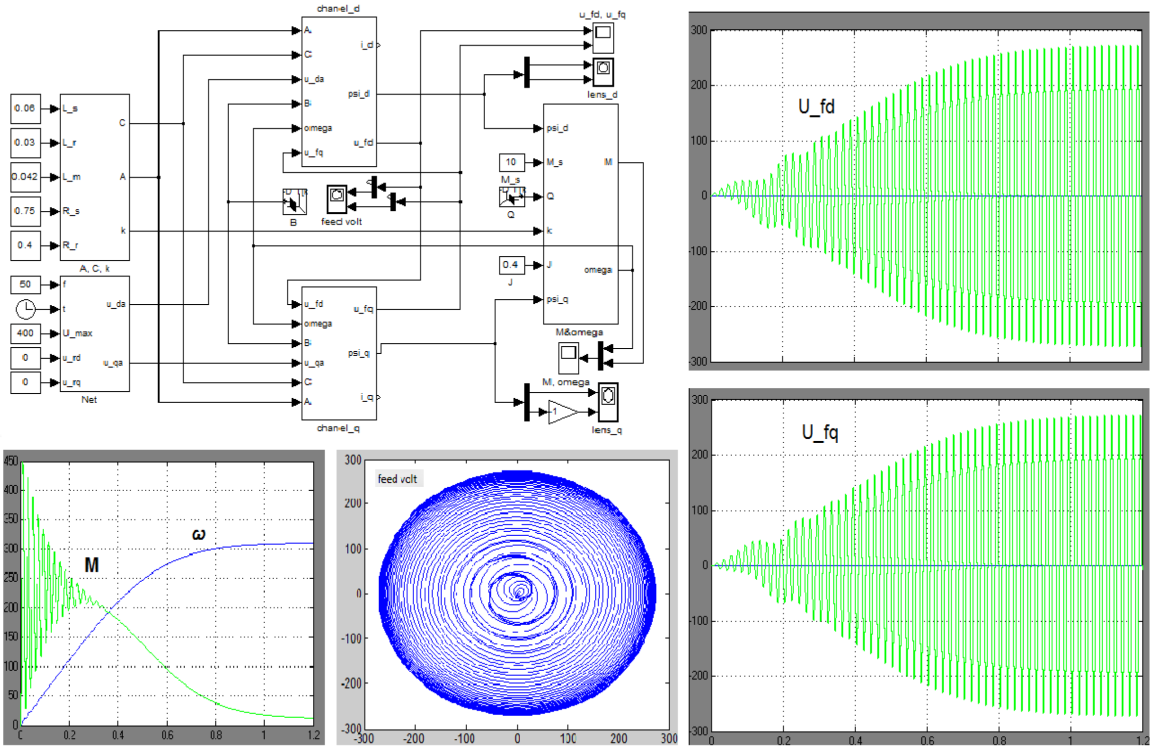

Fig.2 The nonlinear model and simulation results

Differences exist only on the start of the machine because the nonlinearities influences on start are relevant.

The problem to resolve is to find a solution for the matrix equation (1). After this the equations (2), (3) and (4) can be solved without difficulty.

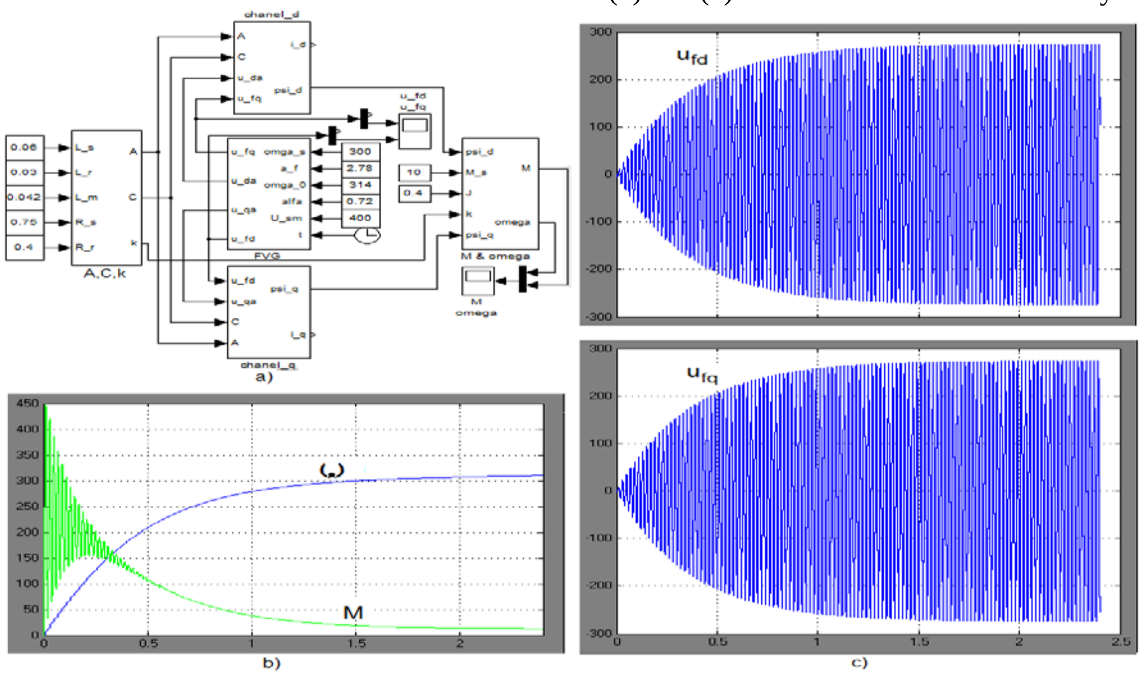

Fig.3 The approximated linear model and simulation results

3 Problem Solution

In this section we propose and solve a mathematical model based on transfer matrices for complex systems behavioral response. It's a complex subject to solve the linear matrix equations (1).

The general solution for the equation is depicted by the equations below.

$$\begin{bmatrix}\Psi_d\\ \Psi_q\end{bmatrix}=\begin{bmatrix}e^{At} & 0\\ 0 & e^{At}\end{bmatrix}\cdot\begin{bmatrix}\Psi_d(0)\\ \Psi_q(0)\end{bmatrix}+\int_0^t e^{A(t-\tau)}\cdot\begin{bmatrix}u_{da}(\tau)-u_{fq}(\tau)\\ u_{qa}(\tau)+u_{fd}(\tau)\end{bmatrix}\cdot d\tau$$

$$e^{At}=L^{-1}\left[(sI-A)^{-1}\right]=\frac{1}{s_2-s_1}\begin{bmatrix}(d-s_1)e^{s_1t}-(d-s_2)e^{s_2t} & -be^{s_1t}+be^{s_2t}\\ -ce^{s_1t}+ce^{s_2t} & (a-s_1)e^{s_1t}-(a-s_2)e^{s_2t}\end{bmatrix}$$

$$\begin{bmatrix}\Psi_d\\ \Psi_q\end{bmatrix}=\begin{bmatrix}\Psi_{sd}\\ \Psi_{rd}\\ \Psi_{sq}\\ \Psi_{rq}\end{bmatrix}=\frac{1}{s_2-s_1}\left\{\begin{bmatrix}A_0e^{s_1t}-B_0e^{s_2t} & -be^{s_1t}+be^{s_2t} & 0 & 0\\ -ce^{s_1t}+ce^{s_2t} & C_0e^{s_1t}-D_0e^{s_2t} & 0 & 0\\ 0 & 0 & A_0e^{s_1t}-B_0e^{s_2t} & -be^{s_1t}+be^{s_2t}\\ 0 & 0 & -ce^{s_1t}+ce^{s_2t} & C_0e^{s_1t}-D_0e^{s_2t}\end{bmatrix}\cdot\begin{bmatrix}\Psi_{sd}(0)\\ \Psi_{rd}(0)\\ \Psi_{sq}(0)\\ \Psi_{rq}(0)\end{bmatrix}+\right.$$

$$\left.+\begin{bmatrix}-\phi_1\cos(\omega_0t-\alpha_1)+\phi_2\sin(\omega_0t-\alpha_2)e^{-a_ft}+A_3e^{s_1t}+A_4e^{s_2t}\\ -\phi_3\cos(\omega_0t-\alpha_3)+\phi_4\cos(\omega_0t-\alpha_4)e^{-a_ft}+B_3e^{s_1t}+B_4e^{s_2t}\\ -\phi_1\sin(\omega_0t-\alpha_1)-\phi_2\cos(\omega_0t+\alpha_2)e^{-a_ft}+C_1e^{s_1t}+C_2e^{s_2t}\\ -\phi_3\sin(\omega_0t-\alpha_3)+\phi_4\cos(\omega_0t-\alpha_4)e^{-a_ft}+C_3e^{s_1t}+C_4e^{s_2t}\end{bmatrix}\right\}$$

The used eigenvalues s_1, s_2 are:

$$s_1=\frac{(a+d)+\sqrt{(a-d)^2+4bc}}{2}$$

$$s_2=\frac{(a+d)-\sqrt{(a-d)^2+4bc}}{2}$$

The s_1, s_2 are the eigenvalues of matrix A, the $A_0, B_0, C_0, D_0, A_i(i=1,6), B_i(i=1,6), C_i(i=1,4)$ are the constants depending of A matrix elements and feedback voltage parameters.

The $\phi_i(i=1,8)$, $\alpha_i(i=1,8)$ depend of A_i, B_i, C_i.

The next step is to calculate the torque using the relation:

$$M=k\cdot\left[\Psi_{rd}\cdot\Psi_{sq}-\Psi_{rq}\cdot\Psi_{sd}\right]$$

The full expression for the torque is presented by the following equation

$$\begin{aligned}M=&M_0+M_1\cdot e^{-a_ft}-M_2\cdot e^{-2a_ft}+M_2\cdot e^{-2a_ft}+M_3\cdot e^{-2s_1t}+M_4\cdot e^{2s_2t}+M_5\cdot e^{(s_1+s_2)t}+\\ &+e^{s_1t}\cdot M_6\cdot\sin(\omega_0t-d_3-d_5)-e^{s_1t}\cdot M_7\cdot\sin(\omega_0t-d_1-d_0)-e^{s_2t}\cdot M_8\cdot\sin(\omega_0t-d_3+d_7)+\\ &+e^{s_2t}\cdot M_9\cdot\sin(\omega_0t-d_1-d_8)+\left[-M_{10}\cdot\sin(\omega_0t-d_4-d_5)-M_{11}\cdot\cos(\omega_0t-\alpha_2-\alpha_0)\right]e^{(s_1-a_f)t}+\\ &+\left[M_{12}\cdot\cos(\omega_0t-d_4+d_7)+M_{13}\cdot\cos(\omega_0t+d_2-d_8)\right]\cdot e^{(s_2-a_f)t}\end{aligned}\tag{7}$$

$$\begin{aligned}M_0&=k\cdot\phi_1\cdot\phi_3\cdot\sin(\alpha_3-\alpha_1)\\ M_1&=k\cdot\phi_2\cdot\phi_3\cdot\cos(\alpha_2+\alpha_3)+\\ &\quad+k\cdot\phi_1\cdot\phi_2\cdot\sin(\alpha_1-\alpha_4)\\ M_2&=k\cdot\phi_2\cdot\phi_4\cdot\cos(\alpha_2+\alpha_4)\\ M_3&=k[B_3\cdot C_3-C_3\cdot A_3]\\ M_4&=k[B_4\cdot C_2-C_4\cdot A_4]\\ M_5&=k\cdot B_3\cdot C_2+k\cdot B_4\cdot C_1-\\ &\quad-k\cdot C_3\cdot A_4-k\cdot C_4\cdot A_3\end{aligned}$$

$$\begin{aligned}M_6&=k\cdot\phi_3\cdot\phi_5\\ M_7&=k\cdot\phi_1\cdot\phi_8\\ M_8&=k\cdot\phi_3\cdot\phi_7\\ M_9&=k\cdot\phi_1\cdot\phi_8\\ M_{10}&=k\cdot\phi_4\cdot\phi_5\\ M_{11}&=k\cdot\phi_2\cdot\phi_6\\ M_{12}&=k\cdot\phi_4\cdot\phi_7\\ M_{13}&=k\cdot\phi_2\cdot\phi_8\end{aligned}$$

Analyzing the magnitude of $M_i(i=1,13)$ six terms can be neglected without losing the accuracy of the result. The final form of the formula to express torque is presented below.

$$M=M_0-M_1\cdot e^{-a_ft}-M_2\cdot e^{-2a_ft}-\left[M_{10}\cdot\sin(\omega_0t-\alpha_4-\alpha_5)+M_{11}\cdot\cos(\omega_0t+\alpha_2-\alpha_6)\right]\cdot e^{(s_1-a_f)t}\tag{8}$$

ISBN: 978-1-61804-171-5

The speed is determined from the equation:

$\frac{d\omega}{dt} = \frac{(M - M_s)}{J}$, where M is expressed by equation (8).

$$\omega = \frac{M_0 - M_s}{J} t + \frac{M_1}{Ja_f}\left(1 - e^{-a_f t}\right) - \frac{M_2}{2Ja_f}\left(1 - e^{-2a_f t}\right) - \frac{M_{10}(s_1 - a_f)}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} e^{(s_1 - a_f)t} \sin(\omega_0 t - \alpha_4 - \alpha_5) -$$
$$- \frac{M_{10}(s_1 - a_f)}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} \sin(\alpha_4 + \alpha_5) + \frac{M_{10}\omega_0}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} e^{(s_1 - a_f)t} \cos(\omega_0 t - \alpha_4 - \alpha_5) -$$
$$- \frac{M_{10}\omega_0}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} \cos(\alpha_4 + \alpha_5) - \frac{M_{11}(s_1 - a_f)\omega_0}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} e^{(s_1 - a_f)t} \cos(\omega_0 t + \alpha_2 - \alpha_6) + \tag{9}$$
$$+ \frac{M_{11}(s_1 - a_f)\omega_0}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} \cos(\alpha_2 + \alpha 6) - \frac{M_{11} \cdot \omega_0}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} e^{(s_1 - a_f)t} \sin(\omega_0 t + \alpha_2 - \alpha_6) +$$
$$+ \frac{M_{11} \cdot \omega_0}{J\left[(s_1 - a_f)^2 + \omega_0^2\right]} \sin(\alpha_2 - \alpha 6)$$

The $M_0 \approx M_s$ for stability of the drive and the last three terms are negligible because of the high value of ω_0^2 from the denominators. The simplified form of the speed is:

$$\omega_0 = \frac{M_1}{J \cdot a_f}\left(1 - e^{-a_f t}\right) - \frac{M_2}{2Ja_f}\left(1 - e^{-2a_f t}\right)$$

The most important thing is the possibility to determine the transfer matrix from voltage to fluxes.

