DISCRETE-TIME ADAPTIVE LQG/LTR CONTROL

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Abstract: An adaptive discrete-time LQG control with loop transfer recovery (LTR) is considered. The control problem is analyzed using state-space model and the parameter estimation problem is implemented for corresponding discrete-time model in transfer function form obtained via ZOH. Thus the direct estimation of model parameters is possible by means of standard RLS or ERLS procedure and the adaptive control is implemented through certainty equivalence principle. Computer simulations of third-order systems modeled by a second-order minimum-phase and nonminimum-phase models are given to illustrate the robustness and performance properties of the adaptive LQG/LTR controller, particularly with respect to the weighting parameter \( \rho \).

Key–Words: LQG control. Loop transfer recovery. Adaptive control

1 Introduction

The problem of adaptive LQG control is not much investigated in the literature, in particular this holds for adaptive LQG control with loop transfer recovery (LTR). Adaptive LQG control has been discussed e.g. in [13, 1, 2, 3, 4, 14], where in [2] an adaptive LQG/LTR problem was solved augmenting the basic estimator-based controller with a stable proper linear system feeding back the estimation residuals. This idea was also used for non-adaptive continuous-time systems in [10] using the \( H^\infty/H^2 \) optimization technique.

The key issue in adaptive LQG control is the closed-loop identifiability. For example in [13] a discrete-time system and a cost-biased least-squares parameter estimation was used in order to achieve overall asymptotic system optimality. In [14] a continuous-time system was considered and a modified weighted least-squares parameter estimation algorithm was used to obtain good properties of estimates. In both papers only a fully state observation case was considered.

In this paper, LQG adaptive control with possible application of LTR technique is considered. The adaptive discrete-time LQG control algorithm is proposed where the controller/filter parameters are tuned on the basis of transfer function model identification. Asymptotic performance and robustness properties are analyzed and simulations for third-order system considered as a second-order model are given. The role of tuning parameter \( \rho \) is underlined.

2 LTR for Discrete-Time System

The discrete-time system obtained with ZOH is given by

\[
\begin{align*}
\dot{x}_{t+1} &= F x_t + G w_t + w_t & (1) \\
\hat{y}_t &= H x_t + w_t & (2)
\end{align*}
\]

where \( \{w_t\} \) and \( \{u_t\} \) are sequences of independent random vector variables with zero mean and covariances \( E w_t w^T_t = \Sigma w_{t,t} \), \( \Sigma v_{t,t} = \Sigma v_{t,t} \).

The Kalman predictor in steady-state is given by

\[
\hat{x}_{t+1/t-1} = F \hat{x}_{t/t-1} + G \hat{w}_t + K_p \hat{y}_t & (3)
\]

where \( \hat{y}_t = y_t - H \hat{x}_{t/t} \) is an innovation of output at time t. The Kalman gain is given by

\[
K_p = F P_f H^T [H P_f H^T + \Sigma_v]^{-1} & (4)
\]

where \( F_f \) is the solution of Riccati equation

\[
P_f = F P_f F + \Sigma w - F P_f H H^T [H P_f H^T + \Sigma_v]^{-1} H P_f F & (5)
\]

The covariance of the innovation \( \hat{y}_t \) is \( \Sigma y = H P_f H^T + \Sigma_v \). Filtered estimate \( \hat{x}_{t/t} \) in terms of \( \hat{x}_{t/t-1} \) is

\[
\hat{x}_{t/t} = \hat{x}_{t/t-1} + K_f \hat{y}_t & (6)
\]

and its recursive version is

\[
\hat{x}_{t+1/t+1} = \hat{x}_{t/t} + (I - K_f H) G w_t + K_f \hat{y}_{t+1} & (7)
\]

where \( \hat{y}_{t+1} = \hat{y}_{t+1} - H F \hat{x}_{t/t} \) and the filter gain

