Report about Solutions of Beam on Nonlinear Elastic Foundation

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Abstract: - This report is focused on the theory and practice of straight beams on elastic (Winkler's) foundation. Nonlinear dependence for the reaction force on displacement in foundation (i.e. data of experiment) can be described via linear or cubic or linear + cubic approximations. This leads to linear or nonlinear differential equations. For solutions of nonlinear problems of mechanics, the Finite Difference Method (i.e. Central Difference Method) in combination with Newton – Raphson Method can be applied. Finally, linear and nonlinear approaches are evaluated and compared.

Key-Words: - beam, elastic foundation, nonlinearity, Finite Difference Method, Newton – Raphson Method, linear and nonlinear approximations

1 Introduction

Beams on elastic foundations are typical problems in mechanics. The basic analysis of bending of beams on an elastic foundation is developed on the assumption that the strains are small, see Fig.1 and [1] to [5].

Fig.1 Element of a straight beam on elastic foundation (undeformed and deformed state)

In this context, an elastic foundation is defined as a support which is continuously or discontinuously distributed along the length of the beam. The reaction force \( q_R = q_R (v, x, \ldots) \) /Nm\(^{-1}\) distributed in a foundation is directly a function of the deflection \( v = v(x) \) /m/ of a straight beam, see Fig.1. Shearing force and bending moments are noted as \( T /N/ \) and \( M_x /Nm/ \).

This article is focused on the solution of the straight beams on elastic foundation with nonlinear behaviour which can be approximated via linear or nonlinear dependencies. Finally, this leads to the solution of linear or nonlinear differential equations via Finite Difference Method (i.e. Central Difference Method), see [6] to [9].

2 Differential Equation for a Beam Rested on Elastic Foundation

The bending of straight beams on elastic foundations, see Fig.1, can be described by ordinary differential equation

\[
\frac{d^4 v}{dx^4} = \frac{N}{EJ_{zz}} \frac{d^2 v}{dx^2} + \frac{\beta}{GA} \frac{d^2 q_R}{dx^2} + \frac{q_R}{EJ_{zz}} = \frac{1}{EJ_{zz}} \left( q - \frac{dm}{dx} \right) + \frac{\beta}{GA} \frac{d^2 q}{dx^2} - \frac{\alpha_1}{h} \frac{d^2 (t_2 - t_1)}{dx^2}
\]  

(1)
where: \( E/\text{Pa} \) is modulus of elasticity of the beam, \( J_{zt}/\text{m}^4 \) is the major principal second moment of area \( A/\text{m}^2 \) of the beam cross-section, \( \beta/\text{m} \) is shear deflection constant of the beam, \( G/\text{Pa} \) is shear modulus of the beam, \( N/\text{N} \) is normal force, \( q = q(x)/\text{Nm}^{-1} \) is distributed load (intensity of force), \( m/\text{N} \) is distributed couple (intensity of moment), \( \alpha_t/\text{deg}^{-1} \) is coefficient of thermal expansion of the beam, \( h/\text{m} \) is depth of the beam and \( t_2 - t_1/\text{deg} \) is transversal temperature increasing in the beam. Equation (1) is derived for the situations when input parameters \( E, h, J_{zt}, N, \beta, G, A \) and \( \alpha_t \) are constant. For more information about the derivation of eq. (1), see references [1] to [5]. In the following example, the variable \( N, \beta, m, q \) and \( t_2 - t_1 \) are zero (i.e. a beam is not exposed to them). Hence eq. (1) is simplified into

\[
\frac{d^4 v}{dx^4} + \frac{q_R}{EJ_{zt}} = 0
\]

(2)

where the reaction force do not depend on longitudinal changes.

