The Change in Impedance of a Single-Turn Coil due to a Cylindrical Flaw in a Conducting Half-Space

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Abstract: - Semi-analytical solution of an eddy current testing problem is obtained in the present paper for the case of a cylindrical flaw in a conducting half-space. The axis of the flaw coincides with the axis of an excitation coil. It is assumed that the vector potential is zero at a sufficiently large distance from the axis of the coil. Method of truncated eigenfunction expansions is used in the paper in order to construct the solution of the Maxwell's equations. The change in impedance of the coil is calculated for different frequencies of the excitation current. The model described in the paper can be used in practice for quality testing of spot welds by eddy current method.

Key-Words: - Eddy current testing, truncated eigenfuction expansions, change in impedance

1 Introduction

Mathematical models of eddy current testing problems developed in the literature [1]-[3] are often based on the assumption that a conducting medium is infinite in one or two spatial dimensions. The method of integral transforms (such as Fourier or Hankel integral transforms) can be used in such cases in order to construct closed-form solutions of the corresponding equations for the vector potential.

Recently a quasi-analytical approach for the solution of eddy current testing problems is suggested in [3]. The authors use the abbreviation TREE (TRuncated Eigenfuction Expansion) method. The main idea of the TREE method is that the vector potential is assumed to be zero at a sufficiently large radial distance $r = b$ from an eddy current coil (provided that there are no other sources of alternating current). Note that for the case of an unbounded medium the vector potential approaches zero at infinity. From a physical point of view the assumption of the TREE method (the vector potential is equal to zero at a large distance from the coil) is quite reasonable. Recommendations on the selection of the value of b are given in [3].

Thus, a solution of an eddy current problem with the TREE method is expressed in terms of a series (rather than integrals). This is the reason the term "TRuncated Eigenfuction Expansion" is used in order to describe the method. The main advantage of the TREE method in comparison with other analytical methods used for infinite domains is that with the TREE method one can also construct quasi-analytical solutions for the cases where a conducting medium has a finite size. Such models are quite important for applications since one can also model the presence of inhomogeneities (flaws) of finite size in a conducting medium.

In this paper we consider a model which can be used to test the quality of spot welding [4]-[6] by eddy current methods. In this case a cast core is formed as a result of the welding process. The properties of the core are close to the properties of the surrounding medium. From a mathematical point of view we consider a symmetric problem where a coil with alternating current is located above a conducting half-space with a flaw in the form of a circular cylinder whose axis coincides with the axis of the coil. Semi-analytical solution of the problem is found by the TREE method. Results of numerical calculations of the change in impedance are discussed.

2 Solution of the Problem

Consider a coil of radius r_0 located at a distance h above a conducting half-space carrying an alternating current with frequency $\omega = 2\pi f$ (see Fig.1).

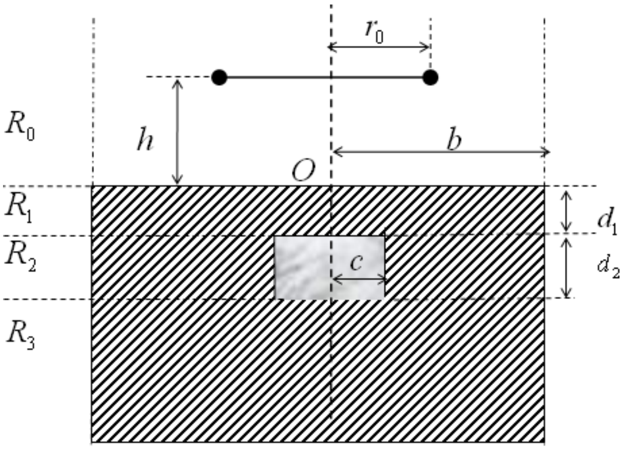

Fig.1. A single-turn coil of radius r_0 above a conducting half-space.

The half-space has a flaw in the form of a circular cylinder with height d_2 and radius c. The distance of the flaw from the surface is d_1. We use the TREE method to solve the problem. The basic assumption of the method is that the vector potential is equal to zero at $r = b$. Recommendations on the choice of b are given in [3]. The system of equations for the components of the vector potential in regions $R_i, i = 0,1,2,3$ has the form

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0)\delta(z - h), \quad (1)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma_1\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (2)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} - j\omega\sigma\mu_0 A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (3)$$