\[
K_f = P_f H^T [H P_f H^T + \Sigma_v]^{-1} & (8)
\]
so $K_p = F K_f$ in view of (4). An alternative equation for (7) is
\[ \dot{x}_{t+1} = F \dot{x}_t + G u_t + K_f \hat{y}_{t+1} \] (9)
The LQG control law
\[ u_t = K_c x_t \] (10)
aims to minimize the performance criterion
\[ J = E \sum_{t=0}^{\infty} \dot{u}_t^T Q \dot{u}_t + \dot{u}_t^T R u_t. \] (11)
where
\[ K_c = -[G^T P_c G + R]^{-1} G^T P_c F \] (12)
and $P_c$ is the solution of Riccati equation
\[ P_c = F^T P_c F - F^T P_c G [G^T P_c G + R]^{-1} G^T P_c F + Q \] (13)
It is worthy noting that the optimal performance is
\[ J_{\text{opt}} = \text{tr}[P_c \Sigma_u + P_c G (G^T P_c G + R)^{-1} G^T P_c F P_f F^T]. \] (14)
When weighting matrices $Q$ and $R$ in the performance criterion (11) take form $Q = H^T H$, $R = \rho I$ and $\det(HG) \neq 0$ (i.e. the system is square) then asymptotically as $\rho \to 0$, one have the case of the simple control with criterion
\[ J = E \sum_{t=0}^{\infty} \dot{u}_t^T \dot{u}_t, \] (15)
and the criterion (14) taking a simple form
\[ J_{\text{opt}} = \text{tr} [H^T H] \Sigma_u + F P_f F^T. \] (16)
Moreover, assuming that the system (1), (2) is stabilizable and detectable it can be shown [7, 5] that $K_c$ takes then very simple form
\[ K_c = -(HG)^{-1} HF. \] (17)
If $G(z) = H(z I - F)^{-1} G$ is minimum-phase (mph) and $K_c$ takes a form (17) then the perfect recovery takes place, that is for $\rho \to 0$
\[ \Delta(z) = G(z) G_f(z) - \Phi(z) = 0, \] (18)
where the transfer function $G_f(z)$ of compensator defined by (7) and (10) can be manipulated into the form
\[ G_f(z) = -z K_c [z I - (I - K_f H)(F + G K_c)]^{-1} K_f = -z K_c [z I - F - G K_c]^{-1} K_f, \] (19)
and the filter’s open-loop return ratio is
\[ \Phi(z) = H(z I - F)^{-1} K_p. \] (20)
Putting (17) into (18) it can be seen that $\Delta(z) = 0$ so the recovery takes place.
When $G(z)$ is nonminimum-phase (nmp) then the perfect recovery is in general not possible, however is recommended and the possibility of recovery is frequently achieved in closed-loop bandwidth. The robustness can be measured by means of the $||H||_\infty$ norm of sensitivity transfer function
\[ S(z) = (I + G(z) G_f(z))^{-1}. \] (21)
For the Kalman predictor feedback, the controller is
\[ u_t = K_c \hat{y}_{t-1} \] (22)
and its transfer function is
\[ G_p(z) = K_c [z I - F + G K_c + K_p H]^{-1} K_p. \] (23)
In this case the perfect recovery cannot be in general obtained.

3 Adaptive Control

The SISO ARMAX model is given by
\[ A(q^{-1}) y_t = B(q^{-1}) u_t + C(q^{-1}) e_t \] (24)
where $A(q^{-1}), B(q^{-1})$ and $C(q^{-1})$ are polynomials in the backward shift operator $q^{-1}$, i.e. $A(q^{-1}) = 1 + a_1 q^{-1} + ... + a_n q^{-n}$, $B(q^{-1}) = b_1 q^{-1} + ... + b_n q^{-n}$, $C(q^{-1}) = 1 + c_1 q^{-1} + ... + c_n q^{-n}$ and $y_t$ is the output, $u_t$ is the control input, and $\{e_t\}$ is assumed to be a sequence of independent variables with zero mean and variance $\sigma_e^2$. Unknown system parameters $\hat{\theta} = (a_1, ... , a_n, b_1, ... , b_n, c_1, ... , c_n)^T$ are estimated on-line to obtain an updated model at time $t$, i.e. $\hat{\theta}_t$.

ARMAX model (24) has an equivalent innovation state space representation
\[ \begin{align*}
\dot{x}_{t+1} &= F x_t + g u_t + k_e e_t \\
y_t &= h^T x_t + e_t
\end{align*} \] (25, 26)
where $g = (b_1, ... , b_n)^T$, $k_e = (c_1 - a_1, ... , c_n - a_n)^T$, $h^T = (1, 0, ... , 0)$
\[ F = \begin{bmatrix} -a_1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & \ldots & 1 \\ -a_n & \ldots & 0 \end{bmatrix}. \]
Kalman predictor (3) associated with eq.(25) is
\[ \hat{x}_{t+1} = F\hat{x}_t + ku + \hat{K}_p \hat{y}_t \]  
(27)
where \( \hat{y}_t = y_t - h^T \hat{x}_t \) and \( \sigma^2_{\hat{y}_p} \) is the variance of \( \hat{y}_t \), for which it holds \( \sigma^2_{\hat{y}_p} = \sigma^2_x \). The predictor gain is now given by
\[ \hat{K}_p = (FP_f h + \sigma^2_e k_e) [h^T P_f h + \sigma^2_e]^{-1} \]  
(28)
where \( P_f \) is the solution of Riccati equation
\[ P_f = FP_f F^T + \hat{K}_p h [\sigma^2_e] - (FP_f h + \hat{K}_p \sigma^2_e) \times \]  
\[ \times [h^T P_f h + \sigma^2_e]^{-1} (FP_f h + \hat{K}_p \sigma^2_e)^T \]  
(29)
The actual model used for LQG/LTR control is obtained for current parameter estimates \( \hat{\theta} \). The LTR control law (17) is especially useful for adaptive control because it does not need solving the Riccati equation (13) for every \( \hat{\theta} \), and the feedback gain \( K_c \) can be tuned directly.

The investigated problem is to check out how the design parameter \( \rho \) used in the LTR technique can influence the performance of adaptive control in case of undermodeling. It is supposed that \( \rho \) used in adaptive LQG/LTR control can be tuned to compromise the LQG/LTR control performance, robustness and parameter estimation quality.

The issue of stability of the proposed adaptive LQG/LTR control system is of course crucial. This depends not only on the magnitude of modeling error but also on the asymptotic convergence of parameter estimates. Particularly, one should take into account that in general the parameter estimation in LQG adaptive control even in the lack of modelling error, does not assure the convergence to the true parameters.

Closed loop stability and good performance cannot be guaranteed especially during the transient stage.

4 Simulations

Consider an example of a third-order actual system
\[ G^*(s) = \frac{-s + 1}{(s + 1)(s + 2)} + \eta \frac{1}{s + 2} \]
whose nominal model \( G(s) \) is mph, so the case \( \eta = 0 \) corresponds to the lack of modeling error and \( \eta = 1 \) is the case of undermodeling.

Discretizing the continuous-time system with ZOH and sampling period \( T_s = 0.5s \) yields the following transfer function in \( q^{-1} \) operator
\[ G^*(q^{-1}) = \frac{-0.1612q^{-1} + 0.2856q^{-2} - 0.259q^{-1}}{1 - 0.9744q^{-1} + 0.223q^{-2} + \eta} \]
(31)
From Fig.4 one can see that for \( \eta = 0 \) parameter estimates again converge to the true parameters of second-order nominal model for all \( \rho \).

The case with \( \eta = 0.5 \) is shown in Fig.5. This time, again the parameter estimates converge to some stationary points, however different in general for various \( \rho \). Fig.6 represents a simulated criterion (11) plotted versus \( \rho \) for known parameters and adaptive control of nmph nominal model with \( \eta = 0 \) and \( \eta = 0.5 \). In case of greater values of \( \rho \) the adaptive system gets unstable while for nominal system the performance is quite good. The criterion values for \( \rho = 0 \) are \( J_0 = 0.1559, 0.1696, 20.7926 \), respectively.

The norms \( ||S(z)||_\infty \), \( ||G(z)||_\infty \), \( ||G^*(z)||_\infty \) and \( ||G^*(z)G^*_f(z)||_\infty \) for known parameters are plotted as shown in Fig.7 for \( \eta = 1 \) where \( G^*_f(z) \) is an additive perturbation, i.e. \( G^* = G + \eta G^*_f \). It can

As already mentioned, a second-order model was taken for identification and certainty equivalence principle was used to implement the adaptive control system to demonstrate the robustness of adaptive LQG/LTR controller with respect to undermodeling. The corresponding RLS parameter estimates of the nominal model are shown in Fig.1, for \( \rho = 0 \) and \( \rho = 1.5 \).

Fig.2 shows the case of undermodeling (\( \eta = 1 \)). From Fig.1 one can see that for \( \eta = 0 \) parameter estimates converge to the true parameters of second-order nominal model. The convergence holds for any value of \( \rho \), however the bigger the value of \( \rho \) the slower the estimation convergence and the worse performance of adaptive control can be expected.

It is also to observe that when \( \eta = 1 \) the parameter estimates converge to some stationary points, however different from the true ones and in general different for various \( \rho \). In this situation a more significant deterioration of control criterion is to be expected. This is illustrated in Fig.3 where the simulated criterion (11) is plotted versus \( \rho \) for known parameters and for adaptive control with \( \eta = 0 \) and \( \eta = 1 \). The criterion values for \( \rho = 0 \) are \( J_0 = 0.0611, 0.0693, 0.1975 \), respectively.
be observed that $\rho$ does influence the sensitivity and stability for mph system is assured for all $\rho$. For considered example of nmph system we can see that the system gets unstable for $\rho < 0.0026$. Obviously, estimation procedure introduces an additional lack of modeling in the form of uncertain estimates that can worsen stability robustness.

System parameters were identified using the RLS algorithm and output noise variance $\sigma_e^2$ was set at 0.1. Figures show that for both mph and nmph nominal systems the smaller the weight $\rho$ the better the estimates convergence rate where this effect is stronger in the case of $\eta = 0$ when the estimates tend to the true nominal values. Obviously, in the case of more general models, the recursive pseudolinear regression (RPLR) or recursive prediction error (RPEM) algorithms should then be applied. The results shown in [8], confirm that RPEM is more suitable in the considered undermodelled situation taking into account the asymptotic properties of the algorithm.

5 CONCLUSIONS

Application of loop transfer recovery technique in the context of adaptive discrete-time LQG control is presented. Parameter estimation of ARMAX model transfer function is used for tuning the discrete-time LQG/LTR compensator. The interplay between robustness, performance and estimation convergence with respect to the weight $\rho$ is underlined, particularly when $\rho = 0$. Examples of third-order actual systems

![Figure 1: MPH: estimates for $\eta = 0$](image1.png)

![Figure 2: MPH: estimates for $\eta = 1$](image2.png)

![Figure 3: MPH: plot of $J$ versus $\rho$](image3.png)

![Figure 4: NMPH: estimates for $\eta = 0$](image4.png)

![Figure 5: NMPH: estimates for $\eta = 0.5$](image5.png)
described by a second-order mph and nmph nominal models are taken for simulation. Simulation results show an effectiveness of the adaptive LQG control with possibility of using the LTR tuning parameter $\rho$ as a way for robustifying the adaptive control.

References:


