Fuzzy Observers for Takagi-Sugeno Models with Local Nonlinear Terms

DUŠAN KROKAVEC, ANNA FILASOVÁ
Technical University of Košice
Department of Cybernetics and Artificial Intelligence
Letná 9, 042 00 Košice
SLOVAKIA
dusan.krokavec@tuke.sk, anna.filasova@tuke.sk

Abstract: The paper deals with the problem of nonlinear fuzzy observers for a class of continuous-time nonlinear systems, represented by Takagi-Sugeno models with local nonlinear terms. On the basis of the Lyapunov stability criterion and incremental quadratic inequalities, the sufficient LMI design conditions are outlined. Numerical example is given to illustrate the procedure and to validate the performances of the approach.

Key–Words: Takagi-Sugeno models, fuzzy observer, linear matrix inequality, incremental quadratic inequality.

1 Introduction

As is well known, observer design is a hot research field owing to its particular importance in the state-space control. The nonlinear system theory principles, using Lipschitz conditions, has emerged as a method capable of use in state estimation for nonlinear systems, although Lipschitz condition is a strong restrictive condition which many classes of systems may not satisfy. Design method for asymptotic observer for nonlinear systems with globally Lipschitz nonlinearities is presented, e.g., in [3], [15], the problem of designing asymptotic observers for the system whose nonlinear time-varying terms satisfy an incremental quadratic inequality, is given in [2].

An alternative to design an observer for nonlinear systems is fuzzy modeling approach, which benefits from the advantages of the approximation techniques approximating nonlinear system model equations [13]. Stability conditions, relying on the feasibility of an associated system of linear matrix inequalities (LMI) and TS fuzzy model based nonlinear state observers, were educed, e.g., in [7], [12].

In the paper, TS fuzzy models with local nonlinear terms are considered in a nonlinear state observer design task. The produced method prefer the methodology given in [4], [5], but focused in the mentioned papers on determining the compensating controllers for continuous and discrete-time systems. By using the properties of incremental quadratic constraints for local nonlinear terms of the TS model and exploiting the Krasovskii theorem, it is demonstrated that an incremental quadratic inequality, parameterized by a multiplier matrix, can be reflected in an extended LMI from the observer design conditions. Since fewer control rules can be exploited, the proposed method mainly reduce computational burden which is often favorable for implementation.

The paper is sequenced in six sections. Following the introduction in Section I, basic nature of the TS fuzzy models with local nonlinear terms is presented in Section II. The preliminary results, focused on the definition of TS fuzzy state observers with local nonlinear model terms and on the incremental quadratic constraint inequality formulation, are presented in Section III. Section IV provides the stability analysis of the nonlinear fuzzy state observer by use of LMIs, and explains the observer design conditions. Section V illustrates the observer design task by a numerical solution and the last Section VI draws some conclusion remarks.

Throughout the paper, the notations is narrowly standard in such way that \( x^T \), \( X^T \) denotes the transpose of the vector \( x \) and matrix \( X \), respectively, \( X = X^T > 0 \) means that \( X \) is a symmetric positive definite matrix, \( \text{rank}(\cdot) \) remits the rank of a matrix, the symbol \( \mathbb{I}_n \) indicates the \( n \)-th order unit matrix, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{R}^{n \times r} \) refers to the set of all \( n \times r \) real matrices.

2 Takagi-Sugeno Fuzzy Model

The systems under consideration devolve to the class of MIMO nonlinear dynamic continuous-time systems, described, using TS approach, as follows

\[
\dot{q}(t) = \sum_{i=1}^{s} h_i(\theta(t))(A_i q(t) + B_i u(t) + G_i p(t))
\]