$$\Psi_d = \begin{bmatrix} \Psi_{sd} \\ \Psi_{rd} \end{bmatrix} = [sI - A]^{-1} \cdot \begin{bmatrix} u_{sd} \\ u_{rd} \end{bmatrix}$$

$$\Psi_d = [sI - A]^{-1} \cdot \begin{bmatrix} u_{sm} \cdot \sin(\omega_0 t) \\ -\left(1 - e^{-a_f t}\right) \cdot u_{rm} \cdot \sin(\omega_0 t) \end{bmatrix}$$

$$L\left\{\left(1 - e^{-a_f t}\right) u_{rm} \sin(\omega_0 t)\right\} = \frac{-2\cos a_f \cdot s}{k_0 \omega_0 \left[(s + a_f)^2 + \omega_0^2\right]}$$

$$u_{rm} = \frac{\omega_s}{\omega_0} \cdot u_{sm}$$

Based on the equations above we get the following transfer matrix:

$\begin{bmatrix} \Psi_d \\ \Psi_q \end{bmatrix} = \begin{bmatrix} G_d(s) \\ G_q(s) \end{bmatrix} \cdot u_{sm}$, where

$$G_d(s) = \begin{bmatrix} G_1 & G_2 \cdot G_5 \\ G_3 & G_4 \cdot G_5 \end{bmatrix}$$

$$G_q(s) = \begin{bmatrix} G_1 & G_2 \cdot G_6 \\ G_3 & G_4 \cdot G_6 \end{bmatrix}$$

Each of the $G_i (i = 1..6)$ terms has an expression depending of the electric and the mechanic parameters.

The parameters from a real induction motor were used for calculating the six terms of the transfer matrix. The values of the motor parameters are depicted in Fig.3 and the resulting values of transfer matrix terms are displayed by the equation set (10).

$$\begin{aligned} G_1(s) &= \frac{s + 667}{s^2 + 1298s + 8250} \\ G_2(s) &= \frac{875}{s^2 + 1298s + 8250} \\ G_3(s) &= \frac{467}{s^2 + 1298s + 8250} \\ G_4(s) &= \frac{s + 625}{s^2 + 1298s + 8250} \\ G_5(s) &= \frac{5.58s + 7.784}{s^2 + 5.58s + 98604} \\ G_6(s) &= \frac{2.79s^2 + 7.784s - 275083}{s^3 + 5.58s^2 + 98604s} \end{aligned} \tag{10}$$

Based on the parameters and transfer matrix a Matlab Simulink simulation model was created in order to validate the proposed mathematical model.

As can be observed in Fig.4 the curves obtained for torque and speed match very accurate the ones obtained from the nonlinear model and linear approximated model [10].

This result allows the implementation of the mathematical model into an intelligent electric inverter and to drive an AC induction motor through all functioning regimes to confirm the results.

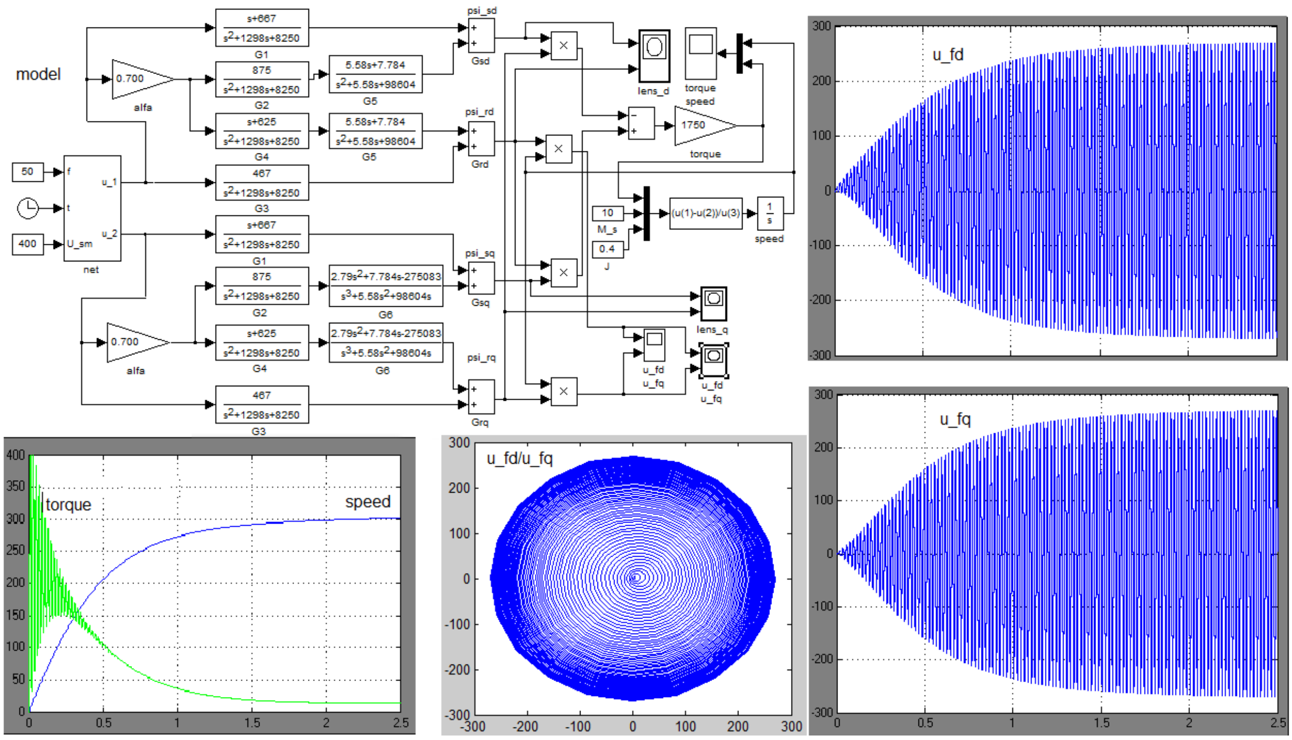

Fig. 4 Induction motor: a) transfer matrices model; b) simulation results.

4 Conclusion

An original approach based on the asymptotic linearization feedback voltages and the system theory was developed in this paper in order to solve the mathematical model of the induction machine.

To ease finding the solution for the proposed model, an interesting matrix system was constructed in order to accurately approximate the system response.

Then the solution equations were modeled and simulated, and the results were compared with the ones of the consecrated model with synthesized feedback voltages.

After determining the solutions for field components the expressions for torque, speed and transfer matrix on channel voltage-field were calculated and then implemented into a simulation model for results analysis.

The results are original and very close with the ones from the nonlinear model validating the method developed here.

References:

[1] Badea A., Pop E., Badea F., VHDL code generation based on Hierarchical node structure, *WSEAS Proceedings*, Malta 2012.

[2] Boldea I., Nasar S.A., *Vector Control of AC Drive*, CRC Press LLC, 1992.

[3] Constantinescu S.L., Soal K., *Electrische Machinen und Antriebssysteme*, Springer, 1999.

[4] Ionescu V., Varga A., *Teoria sistemelor*, Ed. All, Bucuresti 1994.

[5] Kelemen A., *Sisteme de Reglare cu Orientare Dupa Camp*, Editura Tehnica Bucuresti, 1989.

[6] Kovacs P., *Analiza regimurilor tranzitorii ale masinilor electrice*, Ed. Tehnica, Bucuresti, 1980.

[7] Leonhard W., *Control of Electrical Drives*, Springer, 1996.

[8] Mastorakis N., Bulucea C., Nicola D., Modeling of Three-Phase Induction Motors in Dynamic Regimes According to an Ecosystem Pattern, *Proceedings of the 13th WSEAS International Conference on Systems, 2009*

[9] Pop E., Bubatu R., *Teoria sistemelor, educatie prin E-Learning*, Universitas Press, Petrosani, 2012.

[10] Patrascoiu N., *Modeling and Simulation of Systems (Modelarea si simularea sistemelor)*, Editura FOCUS, Petrosani, Romania, ISBN 973-85487-8-0, 2001.