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r}\frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} - j\omega\sigma_1\mu_0 A_3 + \frac{\partial^2 A_3}{\partial z^2} = 0, \quad (4)$$

where A_i is the solution in region $R_i, i = 0,1,2,3$ (note that $\sigma = \sigma_1 - \sigma_2$ and $\sigma = \sigma_1$ in region R_2 where $0 \le r \le c$ and $c \le r \le b$, respectively). The boundary conditions are

$$A_i\,|_{r=b} = 0, \quad i = 0,1,3, \; A_2^{con}\,|_{r=b} = 0, \quad (5)$$

$$A_0\,|_{z=0} = A_1\,|_{z=0}, \frac{\partial A_0}{\partial z}\,|_{z=0} = \frac{\partial A_1}{\partial z}\,|_{z=0}, 0 \le r \le b, \quad (6)$$

$$A_1\,|_{z=-d_1} = A_2^{cc}\,|_{z=-d_1}, \frac{\partial A_1}{\partial z}\,|_{z=-d_1} = \frac{\partial A_2^{cc}}{\partial z}\,|_{z=-d_1}, \quad 0 \le r \le c, \quad (7)$$

$$A_1\,|_{z=-d_1} = A_2^{con}\,|_{z=-d_1}, \quad \frac{\partial A_1}{\partial z}\,|_{z=-d_1} = \frac{\partial A_2^{con}}{\partial z}\,|_{z=-d_1}, \quad c \le r \le b, \quad (8)$$

$$A_2^{cc}\,|_{z=-d_1-d_2} = A_3\,|_{z=-d_1-d_2}, \quad \frac{\partial A_2^{cc}}{\partial z}\,|_{z=-d_1-d_2} = \frac{\partial A_2}{\partial z}\,|_{z=-d_1-d_2}, \quad 0 \le r \le c, \quad (9)$$

$$A_2^{con}\,|_{z=-d_1-d_2} = A_3\,|_{z=-d_1-d_2}, \quad \frac{\partial A_2^{con}}{\partial z}\,|_{z=-d_1-d_2} = \frac{\partial A_3}{\partial z}\,|_{z=-d_1-d_2}, \quad c \le r \le b, \quad (10)$$

$$A_2^{con}\,|_{r=c} = A_2^{cc}\,|_{r=c}, \quad \frac{\partial A_2^{con}}{\partial r}\,|_{r=c} = \frac{\partial A_2^{cc}}{\partial r}\,|_{r=c}. \quad (11)$$

Here we used the abbreviations “cc” and “con” in region R_2 with the reference to conducting cylinder (cast core) and homogeneous conducting region, respectively.

Solution to (1) is obtained by the method of separation of variables in regions $R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$ and is given by (12) and (13), respectively:

$$A_{00}(r,z) = \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)} e^{-\lambda_i (h-z)} J_1(\lambda_i r), \quad (12)$$

$$A_{01}(r,z) = \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)} e^{-\lambda_i (z-h)} J_1(\lambda_i r), \quad (13)$$

where λ_i are the roots of the equation

$$J_1(\lambda_i b) = 0.$$

The general solution to (2) can be written in the form

$$A_1(r,z) = \sum_{i=1}^{\infty} (D_{4i} e^{p_i z} + D_{5i} e^{-p_i z}) J_1(\lambda_i r), \quad (14)$$

where $p_i = \sqrt{\lambda_i^2 + j\omega\sigma_1\mu_0}$.

General solution to (3) is

$$A_2^{cc}(r,z) = \sum_{i=1}^{\infty}[D_{10i}e^{p_{1i}z} + D_{11i}e^{-p_{1i}z}]J_1(p_{2i}r), \quad (15)$$

and

$$A_2^{con}(r,z) = \sum_{i=1}^{\infty}\begin{matrix}[D_{6i}J_1(q_ir) + D_{7i}Y_1(q_ir)]e^{p_{1i}z} + [D_{8i}J_1(q_ir) \\ + D_{9i}Y_1(q_ir)]e^{-p_{1i}z}\end{matrix} \quad (16)$$

where $p_{1i} = \sqrt{q_i^2 + j\omega\sigma_1\mu_0}$ and $p_{2i} = \sqrt{q_i^2 + j\omega\sigma_2\mu_0}$.

General solution to (4) which is bounded as $z \to -\infty$ is written as follows

$$A_3(r,z) = \sum_{i=1}^{\infty} D_{12i}e^{p_i z}J_1(\lambda_i r). \quad (17)$$

Using (16) and the last boundary condition in (5) we obtain the following two equations

$$D_{7i} = -D_{6i}\frac{J_1(q_ib)}{Y_1(q_ib)},\ D_{9i} = -D_{8i}\frac{J_1(q_ib)}{Y_1(q_ib)}. \quad (18)$$

Using continuity of the functions A_2^{cc} and A_2^{con} at $r = c$ (the first condition in (11)) we get

$$\sum_{i=1}^{\infty}[D_{10i}e^{p_{1i}z} + D_{11i}e^{-p_{1i}z}]J_1(p_{2i}c) =$$
$$= \sum_{i=1}^{\infty}\left\{[D_{6i}J_1(q_ic) + D_{7i}Y_1(q_ic)]e^{p_{1i}z} + [D_{8i}J_1(q_ic) + D_{9i}Y_1(q_ic)]e^{-p_{1i}z}\right\}. \quad (19)$$

The following two relationships are obtained from (19):

$$D_{10i}J_1(p_{2i}c) = D_{6i}J_1(q_ic) + D_{7i}Y_1(q_ic), \quad (20)$$
$$D_{11i}J_1(p_{2i}c) = D_{8i}J_1(q_ic) + D_{9i}Y_1(q_ic). \quad (21)$$

Combining equations (18) and (20) we obtain

$$D_{10i} = \hat{D}_{6i}[J_1(q_ic)Y_1(q_ib) - J_1(q_ib)Y_1(q_ic)],$$

where

$$\hat{D}_{6i} = \frac{D_{6i}}{J_1(p_{2i}c)Y_1(q_ib)}. \quad (22)$$

It follows from (18) and (21) that

$$D_{11i} = \hat{D}_{8i}[J_1(q_ic)Y_1(q_ib) - J_1(q_ib)Y_1(q_ic)],$$

where

$$\hat{D}_{8i} = \frac{D_{8i}}{J_1(p_{2i}c)Y_1(q_ib)}. \quad (23)$$

Solutions (15) and (16) can be written in the form

$$A_2^{cc}(r,z) = \sum_{i=1}^{\infty}T_1(q_ic)J_1(p_{2i}r)[\hat{D}_{6i}e^{p_{1i}z} + \hat{D}_{8i}e^{-p_{1i}z}], \quad (24)$$

$$A_2^{con}(r,z) = \sum_{i=1}^{\infty}T_1(q_ir)J_1(p_{2i}c)[\hat{D}_{6i}e^{p_{1i}z} + \hat{D}_{8i}e^{-p_{1i}z}], \quad (25)$$

where

$$T_1(q_ir) = J_1(q_ir)Y_1(q_ib) - J_1(q_ib)Y_1(q_ir). \quad (26)$$

Differentiating (24) with respect to r and evaluating the derivatives at $r = c$ we obtain

$$\frac{\partial A_2^{cc}}{\partial r}\Big|_{r=c} = \sum_{i=1}^{\infty} p_{2i}J_1'(p_{2i}c)T_1(q_ic)(\hat{D}_{6i}e^{p_{1i}z} + \hat{D}_{8i}e^{-p_{1i}z}), \quad (27)$$

$$\frac{\partial A_2^{con}}{\partial r}\Big|_{r=c} = \sum_{i=1}^{\infty} q_iT_1'(q_ic)J_1(p_{2i}c)(\hat{D}_{6i}e^{p_{1i}z} + \hat{D}_{8i}e^{-p_{1i}z}). \quad (28)$$

It follows from (27) and (28) and the second boundary condition in (11) that

$$p_{2i}J_1'(p_{2i}c)T_1(q_ic) = q_iT_1'(q_ic)J_1(p_{2i}c). \quad (29)$$

Equation (29) is used to determine the eigenvalues p_{2i} and the related values q_i.

Thus, the solution in regions R_0, R_1, R_2 and R_3 is given by (12)-(17). The six sets of constants in these formulas, namely, $D_{2i}, D_{4i}, D_{5i}, \hat{D}_{6i}, \hat{D}_{8i}$ and D_{12i} can be obtained from the boundary conditions (6) – (10).

Using (6) we obtain

$$D_{2j} = D_{4j} + D_{5j} - \frac{\mu_0 Ir_0}{b^2\lambda_j}\frac{J_1(\lambda_j r_0)}{J_0^2(\lambda_j b)}e^{-\lambda_j h}, \quad (30)$$

$$(\lambda_j + p_j)D_{4j} + (\lambda_j - p_j)D_{5j} = \frac{2\mu_0 Ir_0 J_1(\lambda_j r_0)e^{-\lambda_j h}}{b^2 J_0^2(\lambda_j b)}, \quad (31)$$

where $p_j = \sqrt{\lambda_j^2 + j\omega\sigma_1\mu_0}$.

Using the first conditions in (6) and (7) we obtain

$$\sum_{i=1}^{\infty}(D_{4i}e^{-p_id_1} + D_{5i}e^{p_id_1})J_1(\lambda_i r) =$$
$$= \sum_{i=1}^{\infty}T_1(q_ic)J_1(p_{2i}r)(\hat{D}_{6i}e^{-p_{1i}d_1} + \hat{D}_{8i}e^{p_{1i}d_1}), \quad 0 \le r \le c, \quad (32)$$

$$\sum_{i=1}^{\infty}(D_{4i}e^{-p_i d_1}+D_{5i}e^{p_i d_1})J_1(\lambda_i r)=$$
$$=\sum_{i=1}^{\infty}T_1(q_i r)J_1(p_{2i}c)(\hat{D}_{6i}e^{-p_{1i}d_1}+\hat{D}_{8i}e^{p_{1i}d_1}),\quad c\le r\le b,\tag{33}$$

In order to determine the coefficients $\hat{D}_{6i}$ and $\hat{D}_{8i}$ the following procedure is used. First, equations (32) and (33) are combined into one equation where the right-hand side of the resulting equation is given by different expressions on the intervals $0\le r\le c$ and $c\le r\le b$. These expressions are defined by the right-hand sides of (32) and (33), respectively. Second, the obtained equation is multiplied by $rJ_1(\lambda_j r)$ and the resulting equation is integrated with respect to r from 0 to b. Third, we use the orthogonality condition

$$\int_0^b rJ_1(\lambda_j r)J_1(\lambda_i r)dr=\begin{cases}0, & i\ne j\\ \dfrac{b^2}{2}J_0^2(\lambda_j b), & i=j.\end{cases}\tag{34}$$

and formulas

$$\tilde{a}_{ij}=\int_0^c rJ_1(\lambda_j r)J_1(q_i r)dr$$
$$=\frac{c}{\lambda_j^2-q_i^2}\left(\lambda_j J_2(\lambda_j c)J_1(q_i c)-q_i J_1(\lambda_j c)J_2(q_i c)\right)\tag{35}$$

$$\tilde{\tilde{a}}_{ij}=\int_c^b rJ_1(\lambda_j r)T_1(p_i r)dr=Y_1(p_i b)\int_c^b rJ_1(\lambda_j r)J_1(p_i r)dr$$
$$-J_1(p_i b)\int_c^b rJ_1(\lambda_j r)Y_1(p_i r)dr=$$
$$=\frac{1}{\lambda_j^2-p_i^2}\left\{\begin{aligned}&bp_i J_1(\lambda_j b)[J_1(p_i b)Y_2(p_i b)-J_2(p_i b)Y_1(p_i b)]+\\&+c\lambda_j J_2(\lambda_j c)[J_1(p_i b)Y_1(p_i c)-J_1(p_i c)Y_1(p_i b)]+\\&+cp_i J_1(\lambda_j c)[J_2(p_i c)Y_1(p_i b)-J_1(p_i b)Y_2(p_i c)]\end{aligned}\right\}\tag{36}$$

Formulas (35) and (36) can be found in [7]. The result is

$$D_{4j}e^{-p_j d_1}+D_{5j}e^{p_j d_1}$$
$$=\frac{2}{b^2J_0^2(\lambda_j b)}\sum_{i=1}^{\infty}(\hat{D}_{6i}e^{-p_{1i}d_1}+\hat{D}_{8i}e^{p_{1i}d_1})a_{ij},\tag{37}$$

where

$$a_{ij}=T_1(q_i c)\tilde{a}_{ij}+J_1(p_{2i}c)\tilde{\tilde{a}}_{ij}.\tag{38}$$

Using the same procedure and applying the second condition in (7) and in (8) we obtain

$$\left(D_{4j}e^{-p_j d_1}-D_{5j}e^{p_j d_1}\right)p_j$$
$$=\frac{2}{b^2J_0^2(\lambda_j b)}\sum_{i=1}^{\infty}p_{1i}(\hat{D}_{6i}e^{-p_{1i}d_1}-\hat{D}_{8i}e^{p_{1i}d_1})a_{ij}\tag{39}$$

Two additional equations are obtained if the same procedure is applied to (24), (25) and (17) using boundary conditions (9) and (10). The result is shown below

$$D_{12j}e^{-p_j d_3}\frac{b^2J_0^2(\lambda_j b)}{2}$$
$$=\sum_{i=1}^{\infty}(\hat{D}_{6i}e^{-p_{1i}d_3}+\hat{D}_{8i}e^{p_{1i}d_3})a_{ij},\tag{40}$$

$$D_{12j}p_j e^{-p_j d_3}\frac{b^2J_0^2(\lambda_j b)}{2}$$
$$=\sum_{i=1}^{\infty}p_{1i}(\hat{D}_{6i}e^{-p_{1i}d_3}-\hat{D}_{8i}e^{p_{1i}d_3})a_{ij},\tag{41}$$

where $d_3=d_1+d_2$.

Multiplying (37) by p_j and adding with (39) we obtain

$$D_{4j}=\frac{e^{p_j d_1}}{p_j b^2J_0^2(\lambda_j b)}$$
$$\sum_{i=1}^{\infty}\left\{\left((p_j+p_{1i})e^{-p_{1i}d_1}\hat{D}_{6i}+(p_j-p_{1i})e^{p_{1i}d_1}\hat{D}_{8i}\right)a_{ij}\right\}\tag{42}$$

Multiplying (37) by $(-p_j)$ and adding with (39) we obtain

$$D_{5j}=\frac{e^{-p_j d_1}}{p_j b^2J_0^2(\lambda_j b)}$$
$$\sum_{i=1}^{\infty}\left\{\left((p_j-p_{1i})e^{-p_{1i}d_1}\hat{D}_{6i}+(p_j+p_{1i})e^{p_{1i}d_1}\hat{D}_{8i}\right)a_{ij}\right\}\tag{43}$$

Using (31), (40)-(43) we obtain the system of equations for the unknowns $\hat{D}_{6i},\hat{D}_{8i}$. Solving the system we then calculate D_{4j},D_{5j},D_{2j} from (42), (43) and (30). The vector potential in each of the regions $R_i, i=0,1,2,3$ is then given by (12)-(17).

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3 The Change in Impedance and Numerical Results

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} A_0^{ind}(r_0, h) \cdot 2\pi r_0, \tag{44}$$

where

$$A_0^{ind}(r, z) = \sum_{i=1}^{\infty} D_{2i} J_1(\lambda_i r_0) e^{-\lambda_i h}. \tag{45}$$

We consider a numerical example. The following parameters of the problem are selected: $\sigma_1 = 18.5\,\text{Ms/m}$, $\sigma_2 = 3\,\text{Ms/m}$, $c = 2.2\,\text{mm}$, $r_0 = 4.5\,\text{mm}$, $h = 1.4\,\text{mm}$, $d_1 = 0.7\,\text{mm}$, $d_2 = 0.3\,\text{mm}$, $b = 55\,\text{mm}$.

The change in impedance is computed by means of (44) for the following seven frequencies: $f = 1,2,3,...,7\,\text{kHz}$. The results of calculations can be seen in Fig.2.

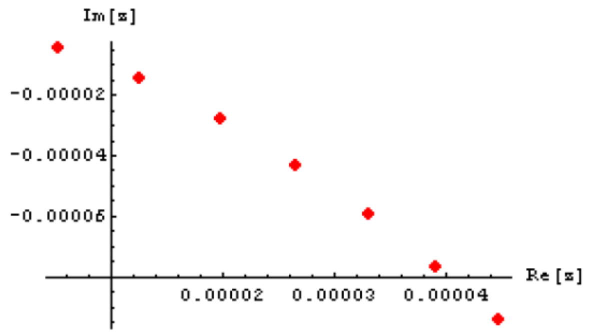

Fig.2. The change in impedance of the coil for seven frequencies ($f = 1,2,3,..,7\,\text{kHz}$, from top to bottom).

The upper limit of the summation index in (44) is fixed at $N = 62$. Comparison of the computational results obtained for other values of N showed that the chosen value of 62 is quite satisfactory in terms of calculation accuracy.

Several computational steps are necessary in order to calculate the induced change in impedance. First, the set of eigenvalues λ_i has to be calculated. This can easily be done in Mathematica using a built-in routine BesselZeros. Second, a set of complex roots of (29) should be computed. Calculations are based on the method described in [8]-[10]. Third, several systems of linear equations have to be solved in order to determine expansion coefficients. Finally, the change in impedance is computed using (44) and (45).

4 Conclusion

The change in impedance of a single-turn coil located above a conducting half-space with a flaw in the form of a conducting cylinder whose axis coincides with the axis of the coil is calculated in the present paper. The method of truncated eigenfunction expansions is used to construct a semi-analytical solution of the problem.

Several generalizations of the method presented in the paper are possible. Using superposition principle one can also construct a solution for the case where a coil with finite dimensions is located above a conducting half-space with a flaw. In addition, other important eddy current testing problems with cylindrical symmetry can be solved by the method described in the paper. Examples include surface defects of a cylindrical shape in plates in order to estimate the effect of corrosion and internal flaws for quality testing of electrically conducting materials.

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