\[
y(t) = C q(t)
\]
where \( q(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^r \), \( y(t) \in \mathbb{R}^m \) are vectors of the state, input, and output variables, respectively, \( C \in \mathbb{R}^{m \times n} \), \( A_i \in \mathbb{R}^{m \times n} \), \( B_i \in \mathbb{R}^{m \times r_i} \), \( G_i \in \mathbb{R}^{m \times r_p} \), \( i = 1, 2, \ldots, s \), are constant matrices, \( t \in \mathbb{R} \) is the time variable, \( h_i(\theta(t)) \) is the weight for \( i \)-th rule, satisfying, by definition, the property
\[
0 \leq h_i(\theta(t)) \leq 1, \quad \sum_{i=1}^{s} h_i(\theta(t)) = 1 \quad \forall i \in (1, \ldots, s)
\]
and
\[
\theta(t) = \left[ \begin{array}{cccc}
\theta_1(t) & \theta_2(t) & \cdots & \theta_v(t)
\end{array} \right]
\]
is the vector of the premise variables, where \( s, v \) are the numbers of fuzzy rules and premise variables, respectively. The nonlinear function \( p(t) \in \mathbb{R}^{rp} \) is continuous and implicitly given by [1]
\[
p(t) = \varphi(V q(t) + W p(t))
\]
and
\[
V \in \mathbb{R}^{m_p \times m}, \quad W \in \mathbb{R}^{m_p \times r_p}
\]
are constant matrices.

It is supposed in the next that all premise variables are measurable and independent on \( u(t) \) (more details can be found, e.g., in [9], [14]).

### 3 Preliminary Results

**Definition 1** Considering (1), (2), and using the same set of membership function, the nonlinear state estimator is defined as
\[
q_e(t) = \sum_{i=1}^{s} h_i(\theta(t)) \left\{ A_i q_e(t) + B_i u(t) + G_i p_e(t) + J_i(y(t) - y_e(t)) \right\}
\]
where \( q_e(t) \in \mathbb{R}^n \) is the estimation of the system state vector, \( p_e(t) \in \mathbb{R}^{rp} \) is the estimation of the nonlinear function \( p(t) \) and \( J_i \in \mathbb{R}^{mxm} \), \( i = 1, 2, \ldots, s \), and \( L \in \mathbb{R}^{m_p \times m} \) is the set of the observer gain matrices, which has to be so designed that the observer is stable.

**Proposition 2** (incremental quadratic constraint) If a matrix \( M \in \mathcal{M} \), where \( \mathcal{M} \) is the set of real incremental multiplier matrices of dimension \((m_p + r_p) \times (m_p + r_p)\), then for given matrices \( V \in \mathbb{R}^{m_p \times m} \), \( W \in \mathbb{R}^{m_p \times r_p} \), \( L \in \mathbb{R}^{m_p \times m} \), and \( C \in \mathbb{R}^{m \times n} \), the incremental quadratic constraint is
\[
\left[ e^T(t) \quad \delta p^T(t) \right] N \left[ e(t) \quad \delta p(t) \right] \geq 0
\]
where
\[
N = \begin{bmatrix}
(V - LC)^T & 0 \\
0 & I_{r_p}
\end{bmatrix}
\]

and
\[
Q = \begin{bmatrix}
I_{m_p} & W \\
0 & I_{r_p}
\end{bmatrix}
\]

Proof: Defining the state estimate error
\[
e(t) = q(t) - q_e(t)
\]
(8) can be written as
\[
p_e(t) = \varphi \left( V q(t) - e(t) + W p_e(t) + L(C q(t) - C(q(t) - e(t))) \right) = \varphi(V q(t) + W p_e(t) - (V - LC)e(t))
\]
Introducing the variables
\[
z_1(t) = V q(t) + W p_e(t)
\]
\[
z_2(t) = V q(t) + W p_e(t) - (V - LC)e(t)
\]
it yields
\[
\delta z(t) = z_1(t) - z_2(t) = (V - LC)e(t) + W \delta p(t)
\]
where
\[
\delta p(t) = p(t) - p_e(t)
\]
Since now (5), (13) implies
\[
\delta p(t) = p(t) - p_e(t) = \varphi(z_1(t)) - \varphi(z_2(t)) = \delta \varphi(t)
\]
writing (16), (18) compactly as
\[
\begin{bmatrix}
\delta z(t) \\
\delta \varphi(t)
\end{bmatrix} = \begin{bmatrix}
V - LC & W \\
0 & I_{r_p}
\end{bmatrix} \begin{bmatrix}
e(t) \\
\delta p(t)
\end{bmatrix}
\]
\[
\begin{bmatrix}
\delta z(t) \\
\delta \varphi(t)
\end{bmatrix} = \begin{bmatrix}
I_{m_p} & W \\
0 & I_{r_p}
\end{bmatrix} \begin{bmatrix}
V - LC & 0 \\
0 & I_{r_p}
\end{bmatrix} \begin{bmatrix}
e(t) \\
\delta p(t)
\end{bmatrix}
\]
respectively, then (20) for a symmetric \( M \in \mathcal{M} \) gives
\[
\begin{bmatrix}
\delta z^T(t) & \delta \varphi^T(t)
\end{bmatrix} M \begin{bmatrix}
\delta z(t) \\
\delta \varphi(t)
\end{bmatrix} = \begin{bmatrix}
e^T(t) & \delta p^T(t)
\end{bmatrix} N \begin{bmatrix}
e(t) \\
\delta p(t)
\end{bmatrix} \geq 0
\]
where, evidently, \( N \) takes the structure (10). This concludes the proof. \( \Box \)

Note, if the nonlinear term \( p(t) \) does not depend on the derivative of a state variable, \( W \) is the zero matrix.
4 Fuzzy Observer Design

To provide an asymptotic estimate of the system state, the design objective is to give conditions on the observer gain matrices $J_i$, $i = 1, 2, \ldots, s$, and $L$, which result in asymptotic decaying the estimate error (12).

**Theorem 3** The observer (6)-(8) is asymptotically stable if there exist symmetric positive definite matrices $P \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{m_p \times m_p}$, $Y \in \mathbb{R}^{r \times r}$, and matrices $Z \in \mathbb{R}^{m_p \times m}$, $Z_i \in \mathbb{R}^{n \times m}$ such that

$$P = P^T > 0, \ X = X^T > 0, \ Y = Y^T > 0 \quad (22)$$

$$[PA_i - Z_i C + A_i^T P - C^T Z_i^T \ * \ * \ ] < 0 \quad (23)$$

for all $i \in \{1, 2, \ldots, s\}$. If the above conditions hold, the observer gain matrices can be found as

$$L = X^{-1} Z, \ J_i = P^{-1} Z_i, \ i = 1, 2, \ldots, s \quad (24)$$

Here, and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

**Proof:** If the Lyapunov function is defined as

$$v(e(t)) = e^T(t) Pe(t) > 0 \quad (25)$$

where $P = P^T > 0, \ P \in \mathbb{R}^{n \times n}$, then evaluation of the time derivative of $v(e(t))$ along a observer trajectory leads to the result

$$\dot{v}(e(t)) = \dot{e}^T(t) Pe(t) + e^T(t) P \dot{e}(t) < 0 \quad (26)$$

Using (1), (2) and (6), (7)

$$\dot{e}(t) = \sum_{i=1}^{s} h_i(\theta(t))((A_i - J_i C)e(t) + G_i \delta p(t)) \quad (27)$$

respectively. Substituting (27) in (26) results in

$$\dot{v}(e(t)) = \sum_{i=1}^{s} h_i(\theta(t))((A_i - J_i C)e(t) + G_i \delta p(t))^T Pe(t) +$$

$$+ \sum_{i=1}^{s} h_i(\theta(t)) e^T(t) P((A_i - J_i C)e(t) + G_i \delta p(t))$$

and with the notation

$$e^{\delta T}(t) = [e^T(t) \ \ \delta p^T(t)] \quad (29)$$

the time derivative of $v(e(t))$ can be written as

$$\dot{v}(e(t)) = \sum_{i=1}^{s} h_i e^{\delta T}(t) T_i^o e^o(t) \quad (30)$$

where

$$T_i^o = \begin{bmatrix} P(A_i - J_i C) + (A_i - J_i C)^T P & PG_i \ G_i^T P & 0 \end{bmatrix} \quad (31)$$

Since (31) is a singular matrix, writing (9) and (29) as

$$e^{\delta T}(t) Ne^o(t) \geq 0 \quad (32)$$

and using the Kravovskii theorem (see, e.g., [8]), then (30) can be defined as

$$\dot{v}(e(t)) = \sum_{i=1}^{s} h_i(\theta(t)) e^{\delta T}(t) T_i^o e^o(t) \leq$$

$$\leq -e^{\delta T}(t) Ne^o(t) < 0 \quad (33)$$

which implies

$$v(e(t)) \leq \sum_{i=1}^{s} h_i(\theta(t)) e^{\delta T}(t)(T_i^o + N)e^o(t) < 0$$

$$T_i^o + N < 0 \ \ \forall i \quad (34)$$

(35) respectively.

Defining the incremental multiplier matrix as

$$M = \text{diag} \left[ X - Y \right] \quad (36)$$

where $X = X^T > 0, \ X \in \mathbb{R}^{m_p \times m_p}$, $Y = Y^T > 0, \ Y \in \mathbb{R}^{r \times r}$, are symmetric positive definite matrices, then (10) implies

$$N = \begin{bmatrix} (V - LC)^T & 0 \ W^T & I_r \end{bmatrix}$$

$$X = \begin{bmatrix} V - LC & W \ W^T(V - LC) & W^T XW - Y \end{bmatrix} \quad (37)$$

where $X = X^T > 0, \ Y = Y^T > 0, \ Y \in \mathbb{R}^{r \times r}$, are symmetric positive definite matrices, then (10) implies

$$N = \begin{bmatrix} (V - LC)^T \ W^T \\ \end{bmatrix} X [-V - LC W - [0 \ I_r] Y [0 I_r]$$

respectively. Introducing the matrix variables

$$Z = XL, \ \ Z_i = PJ_i \quad (39)$$

(35) can be written as

$$T_i + N = \begin{bmatrix} (XV - ZC)^T & W^T X \ W^T & \end{bmatrix} X^{-1} \begin{bmatrix} XV - ZC & XW \ \end{bmatrix} +$$

$$+ \begin{bmatrix} PA_i - Z_i C + A_i^T P - C^T Z_i^T & PG_i \\ G_i^T P & -Y \end{bmatrix} < 0 \quad (40)$$

and applying Schur complement property, (40) implies (23). This concludes the proof. □
5 Illustrative Example

As an illustrative system model, the nonlinear dynamics of the ball-and-beam system, represented by the nonlinear fourth order state-space model, was taken from [6]

\[
\begin{align*}
\dot{q}_1(t) &= q_2(t) \\
\dot{q}_2(t) &= a(q_1(t)q_3(t) - g \sin(q_3(t))) \\
\dot{q}_3(t) &= q_4(t) \\
\dot{q}_4(t) &= u(t) \\
z(t) &= q_1(t) \\
y_1(t) &= q_1(t) \\
y_2(t) &= q_4(t) \\
y_3(t) &= q_3(t)
\end{align*}
\]

where the input variable \( u(t) \) is the angular acceleration of the beam \([\text{rad/s}^2]\), the output variable \( y(t) \) is equal \( q_1(t) \) and the measured variables are \( q_1(t), q_4(t) \) and \( q_3(t) \), while \( q_1(t) \) is the position of the ball \([\text{m}]\), \( q_2(t) \) is the velocity of the ball \([\text{m/s}]\), \( q_3(t) \) is the angle of the beam \([\text{rad}]\) and \( q_4(t) \) is the velocity of the beam \([\text{rad/s}]\).

The nonlinear model parameters are [6],

\[
a = \frac{m}{m + \frac{T}{r^2}} = 0.7143, \quad g = 9.81
\]

where

- \( m \) - the mass of the ball \( 0.11 \text{ kg} \)
- \( J \) - the inertia mom. of the ball \( 1.76 \times 10^{-5} \text{ kgm}^2 \)
- \( r \) - the radius of the ball \( 0.02 \text{ m} \)
- \( g \) - the gravitational constant \( 9.81 \text{ m/s}^2 \)

Introducing the premise variable

\[
\theta(t) = q_1(t)q_4(t)
\]

which is bounded in the prescribed sector \( q_1(t)q_4(t) \in (-d, d) = (-5, 5) \), the associated sector functions, as well as the normalized membership functions, are

\[
w_2(q_1(t)q_4(t)) = w_2(\theta(t)) = \begin{cases} 
1 & \theta(t) \geq d \\
\frac{1}{d} \theta(t) & 0 < \theta(t) < d \\
0 & \theta(t) \leq 0
\end{cases}
\]

\[
w_3(q_1(t)q_4(t)) = w_3(\theta(t)) = \begin{cases} 
0 & \theta(t) \geq d \\
-\frac{1}{d} \theta(t) & -d < \theta(t) < 0 \\
1 & \theta(t) \leq -d
\end{cases}
\]

\[
w_1(q_1(t)q_4(t)) = w_1(\theta(t)) = 1 - w_2(\theta(t)) - w_3(\theta(t))
\]

and the nonlinear function \( p(t) \) is given as

\[
p(t) = \sin(q_3(t)) = \sin\left( \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} q(t) + W p(t) \right)
\]

where

\[
q^T(t) = \begin{bmatrix} q_1(t) & q_2(t) & q_3(t) & q_4(t) \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad W = w = 0
\]

Evidently, since both \( q_1(t) \), \( q_4(t) \) are measured, the premise variable \( \theta(t) = q_1(t)q_4(t) \) can be computed.

Consequently, the representation of the nonlinear differential equations of the system in a TS fuzzy system model gives

\[
\dot{q}(t) = \sum_{i=1}^{3} h_i(\theta(t))(A_i q(t) + b u(t) + g p(t))
\]

\[
y(t) = C q(t), \quad z(t) = c^T q(t)
\]

\[
A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -ad \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \end{bmatrix}, \quad c^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\]

\[
g^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad b^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
\]

Note, the pairs \( (A_i, C) \), \( i = 1, 2, 3 \), are observable.

Solving (22)-(23) for \( P, X, Y, Z \), \( i = 1, 2, 3 \), the task was feasible and parameters were the following

\[
P = \begin{bmatrix} 0.7503 & -0.2604 & 0.0000 & 0.0000 \\
-0.2604 & 0.2523 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.6280 & -0.0848 \\
0.0000 & 0.0000 & -0.0848 & 0.7598 \end{bmatrix}
\]

\[
X = 0.8258, \quad Y = 0.8258
\]

\[
L = \begin{bmatrix} 0.0000 & 0.0046 & 0.9457 \\
2.3545 & 0.0000 & 0.0000 \\
4.9549 & 0.0000 & 0.0000 \\
0.0000 & 0.5301 & 0.7720 \\
0.0000 & 0.5471 & 0.4632 \end{bmatrix}
\]

\[
J_1 = \begin{bmatrix} 2.3551 & 0.9264 & 0.0010 \\
4.9342 & 4.2713 & 0.0136 \\
-0.0798 & 0.5314 & 0.7719 \\
-0.5577 & 0.5601 & 0.4628 \end{bmatrix}
\]

\[
J_2 = \begin{bmatrix} 2.3551 & -0.9264 & -0.0010 \\
4.9342 & -4.2713 & -0.0136 \\
0.0798 & 0.5314 & 0.7719 \\
0.5577 & 0.5601 & 0.4628 \end{bmatrix}
\]

\[
J_3 = \begin{bmatrix} 2.3551 & 0.9264 & 0.0010 \\
4.9342 & -4.2713 & -0.0136 \\
0.0798 & 0.5314 & 0.7719 \\
0.5577 & 0.5601 & 0.4628 \end{bmatrix}
\]
by which the stable global observer was obtained, while the set of stable eigenvalue spectrum of subsystems is

$$\rho(A_{e1}) = \{-0.6595 \pm 0.4528i, -1.1773 \pm 1.8892i\}$$

$$\rho(A_{e2}) = \rho(A_{e3}) = \{-0.6771 \pm 0.4268i, -1.1665 \pm 2.0191i\}$$

where $A_{ei} = A_i - J_iC$, $i = 1, 2, 3$.

Compared with the standard algorithm, the number of premise variables is one smaller and the number of rules was reduced from six to three. In addition, the solution is not loaded with fuzzy approximation errors of the function sine.

6 Concluding Remarks

Newly introduced nonlinear fuzzy observer design method, based on the TS state-space models with nonlinear terms, is presented in the paper. This is achieved by application of an enhanced Lyapunov inequality, reflecting the incremental quadratic constraint parameterized by a symmetric multiplier matrix, as well as the Krasovskii theorem. In the presented version, the observer stability problem is solved considering the incremental quadratic constraint parameterized by application of an enhanced Lyapunov inequality, linear terms, is presented in the paper. This is achieved by application of an enhanced Lyapunov inequality, reflecting the incremental quadratic constraint parameterized by a symmetric multiplier matrix, as well as the Krasovskii theorem. In the presented version, the observer stability problem is solved considering the incremental quadratic constraint parameterized by application of an enhanced Lyapunov inequality, linear terms, is presented in the paper.

Acknowledgements: The work presented in the paper was supported by VEGA, the Grant Agency of the Ministry of Education and the Academy of Science of Slovak Republic, under Grant No. 1/0256/11. This support is very gratefully acknowledged.

References:


