Spatial stability analysis of curved shallow mixing layers

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Abstract: - Spatial stability analysis of slightly curved shallow mixing layers is presented in the paper. Linear stability problem is solved by means of the collocation method based on Chebyshev polynomials. Since spatial stability problem is nonlinear, a computational scheme based on the combination of bisection method and the solution of linear generalized eigenvalue problem is used to calculate growth rates for the most unstable mode. Results of numerical calculations show that both bottom friction and curvature have a stabilizing influence on the flow.

Key-Words: - Spatial stability, mixing layer, shallow water, collocation method

1 Introduction

Shallow mixing layers are widespread in nature and engineering. Examples include flows at river junctions and flows in compound and composite channels. Three basic methods are usually used in order to analyze the development of shallow mixing layers: numerical methods, experimental investigations and stability analyses [1]. Experiments reported in [2]-[4] show that (i) bed friction has a stabilizing influence on the flow; (ii) development of three-dimensional instabilities is prevented; (iii) growth of the mixing layer is affected by bottom friction.

Linear stability analysis is also often used to predict critical values of the parameters of the problem [5]-[9]. In some cases the results of linear stability can be used as a starting point for weakly nonlinear theories [10]-[13].

There are two basic approaches for the analysis of linear stability of a base flow in fluid mechanics: (a) temporal stability analysis and (b) spatial stability analysis [14]. In both cases the analysis is performed using the method of normal modes: perturbations are assumed to be proportional to $\exp[i(\alpha x-\beta t)]$, where both parameters α and β may be complex: $\alpha=\alpha_r+i\alpha_i$, $\beta=\beta_r+i\beta_i$. In case (a) the wave number $\alpha=\alpha_r$ is real while β is complex. For the case of spatial stability analysis $\beta=\beta_r$ is real and the wave number α is complex: $\alpha=\alpha_r+i\alpha_i$. From a computational point of view temporal stability analysis is simpler since the corresponding eigenvalue problem is linear with respect to eigenvalue β. On the other hand, spatial eigenvalue problem is nonlinear in α. However, spatial growth rates are usually evaluated experimentally so that spatial stability characteristics should be calculated for a proper comparison with experimental data.

Gaster [15] suggested a transformation which can be used to approximate spatial growth rates if temporal growth rates are known. However, Gaster's transformation can be used only in the vicinity of the marginal stability curve.

In the present paper we solve a spatial stability problem for the case of slightly curved shallow mixing layers. Spatial growth rates are calculated for different values of the parameters of the problem. The effect of curvature on the stability of the base flow is analyzed.

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2 Formulation of the Problem

The two-dimensional shallow water equations have the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=0, \quad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2}+\frac{1}{R}u^2=0, \quad (3)$$

where u and v are the depth-averaged velocity components in the x and y directions, respectively, p is the pressure, c_f is the friction coefficient, h is water depth and R is the radius of curvature ($1/R<<1$).

Introducing the stream function by the relations

$$u=\frac{\partial \psi}{\partial y},\ v=-\frac{\partial \psi}{\partial x} \quad (4)$$

we rewrite (1)-(3) in the form

$$\begin{aligned}&(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y\\&+\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}+\frac{2}{R}\psi_y\psi_{xy}\\&+\frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}\left(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}\right)=0\end{aligned} \quad (5)$$

Consider a perturbed solution to (5) of the form

$$\psi(x,y,t)=\psi_0(y)+\varepsilon\psi_1(x,y,t)+... \quad (6)$$

where $\psi_0(y)$ is the base flow solution and ψ_1 is a small unsteady perturbation. Substituting (6) into (5) and linearizing the resulting equation in the neighborhood of the base flow we obtain

$$L_1\psi_1=0, \quad (7)$$

where

$$\begin{aligned}L_1\psi &\equiv \psi_{xxt}+\psi_{yyt}+\psi_{0y}\psi_{xxx}+\psi_{oy}\psi_{yyx}-\psi_{0yyy}\psi_x\\&+\frac{c_f}{2h}(\psi_{0y}\psi_{xx}+2\psi_{oyy}\psi_y+2\psi_{0y}\psi_{yy})+\frac{2}{R}\psi_{0y}\psi_{xy}\end{aligned}$$

Method of normal modes is used to solve (7), that is, the perturbation ψ_1 is represented in the form

$$\psi_1(x,y,t)=\varphi(y)\exp[i(\alpha x-\beta t)], \quad (8)$$

where $\varphi(y)$ is the amplitude of the normal perturbation. Since spatial stability analysis is used in the present study, we assume that $\beta=\beta_r$ is the real frequency of the perturbation and $\alpha=\alpha_r+i\alpha_i$ is a complex number.

Substituting (8) into (7) we obtain the following differential equation

$$\begin{aligned}&\varphi_{yy}\left(\alpha u_0-\beta-iSu_0\right)-iSu_{0y}\varphi_y+2u_0\alpha/R\varphi_y\\&+\varphi\left(\alpha^2\beta-\alpha^3u_0-\alpha u_{oyy}+i\alpha^2u_0S/2\right)=0\end{aligned} \quad (9)$$

with the boundary conditions

$$\varphi(\pm\infty)=0, \quad (10)$$

where $S=c_f b/h$ is the bed-friction number and b is a characteristic length scale (in this case width of the mixing layer).

Problem (9), (10) is an eigenvalue problem. Base flow $u_0(y)$ is said to be linearly stable if all $\alpha_i>0$ and unstable if at least one $\alpha_i<0$.

3 Numerical method

Problem (9), (10) is solved numerically by means of collocation method based on Chebyshev polynomials [16]. Using the transformation $r=\frac{2}{\pi}\arctan y$ we transform the interval $-\infty<y<+\infty$ into the interval $-1<r<1$. The solution is sought in the form

$$\varphi(r)=\sum_{j=0}^{N-1}a_j(1-r^2)T_j(r), \quad (11)$$

where $T_j(r)=\cos j\arccos r$ is the Chebyshev polynomial of the first kind of degree j. The collocation points are

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$$r_m = \cos\frac{\pi m}{N}, \quad m = 1,2,\ldots,N-1. \tag{12}$$

Substituting (11) into (9) and evaluating the function $\varphi_1(r)$ and its derivatives at collocation points (12) we obtain a discretized eigenvalue problem. As it is mentioned above, the corresponding problem is linear with respect to β but nonlinear with respect to α. Hence, the following computational procedure is suggested for the solution of the problem. Assuming that both α and β are complex of the form $\alpha = \alpha_r + i\alpha_i$, $\beta = \beta_r + i\beta_i$, for each fixed S, α_r and β_r we calculate α_i such that $\beta_i = 0$. This is achieved by solving linear generalized eigenvalue problem and selecting the new approximation to β_i using bisection method. Then we change α_r (for the fixed value of S) and repeat the calculation. The region of spatial instability is described by the relation $\alpha_i < 0$.

4 Numerical Results

The base flow is selected in the form

$$u_0(y) = \frac{1}{2}\left(1 + \tanh y\right) \tag{13}$$

The first set of calculations is performed for the case without bottom friction ($S = 0$). The growth rates $-\alpha_i$ versus β_r are shown in Fig.1.

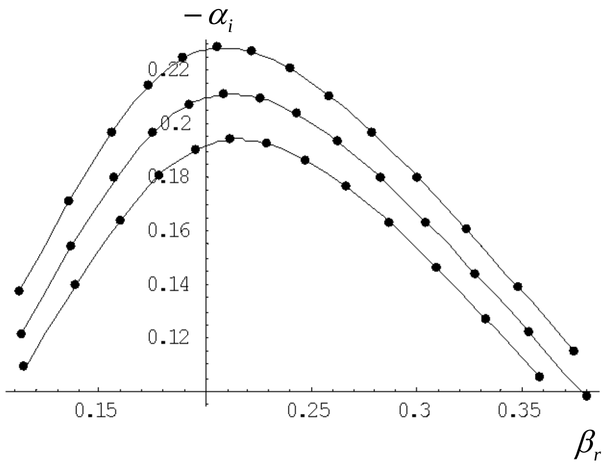

Fig.1. Growth rates $-\alpha_i$ versus β_r for three values of $1/R$=0, 0.025 and 0.05 (from top to bottom).

It follows from Fig.1 that curvature has a stabilizing influence on the flow (the growth rates decrease as the curvature increases).

The growth rates $-\alpha_i$ versus β_r are shown in Fig.2 for the case $1/R$=0.025 and three values of S, namely, S=0,0.05 and 0.1 (from top to bottom).

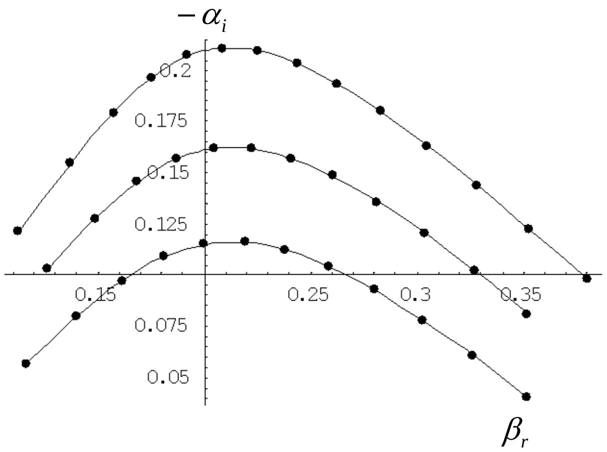

Fig.2. Growth rates $-\alpha_i$ versus β_r for three values of S=0,0.05 and 0.1 (from top to bottom).

As can be seen from Fig.2, the increase of the values of S also leads to more stable flow – the growth rates decrease as the parameter S grows.

5 Conclusion

Spatial stability of slightly curved shallow mixing layers is analyzed in the present paper. Linear stability problem is solved by collocation method based on Chebyshev polynomials. Spatial growth rates are calculated for different values of the parameters of the problem. It is shown that both the bottom friction and flow curvature have a stabilizing influence on the flow.

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