The Electrical Circuits Theory and Groups of Algebraic Topology

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Abstract: It is proved that geometrical structure of a circuit naturally generates groups of homologies and cohomologies. The conception of homologically and cohomologically independence was introduced. It is proved that the ranks of one-dimensional homologic group and the rank of the one-dimensional group of coboundaries are equal to the number of homologically independent loops and to the number of cohomologically independent nodes pairs, respectively. Two pairs of vector spaces are assigned isomorphically to the groups. Also it is proved that the invariance of the input and output powers turns out to be a natural consequence of the topological nature of the circuit, which enables one to construct a tensor model of electric circuits.

Key Words: pure-loop, purer-node, orthogonal circuits, homologies and cohomologies groups, topological invariants, tensorial transformations

1 Definition of the problem.

This paper is devoted to the substantiation of the theoretical foundations of multidimensional tensor theory of linear multiloop electrical circuits. The representation of electrical circuits as a multidimensional tensorial object was introduced by G. Kron [1] in 30-th of XX century. The results obtained by him led to rising of absolutely new and deep methods of analysis and synthesis of electrical circuits. First of all we have to mention important result, obtained within the frame of the circuits' tensor theory, which consists in the proof of equivalency of various possible reconnections of the circuits and tensorial transformations of their basic matrices. It allowed to develop efficient methods of automated design of the complex circuit and, more general, large scale networks. Also we have to mentioned the methods of solution of the problem of computation of full spectra of multiloop circuits. One of such methods, based on tensorial model of electrical circuits, was represented in our previous publications [2, 3].

Despite of that G. Kron's seminal works do not have proper recognition. We may suppose that one of the reasons of such circumstances is that G. Kron did not give a strict substantiation of the tensor model for a circuit. The following main points remain obscure: how a given circuit can be associated with the corresponding linear space, how the basis of this space is defined, and how the basis is connected with currents and node potentials of a circuit. Such substantiation cannot be provided by the algebraic-graph approach either because of a certain incongruity of discrete-algebraic methods of the graph theory, by which discrete systems (circuits) are presently described, and continual by nature tensor-geometric methods. Thus there is a need to give a correct substantiation for the tensor-geometric notion of an electric circuit.

2 Topological Fundamentals of the Tensor Circuit Theory

We have firstly to state shortly the basic notions of tensorial theory of electrical circuits: pure-loop, pure-node, orthogonal and primitive circuits (introduced by G. Kron) which are used in the present paper. The first one is a circuit which consists only of loops, on the contrary a pure-node circuit consists only of node pairs, orthogonal circuits are ordinary
circuits with both loops and node pairs and primitive circuit is a circuit consisting of disconnected branches [1].

A pure-loop circuit can be easily obtained from an ordinary, i.e. orthogonal circuit: if we have a k loop circuit, then we should shortcircuit n-k= m-1 node pairs. However in the case of node analysis leading to pure-node circuits, we are to do a dual operation: to open k loops.

From the methological standpoint the difference between the geometric circuit structure (which is described by means of the graph theory) and tensor geometric methods (continual by nature) demands a special justification of the use of the latter methods for solving problems of the circuit theory. Methods of algebraic topology and, in particular, the basic notions of the theory of homological and cohomological groups, simplicial complexes and their corresponding isomorphic vector spaces may serve as a connecting bridge between graph-analytic and tensor-geometric methods [3].

Let \( a_1, a_2, ..., a_r \) be the system of independent points of an n dimensional Euclidean space \( \mathbb{R}^n \). The set \( t' \) of points x of the space \( \mathbb{R}^n \) that are written in the form \( x = \sum_{i=1}^{r} \eta_i a_i \) where \( \eta_i (i = 1, 2, ..., r) \) are real numbers satisfying the condition \( \sum_{i=1}^{r} \eta_i = 1 \), is called the r-dimensional simplex. The points \( a_i \) are called the simplex vertices, while their totality is called the simplex skeleton.

Each set of k r-dimensional simplexes is called an r-dimensional complex.

The following algebraic objects corresponding to dimensions \( r \leq n \) (\( r = 0, 1, 2, ..., n \)) are connected with each complex K of dimension \( r \) [3]:

1. The \( C_r(X), Z_r(X), B_r(X), H_r(X) \) - groups: of r-dimensional chains of the complex \( K' \), of r-dimensional cycles of the complex \( K' \), of r-dimensional boundaries of the complex \( K' \), of r-dimensional homologies of the complex \( K' \), which is the factor group of the group \( Z_r(X) \) with respect to \( B_r(X) \): \( H_r(X) = Z_r(X) / B_r(X) \);

2. The \( C'(X), Z'(X), B'(X), H'(X) \) groups: of r-dimensional cochains of the complex \( K' \), of r-dimensional cocycles of the complex \( K' \), of r-dimensional coboundaries of the complex \( K' \), of r-dimensional cohomologies of the complex \( K' \), which is a factor group of the group \( B'(X) \) with respect to \( B'(X) \):

\[
H'(X) = Z'(X) / B'(X).
\]

It is obvious that the plane graph is the one-dimensional complex \( K' \) consisting of m zero-dimensional (nodes) and n one-dimensional (branches) simplexes. Therefore the group \( B_1(X) \), which is, by definition, the image of the homomorphism \( \partial_2 \) of the group \( C_2(X) \) into the group \( C_1(X) \), is the zero group from which we have \( H_1(X) = Z_1(X) \), i.e. the one-dimensional homological group of the graph coincides with the one-dimensional group of cycles. In other words, none of the cycles of the graph is the bounding one. The boundary of the zero-dimensional chain is zero (there are no chains of dimension -1) and therefore all zero-dimensional simplexes (nodes) are cycles, i.e. \( C_0(X) = Z_0(X) \), and \( H_0(X) = C_0(X) / B_0(X) \).

As to the cohomological object of the graph, we can say the following: since there are no objects of dimension 2 and larger, each one-dimensional simplex is a cocycle, i.e. \( C^1(X) = Z^1(X) \) and therefore \( H^1(X) = C^1(X) / B^1(X) \). For zero-dimensional objects \( B_0(X) = 0 \) (there are no objects of dimension -1 and therefore there are no zero dimensional cobounding ones either), and \( H^0(X) = Z^0(X) \).

Let us now define the rank of the group \( H_1(X) \).

**Definition 1.** Closed broken lines without one-dimensional simplexes (branches) inside them are called basis loops. For example, in Fig. 1 \( e_1^o, e_2^o, e_6^o, e_5^o \) is a basis loop, while the loop \( e_1^o, e_2^o, e_3^o, e_7^o, e_6^o, e_5^o \) is not because it contains the simplex \( t_5^1 \).
where the coefficients $\alpha_i$ are equal either to 1 or to 0, $k$ is the number of basis loops.

Now we can introduce notion of homologically independent cycles. Cycles are homologically independent if $\sum_{i=1}^k \alpha_i Z_i = 0$ implies that all $\alpha_i = 0$.

If we take into account the fact that chains on the closed loops are homologically independent, then we come to

**Proposition 2.**

Any cycle of a one-dimensional complex $K$ (graph) is a linear combination of cycles defined on basis loops.

These basis cycles form the basis of an abelian group of rank $k$, whence follows

**Proposition 3.**

The rank of a one-dimensional homologic group $H_1(X)$ is equal to $k$, and $H_1(X)$ is isomorphic to a linear vector space of the same dimension.

Indeed, Proposition 2 establishes the one-to-one correspondence between basis cycles of the group $Z_i(X)$ (their number is equal to $k$) and an arbitrary system of basis vectors of a $k$-dimensional linear space. Therefore the group $Z_i(X)$ is isomorphic to this space, which, in view of the equality $Z_i(X) = H_1(X)$, proves Proposition 3. Q.E.D.

**Proposition 4.**

The group $B^1(X)$ of all one-dimensional cocycles cohomological to zero is isomorphic to a linear vector space of dimension $m-1$, where $m$ is the number of zero-dimensional simplexes (nodes) of the graph.

Indeed, the theory of duality of homological and cohomological groups implies that the groups $H_1(X)$ and $B^1(X)$ are mutual annihilators and therefore the sum of dimensions of spaces isomorphic to them must be equal to $n$, i.e. to the rank of the group $C_1(X)$. Since both $H_1(X)$ and $B^1(X)$ are subgroups (subspaces) of $C_1(X)$, and $\dim C_1(X) = n$ and $\dim H_1(X) = k$, from the Euler equality it follows that the rank $B^1(X) = n-k = m-1$, i.e. $B^1(X)$ is isomorphic to a vector space whose dimension is equal to $m-1$. Q.E.D.
The spaces isomorphic to the groups $C_1(X)$, $H_1(X)$ and $B^1(X)$ are denoted by $L^n$, $HL^k$ and $CL^{m-1}$, respectively.

Thus, the topological structure of the graph generates in a natural way the above-considered groups which enable us to associate with the graph the vector spaces $HL^k$ and $CL^{m-1}$. If we consider their conjugate spaces $HL^*k$ and $CL^{m-1}$, then we can associate with the circuit two pairs of spaces $HL^k$, $HL^*k$ and $CL^{m-1}, CL^{m-1}$, which, in view of their origin, can be called homological and cohomological, respectively.

The dimension of the homological pair of spaces is equal to the number of homologically independent cycles $k$ or, in terms of the theory of electric circuits, to the number of independent loops. The dimension of the cohomological pair of spaces turns out to be equal to a minimal number of node pairs $m-1$ (in defining a minimal number of node pairs, each node must be contained in one pair at least).

We call elements of the space $HL^k$ vectors of loop currents $i$ with components $i_1, ..., i_k$, and vectors of the space $HL^*k$ - vectors of loop voltages $e$ with components $e_1, ..., e_k$ ($e_i$ are voltage sources serially connected to circuit branches). This is motivated by the fact that the spaces $HL^k$ and $HL^*k$ are generated by the one-dimensional homological group which is in turn generated by the set of one-dimensional cycles of the graph, i.e. by loops. Hence we can associate with the spaces $HL^k$ and $HL^*k$ the loop values $i$ and $e$.

Analogously, we define the following vectors of the cohomological pair of vectors: the vector $E \in CL^{n-k}$ with components $E_1, ..., E_{n-k}$ equal to differences of potentials on $m-1$ node pairs, and the current vector $I \in CL^{n-k}$ with components $I_1, ..., I_k$ equal to currents of supply sources connected in parallel to $m-1$ node pairs. The latter is due to the fact that the spaces $CL^{m-1}$ and $CL^{m-1}$ generated by the one-dimensional group of cocycles, each of which, as proved above, is a one-dimensional simplex, i.e. a branch. But each branch is uniquely connected with the pair of nodes and therefore we can associate with the spaces $CL^{m-1}$ and $CL^{m-1}$ the node values $I$ and $E$.

Using homological and cohomological groups we have established the correspondence between circuit graphs and linear vector spaces. The nature of these groups, which establish a functional and algebraic relation between zero- and one-dimensional geometric objects of one dimensional simplexes (graphs), explains the nature of circuit duality.

Thus, the loop values $i$ and $e$ and not the currents $i$ and $I$ (as Kron asserts) are homological variables and, correspondingly, the node values $I$ and $E$ and not the voltages $e$ and $E$ (as Kron asserts) are cohomological variables. Moreover, the variables of each of these pairs are mutually conjugate because they are covariant (voltages $e$ and $E$) and contravariant values (currents $i$ and $I$). It should be emphasised that these conclusions differ from Kron’s nomenclature of dual objects.

3 Topological Invariants (Power Invariants) of Electrical Circuits

3.1 of Pure-loop and Pure-node Circuits (Power Invariants)

A pure-loop circuit is obtained from the initial primitive circuit by connecting the previously disconnected branches. Then a pure-loop circuit can be defined as a circuit (graph) for which and \(\dim HL^k = \dim HL^*k = n\) and \(\dim CL^{m-1} = \dim CL^{m-1} = 0\). It is obvious that for a pure-node circuit the following dual relations are fulfilled: \(\dim HL^k = \dim HL^*k = 0\) and \(\dim CL^{m-1} = \dim CL^{m-1} = n\).

The coordinates of the vector $i_j$ of the primitive circuit are related to the coordinates of the new vector $i'_j$ of pure-loop circuit currents by a nonsingular transformation with a matrix $Q$

$$i'_j = \sum_{i=1}^{n} Q_{ji} i_j \quad (j' = 1, 2, \ldots, n).$$

We write analogous relations for the vector of impact voltages $e$ of the conjugate space $HL^n$ (the covariant law):

$$e'_i = \sum_{j=1}^{n} P_{ij} e_j \quad (i' = 1, \ldots, n).$$
The matrices $Q$ and $P$ are related by means of $Q = (P^T)^{-1}$.

**Proposition 5.** The input power of a pure-loop circuit (i.e. $\sum_{k=1}^{n} i_k e_k$) is invariant with respect to transformations (1) and (2).

The proof of this proposition immediately follows from the following sequence of equalities [4]:

$$
\sum_{k=1}^{n} i_k e_k = \sum_{k=1}^{n} (\sum_{l=1}^{m} P_{kl} e_l) = \sum_{l=1}^{m} \sum_{k=1}^{n} P_{kl} e_l = \sum_{l=1}^{m} \sum_{k=1}^{n} i_k P_{kl} e_l = \sum_{l=1}^{m} i_l e_l.
$$

Q.E.D.

Note that relation (3) plays here an essential role, i.e. the power invariance is a direct corollary of the tensor nature of a circuit.

An analogous assertion can also be formulated for the power of the pure-loop circuit (output power).

**Proposition 6.** The output power of a circuit $\sum_{i=1}^{n} i_l E_l$ is invariant with respect to transformations $I$ and $E$.

In this case, the variables $I$ and $E$ of a primitive circuit get transformed to the variables $I'$ and $E'$ of a pure-node circuit in the cohomological spaces $CL^n$ and $CL^m$ when the matrices of contravariant transformations $Q$ (impact currents $I$) and covariant transformations $P$ (node pair voltages $E$) are used.

Remarks:

1. The matrix $Q^{-1}$ is the Kron matrix of connections $C$ obtained by using equations of the first law of Kirchhoff, and the matrix $P^{-1}$, inverse to the covariant matrix of connections $P$, is G.Kron's matrix $A$ obtained by the second law of Kirchhoff;
2. Only after introducing the notions of pure-loop and pure-node circuits it became possible to speak of the tensor nature of the above-mentioned transformations because for orthogonal (ordinary) circuits the corresponding transformation matrices are not square and in that case relation (3) does not hold a priori.

3.2 Topological invariants of orthogonal circuits.

As has already been mentioned, G. Kron applies the term “orthogonal” to ordinary circuits consisting of $k$ loops and $m-1$ node pairs ($m$ is the number of zero-dimensional simplexes – nodes). We can define them as circuits (one-dimensional complexes) for which $\dim HL^k = \dim HL^{m-k}$ and $\dim CL^{m-1} = \dim CL^{*m-1} = m-1$, i.e. as circuits that can be simultaneously described in terms of both homological and cohomological spaces (note that pure-loop circuits lack the cohomological variables $\dim CL^{m-1} = \dim CL^{*m-1} = 0$) and pure-node circuits lack the homological variables $\dim HL^k = \dim HL^{m-k} = 0$.

To an orthogonal circuit consisting of $k$ loops and $m-1$ node pairs we can assign the following four variables:

1. A homological pair consisting of the contravariant vector of loop currents $I$ and the covariant vector of loop impact voltages $E$ (the pair of spaces $HL^k$ and $HL^m$);
2. A cohomological pair consisting of the contravariant vector of impact node currents $I$ and the covariant vector of voltages of node pairs $E$ (a pair of spaces $CL^{m-1}$ and $CL^{*m-1}$).

The transformation of the variables of a primitive circuit to the variables of the considered orthogonal circuit also gives rise to the transformation matrices $P$ and $Q$ which are no longer square since these transformations are defined for spaces of different dimensions: in the case of a homological pair the variables of $n$-dimensional spaces $HL^n$ and $HL^m$ are related to the variables of $k$-dimensional spaces $HL^k$ and $HL^{m-k}$, while in the case of a cohomological pair the variables of $m$-dimensional spaces $CL^m$ and $CL^{*m}$ are related to the variables of $1$-dimensional spaces $CL^{m-1}$ and $CL^{*m-1}$.

Now it is not difficult to prove two dual propositions on the invariance of input and output powers of an orthogonal circuit.

**Proposition 7.** The input power of an orthogonal circuit is invariant with respect to transformations of a homological pair of variables $i$ and $e$.

The proof is reduced to writing the following sequence of equalities:

$$
i' e' = i' C e = Ci'e = ie.$$

Q.E.D.

**Proposition 8.** The output power of an orthogonal circuit is invariant with respect to
transformations of a cohomological pair of
variables $I$ and $E$.

The proof is reduced to writing the following
sequence of equalities:
$$ I'E' = A'I' E' = IAE' = IE \cdot \text{Q.E.D.} $$

4 Tensor objects of pure-loop and
pure-node circuits.

The invariance of input and output powers,
which is a natural consequence of the topological
structure of a circuit, makes it possible to construct the
tensor model of electric circuits.

In the spaces $HL^n$ and $CL^n$ we define the
bilinear forms

$$ i_1 L_i = \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} i_1^i i_2^j, $$

$$ E^1 C E^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} C^i E_1^i E_2^j, $$

where $L_{ij}$ and $C^{ij}$ are respectively the
inductance and capacitance matrices of the circuit
branches. If $L_{ij}$ and $C^{ij}$ matrices of bilinear forms are
symmetrical, $i_1 = i_2$ and $E^1 = E^2$, then (4) and (5)
transform to the quadratic forms $iLi$ and $ECE$ which
represent kinetic (magnetic) and potential (electric)
energy of the circuit.

To the bilinear form (4) we assign in a standard
manner the covariant tensor $T^0_2 = HL^n \otimes HL^n$ ($\otimes$ -
is the sign of a tensor product), the components of
which transform according to the covariant law for
each index:

$$ L_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} p_i^i p_j^j. $$

Analogously, to the bilinear form (5) we assign the
contravariant tensor $T^0_2 = CL^n \otimes CL^n$, the
components of which transform according to the
contravariant law for each index:

$$ C^{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} C^{ij} q_i^i q_j^j. $$

Let us define the impedance tensor $Z$. To this
end, we write the inverse matrix $C_{ij}$ to the
conductance tensor matrix $C^{ij}$:

$$ \sum_{i=1}^{n} \sum_{j=1}^{n} C^{ik} C_{ij} = \delta^i_j $$

Thus, the defined matrix $C_{ij}$ corresponds to the
two-valence tensor, covariant with respect to each one
of the indexes. Hence the impedance tensor $Z$, which
is a linear combination of tensors $L_{ij}$ and $C^{ij}$, also
turns out to be twice covariant:

$$ Z_{ij} = oL_{ij} + 1/(oC_{ij}) \quad (Z \in HL^n \otimes HL^n). $$

Therefore, for the new basis the tensor $Z$ transforms by the law (6)

$$ Z_{i' j'} = \sum_{i=1}^{n} \sum_{j=1}^{n} Z_{ij} p_{i'}^i p_{j'}^j. $$

The tensor form of Ohm’s law turns out to be a
convolution of the contravariant vector of current $i$
with the two-valent covariant tensor $Z$, which gives the
covariant vector of voltages $e$

$$ e_i = \sum_{j=1}^{n} Z_{ij} i^i. $$

Analogously, we write the dual relations for the
twice contravariant tensor of conductances $Y$:

$$ Y_{i' j'} = \sum_{i=1}^{n} \sum_{j=1}^{n} Y_{ij} q_{i'}^i q_{j'}^j; $$

$$ Y' = QYQ'^T; $$

$$ I_i = \sum_{j=1}^{n} Y_{ij} E_j. $$

It should again be emphasised that all the
above transformations may take place only in the
presence of an invariant which in the case of circuits
is power.

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