### 3 Behaviour of Elastic Foundation

According to Fig.2, the measured reaction force \( q_{RE} \) can be evaluated in the interval \( v \in (0; 0.002) \text{m} \) or \( v \in (-0.002; 0) \text{m} \). The nonlinear behavior of reaction force on displacement \( v \) in foundation (i.e. \( q_R = q_R(v) \)) can be approximated by linear or cubic or linear + cubic functions, see Table 1. These approximations (noted as \( q_{R1}, q_{R3} \) and \( q_{R13} \)) are based on the assumption that areas beneath curves are equal

\[
\int_{0}^{0.002} q_{RE} \, dv = \int_{0}^{0.002} q_{R1} \, dv = \int_{0}^{0.002} q_{R3} \, dv = \int_{0}^{0.002} q_{R13} \, dv.
\]

(3)

Integral \( \int_{0}^{0.002} q_{RE} \, dv \) was solved numerically by trapezoidal rule from 5 points. Variables \( k_1/\text{Nm}^2 \) and \( k_3/\text{Nm}^4 \) (see Table 1) are stiffnesses (i.e. spring constants) of elastic foundation.

Hence, general form of governing equation is given by nonlinear differential equation

\[
\frac{d^4 v}{dx^4} + \frac{k_1v + k_3v^3}{EJ_{zt}} = 0.
\]

(4)

### 4 Solved Example

Beam of length \( L/\text{m} \) with rectangular cross-section \( b \times h \) is resting on elastic foundation. The beam is loaded by force \( F = 10^5 \text{N} \), see Fig.3. Material and cross-sectional properties are \( E = 2\times10^{11} \text{Pa}, \)
4m 101.06666712
0.4 0.2
12
h b
−×=×=
ZTJ
and foundation properties (i.e. \( k_1 \) and \( k_3 \)) are described in Table 1.

![Beam of length 2L resting on elastic foundation and loaded by force F.](image)

The beam is described by differential eq. (4) and is symmetric. Hence, it is sufficient to solve \( \frac{1}{2} \) of the beam (i.e. interval \( x \in (0; L) \)).

Boundary conditions are

\[
\begin{align*}
\frac{dv(x=0)}{dx} &= 0, \\
v(0) &= -\frac{F}{2}, \\
M_o(x=L) &= 0, \\
v(L) &= 0.
\end{align*}
\]

(5) \( T(x=0) = -\frac{F}{2} \) \( T(x=L) = 0 \) \( M_o(x=L) = 0 \)

(6) \( T(x=0) = \frac{dv(x=0)}{dx} = \frac{-F}{2}EJ_{iZ} \) \( T(x=L) = 0 \) \( M_o(x=L) = 0 \)

(7) \( M_o(x=L) = 0 \) \( T(x=L) = 0 \) \( \frac{dv(x=L)}{dx} = 0 \)

(8)

\[
\frac{dv(x)}{dx} = 0,
\]

\[
T(x) = -\frac{F}{2}
\]

\[
M_o(x=L) = 0
\]

\[
T(x=L) = 0
\]

\[
\frac{dv(x=L)}{dx} = 0
\]

\[
\frac{dv(x=0)}{dx} = 0
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\[
\frac{dv(x=L)}{dx} = 0
\]

\[
\frac{dv(x=0)}{dx} = 0
\]

5 Central Difference Method (CDM)

![Divisions of the beam (CDM)](image)

According to the theory of Central Difference Method (CDM) described in [4], [6], [7], [8] and [9], the beam and its surroundings can be divided into \( n+5 \) nodes with step \( \Delta = \frac{L}{n} \), see Fig.4.

Denote for simplicity \( a_{1,3} = \frac{k_1 4}{EJ_{iZ}} \) and \( b = \frac{F}{2EJ_{iZ}} \). Boundary conditions (5) to (8) can be approximated by central differences as

\[
v_1 - v_{i-1} = 0,
\]

\[
v_{i+1} - 2v_i + v_{i-1} = b,
\]

\[
v_{i+1} - 2v_i + v_{i-1} = 0,
\]

\[
v_{i+1} - 2v_i + v_{i-1} + v_{i+2} = 0.
\]

(10) \( v_{i-1} - 2v_i + v_{i+1} = 0 \) \( v_{i+1} - 2v_i + v_{i-1} = 0 \)

