Modification of Intermediate Problems Method for Electrical Circuits

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Abstract: Weinstein method of intermediate problems has been modified for a system with concentrated parameters – electrical circuits. Basis (initial) problem for this case of the intermediate problems method is defined. A relationship between eigenvalues (proper frequencies) of impedances of separate branches of the circuit and loop impedances is established. A concrete forms for resolvents of circuit operators and corresponding Weinstein functions are obtained. Finite steps recurrent process of intermediate problems of eigen values is determined. A new algorithm of many loop electrical LC-circuits’ full spectrum of eigen values (proper frequencies) calculation is elaborated.

Key Words: eigen values spectrum, electrical circuit, pure-loop, pure-node, orthogonal circuits, intermediate problems, Weinstein’s Function, roots multiplicity

1. Introduction

We have firstly to state shortly basic conceptions of tensorial theory of electrical circuits [1,2] and some our previous results connected with the theory [3,4].

Four types of circuits (introduced by G. Kron [1]) are used in the present paper: pure-loop, pure-node, orthogonal and primitive. The first one is a circuit which consists only of loops, on the contrary a pure-node circuit consists only of node pairs, orthogonal circuits are ordinary circuits with both loops and node pairs and primitive circuit is a circuit consisting of disconnected branches [1].

A pure-loop circuit can be easily obtained from an ordinary, i.e. orthogonal circuit: if we have a k loop circuit, then we should shortcircuit n-k= m-1 node pairs. However in the case of node analysis leading to pure-node circuits, we are to do a dual operation: to open k loops.

In [3,4] it has been shown that to each circuit, one can assign two pairs of conjugate linear vector spaces $HL_n^\alpha$, $HL_n^\nu$ and $CL_n^\alpha$, $CL_n^\nu$ one of which has a homological origin, while the other one—cohomological. Four spaces generate two pairs of conjugate variables e, i and E, I. Also invariance of input (homological) and output (cohomological) powers was proved. The latter allowed us to substantiate tensorial model of multiloop electrical circuit. From this point of view one can consider the mesh current method as the tensor form of Ohm’s law written for k-dimensional homological spaces $HL_k$ and $HL_k'$, while the node voltage method is the tensor form of Ohm’s law written for m−1=n-k-dimensional cohomological spaces $CL_{n-k}$ and $CL_{n-k}'$. The kinetic (magnetic) energy of the circuit is a bilinear form to which there corresponds a twice covariant inductance (mass) tensor. The potential (electric) energy of the circuit is a bilinear form to which there corresponds a twice contravariant capacitance (elasticity) tensor [3,4].

Another result important for the following is that to a given primitive circuit, one can assign the group $G_C$ of transformations C, which completely describes all possible kinds of pure-loop circuits can be obtained from the initial primitive circuit.

Hereafter we use notation for eigen values $\lambda$, which is equal to the second power of angular frequency $\lambda = \omega^2$. 
2. Problem Formulation

Weinstein’s method of intermediate problems was developed for infinite-dimensional problems, for which it proved to be sufficiently effective, especially for problems connected with oscillations of membranes of various configurations [5]. However the part of the method that was developed for finite-dimensional problems had no practical importance. The reason is obvious: the application of Weinstein’s function and especially of Aronszajn’s lemma require that the resolvent \( R \) be calculated for each tested value, which is absolutely impossible to do in the case of large (many-loop) circuits. Another point, which is probably the main one, is that if in the infinite-dimensional case we manage to construct a series of intermediate problems (by changing the boundary conditions, which is equivalent to imposing successively constraints) converging to the initial one and to estimate the eigenvalues from below (the classical Rietz and Galerkin methods, collocation method and others give estimates from above), while in the finitedimensional case it is not clear how to use Aronszajn’s lemma in general, since the method of intermediate problems does not give any clues as to how one can construct the so-called basic problem.

All the mentioned problems associated with Weinstein’s function and Aronszajn’s lemma become solvable in next to no time and lead to new significant results if this method is modified so as to conform to the notions of primitive and pure-loop (pure-node) circuits.

3. Problem Solution


Let us transform the initial K-loop circuit to a pure-loop circuit by shorting \( n-k \) node pairs. To this circuit there corresponds the \( n \)-dimensional operator \( Z^{(n)} \). Moreover, we have a primitive circuit, to which there corresponds also an \( n \)-dimensional operator \( Z_D \), which matrix is diagonal with the diagonal \( \lambda_{l_j} = 1/c^* \) (\( i=1,2,\ldots,n \)).

The matrices \( Z^{(n)} \) and \( Z_D \) are related through

\[
Z^{(n)}(\omega) = C^T Z_D(\omega) C,
\]

where \( C \) is an \( n \times n \) matrix of transformation of the initial primitive circuit to the connected pure-loop one.

The inverse matrix to (1) is the resolvent \( R^*_\lambda \) for a primitive circuit with diagonal elements inverted to diagonal elements \( Z_D : 1/(\lambda l_j - 1/c^*) \).

Using (1) it is easy to obtain the resolvent for \( Z^{(n)} \)

\[
R^{(n)}_\lambda = C^{-1} R^{(1)}_\lambda (C^{-1})^T
\]

We would like to emphasize the fact that (2) is in fact the resolvent obtained in a general form so that we need not to calculate it anew (i.e. to transform the matrix) for each tested \( \lambda \).

The eigenvalues of the diagonal operator \( Z_D \) are obviously equal to \( \lambda_i = 1/(\lambda_{l_j} c^*) \) (\( i=1,2,\ldots,n \)). Among them there may be multiple eigenvalues too, which from the engineering standpoint means that among the elements used to construct a k-loop circuit there are groups of elements having the same impedance values and the quantities of these groups are equal to the eigen values corresponding multiplicities.

**Proposition 1.** All pure-loop circuits contained in the group \( G_c \) of the initial primitive circuit possess pairwise equal eigenvalues equal in their turn to the eigenvalues of the primitive circuit.

Proof. From (2) it follows that

\[
\det(R^{(n)}_\lambda) = \det(C^{-1}) \det(R^{(1)}_\lambda) \det(C^{-1})^T.
\]

But \( \det C^T \) and \( \det C \) are constant values and therefore the respective determinants are equal to zero only for equal \( \lambda \).

Analogously, for two arbitrary pure-loop circuits, each of which is obtained by means of a nonsingular transformation \( C \) from a given primitive circuit, we can write

\[
\det(Z^{(n)}_j) = \det(C_{ji}^T Z^{(n)}_i C_{ij}),
\]

\( Z^{(n)}_i \) is the impedance tensor of the i-th pure-loop circuit; \( Z^{(n)}_j \) is the impedance tensor of the j-th pure-loop circuit; \( C_{ji} \) is the tensor of transformation of the basis of the i-th pure-loop circuit to that of the j-th circuit.

From (3) it follows that the equality

\[
\det(Z^{(n)}_j) = \det(Z^{(n)}_i) = \det(Z^{(n)}_0) = 0
\]

is again fulfilled for equal \( \lambda \), Q.E.D.
Thus we have obtained two important results: – the resolvent of the operator $R^{(n)}_\lambda$ of a pure-loop circuit can be obtained directly from (2) without transforming the matrix $Z^{(n)}$ and the eigenvalues of the initial primitive system are equal to the eigenvalues of a pure-loop circuit or, in other words, the eigenfrequencies of individual elements, by which the circuit is constructed, are equal to the eigenfrequencies of the constructed circuit where n-k node pairs are shorted.

The above reasoning has been carried out using the terms of the method of loop currents. The same can also be done in terms of node voltages. We are omitting the consideration of the case due to lack of space, but only note that in this case the admittance tensor $Y^n$ should be used.

Propositions 1 implies that in the method of intermediate problems we should consider as basic problems either a pure-loop circuit or a pure-node circuit because the resolvents of these problems are easily defined in a general form, and the eigenvalues are likewise easily calculated. As the basis operator we should consider the impedance tensor $Z^{(n)}$ of a pure-loop circuit (the admittance tensor $Y^{(n)}$ in the case of a pure-node circuit). Consecutive imposing of constraints on the pure-loop circuit generates sequence of respective intermediate operators $Z^{(n-1)}, Z^{(n-2)},..., Z^{(k)}$. Eigenvalues of the latter operator represent our original problem.

3.2. The Weinstein function for oscillatory circuits.

We proceed from the fact that the operator $Z^{(n-k)}$ can be obtained from the operator $Z^{(n)}$ by opening successively n-k short-circuited node pairs of a pure-loop circuit, which is equivalent to imposing n-k constraints.

If one numbers all n loops so that fictious n-k loops would get the last n-k numbers, then the opening of the j-th loop obviously leads to the constraint equation

$$i_j = 0$$

To this equation there corresponds the constraint vector $p_j = (0; 0; \ldots ; 1; \ldots , 0)$, where 1 is in the j-th position. Thus the k-loop circuit is obtained from the corresponding pure-loop circuit by imposing successively (or simultaneously) n-k constraints to which there correspond n-k mutually orthogonal, unit basis constraint vectors $p$.

From the geometric standpoint, the process of imposing r constraints corresponds to the transformation of the operator $Z^{(n)}$ to its part $Z^{(n-r)}$, which is defined on the subspace $LH^{n-r}$ of the space $LH^n$. Now we can obtain the concrete representation of the part of the operator $Z^{(n)}$.

**Proposition 2.** The operator $Z^{(n-r)}$ which is a part of the operator $Z^{(n)}$ and defined on the subspace $LH^{n-r}$ is represented in the coordinate from as a principal submatrix of order n-r of the matrix $Z^{(n)}$.

We omitting the proof of the proposition 2 which can be found in [3].

Thus, when constraints of type (4) are successively imposed on a pure-loop circuit, we obtain a number of intermediate problems on eigenvalues for a chain of operators $Z^{(n-1)}, Z^{(n-2)},..., Z^{(k)}$ each of which is in coordinate terms a principal submatrix (of order smaller by one) of the preceding operator: $Z^{(n-1)}, Z^{(n-2)},..., Z^{(n-i)}, Z^{(n-(i+1))},..., Z^{(k)}$.

Similar to (1), one can write down a transformation for pure-node circuits

$$Y^{(n)}(\omega) = A^TY_D(\omega)A,$$

where - covariant tensor $A$, which connects the conductance tensor $Y_D(\omega)$ of a primitive circuit with $Y^{(n)}(\omega)$ the tensor of an orthogonal pure-node circuit.

It has been shown that the tensors $A$ and $C$ are related by

$$A^T = C^{-1} \quad \text{(5)}$$

If the matrices $A$ and $C$ are divided into blocks in accordance with the division of circuit variables into k loop (contravariant) and n-k node variables (covariant), then it turns out that the matrices of the tensors $C$ and $A$ have the following block structures:

$$C = [C_k C_{n-k}] \quad A = [A_k A_{n-k}]$$

where $C_k$ and $A_{n-k}$ coincide with the loop and structural matrices of the circuit.

**Proposition 3.** The matrices $Z^{(n)}(\omega)$ and $Y^{(n)}(\omega)$ are reciprocal, i.e. $Y^{(n)}(\omega)$ is the resolvent for $Z^{(n)}(\omega)$, and vice versa.

1 A matrix located at the intersection of the first r rows and r columns is called a principal submatrix of order r of an arbitrary square matrix $A$ of order n.
Indeed, taking into account \( Z_D(\lambda) = (Y(\lambda))^{-1} \) and also equality (1) one can write
\[
(Y^{(n)}(\lambda))^{-1} = (A^T Y_D(\lambda) A)^{-1} = A^{-1} Y_D^{-1}(\lambda)(A^T)^{-1} = C^T Z_D(\lambda) C = Z^{(n)}(\lambda)
\]

Q.E.D.

Now one can determine the shape of Weinstein function for finite dimensional discrete system – k-loop circuits. If as a basis operator we take \( Y^{(n)}(\omega) \), then by Proposition 2 its resolvent is \( Y^{(n)}(\omega) \) and for an arbitrary LC-circuit we can write the Weinstein function as follows:
\[
W(\lambda) = \left| Y^{(n)}(\lambda) p_i, p_j \right| (i,j=n,n-1,\ldots,k+1), (6)
\]
where \( p_i \) – a vector of imposed i-th constraint.

Performing all multiplication operations in (6) we obtain the determinant of the matrix of (n-k) order, lying at the intersection of the last n-k rows and columns of the matrix \( Y^{(n)} \). This gives rise to

**Proposition 4.** The Weinstein function for the LC-circuit described by the loop matrix \( Z^{(k)} \) is the determinant of a lower right submatrix of order (n-k) of the resolvent \( Y^{(n)} \).

The dual statement is also valid.

**Proposition 4’.** The Weinstein function for the LC-circuit described by the node matrix \( Y^{(n-k)} \) is the determinant of an upper left submatrix of order k of the resolvent \( Z^{(n)} \).

One can easily establish a relation between these submatrices.

**Lemma 1.** A lower right submatrix of order (n-k) of the resolvent \( Y^{(n)} \) is a node conductance matrix \( Y^{(n-k)} \).

Indeed, rewriting (1’) in the block form and performing multiplication, we have
\[
Y^{(n)}(\lambda) = \left| \begin{array}{c|c}
A_k^T & Y_D^{(n-k)}(\lambda) A_{n-k} \\
\hline
A_k^T Y_D^{(n-k)}(\lambda) A_k & A_{n-k}^T Y_D^{(n-k)}(\lambda) A_{n-k}
\end{array} \right| =
\]

The block located in the right lower corner is the node conductance matrix \( Y^{(n-k)} \) by virtue of the fact that \( A \) coincides with the structural matrix of the circuit. Q.E.D.

The dual statement is proved analogously.

**Lemma 1’.** An upper left upper submatrix of order k of the resolvent \( Z^{(n)} \) is a loop impedance matrix \( Z^{(k)}(\lambda) \).

Propositions 4 and 4’, Lemmas 1 and 1’ immediately imply

**Proposition 5.**

The determinant of the conductance node matrix of an arbitrary k-loop LC-circuit is the Weinstein function for the loop impedance matrix of this circuit, and vice versa.

The latter proposition establishes a deep relationship of the classical loop current and node potential methods with the operator methods of many-dimensional geometry. Thus the determinant \( A_{n-k}^T Y_D(\lambda) A_{n-k} \) is, on the one hand, the Weinstein function obtained by imposing n-k constraints on the resolvent \( Y^{(n)}(\lambda) \) of the operator \( Z^{(n)}(\lambda) \) and, on the other hand, its matrix is the node conductance matrix of the considered circuit. Conversely, if the circuit is considered in terms of node analysis, then the Weinstein function \( C_k^T Z_D(\lambda) C_k \) obtained by imposing k constraints on the resolvent \( Z^{(n)}(\lambda) \) of the operator \( Y^{(n)}(\lambda) \) is the loop resistance matrix of the analyzed circuit.

### 3.3. Roots Multiplicity and Separation

It is obvious that when \( i \) constraints are imposed on a pure loop circuit we obtain new eigenvalues, which allows us to introduce

**Definition 1.**

Eigenvalues of the base oscillatory system are called eigenvalues of zeroth order (they correspond to the operator \( Z^{(n)}(\lambda) \)), while eigenvalues obtained by imposing \( i \) constraints on a pure loop circuit (they correspond to the operator \( Z^{(n-i)}(\lambda) \)) are called eigenvalues of i-th order.

It is obvious that in the light of the introduced terminology the eigenvalues of the considered k-loop circuit are eigenvalues of n-k-th order (they correspond to the operator \( Z^{(n-k)}(\lambda) \)). Thus to each k-loop circuit there correspond n-k series of eigenvalues
\[
\lambda_1^{(n-k)}, \lambda_2^{(n-k)}, \ldots, \lambda_{n-k-1}^{(n-k)}, \lambda_{n-k}^{(n-k)}, \ldots, \lambda_k^{(n-k)} \text{ (the last n-k-th series consisting of eigenvalues in the usual sense)}
\]
Note that eigenvalues of zeroth order $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ are in fact eigenvalues of individual impedances that make up a k-loop circuit and therefore in the general case the notion of an eigenvalue of zero order does not coincide with the notion of a partial frequency. However eigenvalues of individual impedances are partial frequencies for a pure loop circuit.

Also note that the presence of multiple eigenvalues of zeroth order testifies to the existence of groups of impedances of the same kind in a primitive circuit, the number of elements of each group being equal to the corresponding multiplicity.

Using Rayleigh’s theorem one can prove

**Proposition 6.** The eigenvalues of the operator $Z^{(1)}(\lambda)$ separate the eigenvalues of the operator $Z^{(0)}(\lambda)$.

Indeed, the operator $Z^{(1)}(\lambda)$ is obtained from the operator $Z^{(0)}(\lambda)$ by imposing one constraint (by opening one fictitious loop). By Rayleigh’s theorem this means that the eigenvalue of both operators satisfy inequalities

$$\lambda_j^{(n-i)} \leq \lambda_j^{(n-i+1)} \leq \lambda_j^{(n-i)}$$

Q.E.D.

From the Proposition 6 it follows that eigenvalue of a k-loop circuit (eigenvalues of order n-k) and eigenvalues of the corresponding pure-loop circuit (eigenvalues of zeroth order) are related by inequalities (this case corresponds to imposing simultaneously or searially k constraints)

$$\lambda_j^{(n)} \leq \lambda_j^{(n-k)} \leq \lambda_j^{(n)}$$

Inequalities (7) and (8) provide a simple technique of separating the roots of characteristic polynomials of operators $Z^{(0)}(\lambda)$, which enables us to construct a simple effective algorithm of defining a full range of eigenvalues of an arbitrary LC-circuit with a great number of degrees of freedom (with a great number of loops).

It should be said that, as different from the traditional approach consisting in attempts to connect eigenfrequencies and partial ones, inequalities (7), (8) and the expression obtained for a Weinstein function (Proposition 3) make it possible to connect eigenfrequencies of individual impedances with eigenfrequencies of a k-loop circuit.

A question naturally arises what happens to eigenvalues in passing from the operator $Z^{(1)}(\lambda)$ to the operator $Z^{(i)}(\lambda)$. An answer is provided by the Aronszajn’s lemma: when one constraint is imposed, the eigenvalue either may be preserved (and even its multiplicity may increase) or vanish (the latter corresponds to the case where the initial multiplicity equal to one decreases by one). This reasoning leads to

**Definition 2.** An eigenvalue of zeroth order is called i-conservative if it is preserved when i constraints are imposed and vanishes when i + 1 constraints are imposed. An eigenvalue of zeroth order $\lambda_j^{(n)}$ is called conservative if it is preserved when n-k constraints are imposed, i.e. it is an eigenvalue of both a pure loop circuit and a finite k-loop circuit.

A corollary of the definition 2 is

**Proposition 7.** Eigen values of zero-th order of a pure loop circuit (of a base oscillatory system), the multiplicity m of which is greater than or equal to the number of node pairs in a k-loop circuit (to the number n-k of fictitious loops), are conservative.

The proof of the proposition is almost obvious, and we omit it.

Proposition 7, seemingly so simple, proves to be rather effective, since when a primitive circuit has a sufficiently great number m of equal impedance (recall that their number is equal to the multiplicity of an eigenvalue of zeroth order), there is no need to calculate the corresponding eigenvalue of a k-loop circuit – it is enough only to verify the fulfilment of a simple inequality $m > (n-k)$. The latter circumstance can be used in the synthesis of circuits with a given range of eigenfrequencies by simply choosing the required number of elements (impedances) of the same kind in a primitive circuit.

### 4. The algorithm of calculation of a full spectrum of eigenvalues of an electrical circuit with finite number of degrees of freedom

Based on the foregoing results and conclusions, the algorithm of calculation of a full spectrum of eigenvalues of an electrical circuit with a finite number of degrees of freedom can be presented in the following form [6].

**Step 1.** Using the transformation $Z^{(n)} = C^T Z C$, we pass from a primitive circuit to a pure-loop circuit, where the eigenvalues of both
circuits are equal to each other. For this we introduce additional fictive $n-1$ loops.

Step 2. The first constraint (the opening of one of the introduced fictive loops) is imposed on the resulting pure-loop circuit (its eigenvalues are known).

Step 3. A Weinstein function of first order is defined.

Step 4. The roots are separated using Proposition mentioned above, i.e., we use inequalities $\lambda_{i}^{(n-i)} \leq \lambda_{j}^{(n-i)} \leq \lambda_{j+i}^{(n-i)}$, where $\lambda_{j}^{(n-i)}$ is the i-th order eigenvalue obtained by imposing $i$ constraints.

Step 5. By any numerical method, we define the roots of the resulting oscillatory system using the Weinstein function of first order obtained at Step 3. In doing so, we take into account the previously found intervals for the sought roots. It should be noted that some part of the previous roots are preserved due to the conservation property, so that there is no need to calculate them.

Step 6. Another constraint is imposed and the equality $i \leq n - k$ is verified. If it is fulfilled, we pass to Step 2, increasing by one the number of imposed constraints. Otherwise, we complete the process of solution.

Unfortunately, due to lack of space we didn’t represent concrete examples of realization of above defined algorithm of calculation of full spectrum of eigenvalues (proper frequencies) of multi-loop circuit. It is possible to find the examples in [4,6].

5. Conclusion.

Weistain method of intermediate problems has been modified for a system with concentrated parameters – electrical circuits. Basis (initial) problem is defined as pure-loop (pure-node) circuit.

A relationship between eigenvalues (proper frequencies) of impedances of separate branches of the circuit and loop impedances is established. A concrete forms for resolvents of circuit operators and corresponding Weinstein functions are obtained: the determinant of the conductance node matrix of an arbitrary k-loop LC-circuit is the Weinstein function for the loop impedance matrix of this circuit, and vice versa.

A simple technique of separating the roots of characteristic polynomials is elaborated. Finite steps recurrent process of intermediate problems of eigen values is determined. A new algorithm of many loop electrical LC-circuits’ full spectrum of eigen values (proper frequencies) calculation is elaborated.

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