(11) \( v_{i+1} - 2v_i + v_{i-1} + v_{i+2} = 0 \)

(12)

Similarly, eq. (4) can be approximated as

\[
v_{i-2} - 4v_{i-1} + (6 + a_i)v_i - 4v_{i+1} + a_3v_{i+2} = 0
\]

for \( i = 0, 1, 2, 3, \ldots, n \)

(13)

Now express the variables \( v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2} \) from (9)-(12) and substitute them into (13). We get a system of \( n+1 \) nonlinear equations (if \( k_3 = 0 \), then system is linear)

\[
(6 + a_i)v_0 - 8v_1 + 2v_2 + a_3v_0^3 = 0 - \cdots - f_0
\]

\[
-4v_i + (7 + a_i)v_{i-1} - 4v_{i+1} + 2v_{i+2} + a_3^2v_i^3 = 0 - \cdots - f_i
\]

\[
v_0 - 4v_1 + (6 + a_i)v_2 - 4v_3 + v_4 + a_3v_1^3 = 0 - \cdots - f_2
\]

\[
v_1 - 4v_2 + (6 + a_i)v_3 - 4v_4 + v_5 + a_3v_2^3 = 0 - \cdots - f_3
\]

\[
\vdots
\]

\[
v_{n-1} - 4v_{n-2} + (6 + a_i)v_{n-3} - 4v_{n-1} + v_n + a_3v_{n-2}^3 = 0 - \cdots - f_{n-2}
\]

\[
v_{n-1} - 4v_{n-2} + (5 + a_i)v_{n-3} - 2v_{n-1} + a_3v_{n-3}^3 = 0 - \cdots - f_{n-1}
\]

\[
2v_{n-2} - 4v_{n-1} + (2 + a_i)v_n + a_3v_n^3 = 0 - \cdots - f_n
\]

6 Newton Method for nonlinear system (NM)

This system of nonlinear equations \( \mathbf{f}(v) = 0 \), where \( \mathbf{f} = (f_0, f_1, f_2, \ldots, f_n)^T, \mathbf{v} = (v_0, v_1, v_2, \ldots, v_n)^T \), can be solved by Newton’s (Newton-Raphson) Method

\[
\mathbf{v}^{i+1} = \mathbf{v}^i - [J(\mathbf{v}^i)]^{-1} \mathbf{f}(\mathbf{v}^i)
\]

(14)

where \( \mathbf{v}^i \) and \( \mathbf{v}^{i+1} \) are old and new iterations,

\[
J(\mathbf{v}) = \frac{\partial \mathbf{f}}{\partial \mathbf{v}}
\]

is matrix of derivatives. The initial approximation \( \mathbf{v}^0 \) must be close to the expected solution to Newton’s Method. In our example, the values \( \mathbf{v}^0 = 0.001 \). For more information about NM, see [10].

7 Results

According to the chapters 3 to 6 was solved beam on elastic foundation with linear, cubic and linear + cubic dependencies of foundation (Table 1).
Some results are presented in Fig. 5 and 6 (i.e. maximal values of displacement $v_{\text{MAX}}$ and bending moment $M_{\text{OMAX}}$ – which are at the point $x = 0$).

Fig. 5 Dependence of maximal values of displacement on length $L$ of the beam

Fig. 6 Dependence of maximal bending moments on length $L$ of the beam

8 Conclusion

The beam on elastic linear and nonlinear foundation was solved via Central Difference Method (CDM) and Newton’s Method (NM). Solutions of linear and nonlinear differential equations were performed.

From the results, it is evident that combination of linear + cubic approximation for behavior of elastic foundation fits very well. However, application of NM is quite complicated but possible.

Linear approximation for behavior of elastic foundation gives worse but acceptable results. This leads to the easy solution of linear differential equation. For more information about linear solution, see [1] and [2].

On the other side, in the solved example, the cubic approximation of reaction force in foundation gives worse results than linear approximation. From this point of view, cubic approximation is not recommended.

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References: