Numerical Methods using the Successive Approximations for the Solution of a Fredholm Integral Equation

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Abstract: This paper presents two methods for approximating the solution of a Fredholm integral equation, using the successive approximations method with the trapezoids formula and the rectangles formula, respectively, for approximate calculation of integrals that appear in the terms of the successive approximations sequence. These approximation methods will be established under the conditions of theorem of existence and uniqueness of the solution of this integral equation in the sphere $\overline{B}(f;r)\subset C[a,b]$, that has been presented in [7].

Key-Words: - Fredholm integral equation, approximation of the solution, successive approximations method, trapezoids formula, rectangles formula

1 Overview

We consider the Fredholm integral equation:

$$x(t)=\int_a^b K(t,s)\cdot h(s,x(s),x(a),x(b))ds+f(t)\,, \qquad (1)$$

where a, $b\in \boldsymbol{R}$, $a<b$, $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times\boldsymbol{R}^3)$ or $h\in C([a,b]\times J^3)$, $J\subset\boldsymbol{R}$ is a closed interval and $f\in C[a,b]$.

In the paper [7] has been studied the existence and uniqueness of the solution of this integral equation, in the space $C[a,b]$ and in the sphere $\overline{B}(f;r)\subset C[a,b]$, respectively. Thus, have been obtained two theorems, that we present below.

First of all, we denote by M_K a positive constant, such that

$|K(t,s)|\le M_K$, *for all* $t, s\in[a,b]$.

Theorem 1 (of existence and uniqueness in the space $C[a,b]$, [7])

For the Fredholm integral equation (1) *we assume that*:

(i) $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times\boldsymbol{R}^3)$, $f\in C[a,b]$;

(ii) *there exists* $\alpha, \beta, \gamma>0$ *such that*

$$|h(s,u_1,u_2,u_3)-h(s,v_1,v_2,v_3)|\le$$
$$\le \alpha|u_1-v_1|+\beta|u_2-v_2|+\gamma|u_3-v_3|,$$

for all $s\in[a,b]$, $u_i, v_i\in\boldsymbol{R}$, $i=1,2,3$;

(iii) $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a)<1$.

Under these conditions the integral equation (1) *has a unique solution* $x^*\in C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0\in C[a,b]$.

Moreover, if x_m *is the m-th successive approximation, then we have the following estimation*:

$$\|x^*-x_m\|_{C[a,b]}\le$$
$$\le\frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\|x_1-x_0\|_{C[a,b]}. \qquad (2)$$

Theorem 2 (of existence and uniqueness in the sphere $\overline{B}(f;r)\subset C[a,b]$, [7])

For the Fredholm integral equation (1) *we assume that*:

(i) $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times J^3)$, $J\subset\boldsymbol{R}$ *is a closed interval*, $f\in C[a,b]$;

(ii) *there exists* $\alpha, \beta, \gamma>0$ *such that*

$$|h(s,u_1,u_2,u_3)-h(s,v_1,v_2,v_3)|\le$$
$$\le \alpha|u_1-v_1|+\beta|u_2-v_2|+\gamma|u_3-v_3|,$$

for all $s\in[a,b]$, $u_i, v_i\in J\subset\boldsymbol{R}$, $i=1,2,3$;

(iii) $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a)<1$.

If there exists $r>0$ *such that*

$$\left[x\in\overline{B}(f;r)\right]\Rightarrow\left[x(t)\in J\subset R\right] \qquad (3)$$

and the following condition is fulfilled:

(iv) $M_K \cdot M_h \cdot (b-a) < r$,

where M_h *is a positive constant such that, for the restriction* $h\big|_{[a,b]\times J^3}$, $J\subset \mathbf{R}$ *a closed interval, we have*:

$$|h(s,u,v,w)| \le M_h, \quad \text{for all } s\in[a,b],\ u,v,w\in J,$$

then the integral equation (1) *has a unique solution* $x^*\in \overline{B}(f;r)\subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0\in \overline{B}(f;r)\subset C[a,b]$.

Moreover, if x_m *is the m-th successive approximation, then the estimation* (2) *is met.*

The integral equations of similar type have been studied in [1], [2], [4], [5], [7], [8], [12].

The numerical analysis of an integral equation consists in establishment of a method for approximating the solution of the studied equation.

Under the conditions of theorem 2, presented above, the purpose of this paper is to develop two methods for approximating the solution of the integral equation (1), using the successive approximations method together with the following two quadrature formula: the trapezoids formula and the rectangles formula, that were used in the approximate calculation of the integrals from the terms of the successive approximations sequence.

2 The statement of the problem

To establish these procedures for approximating the solution of the integral equation (1) were used the results given by Gh Coman, I. Rus, G. Pavel and I. A. Rus [3], D. V. Ionescu [9], I. A. Rus [15], V. Mureșan [11] and Gheorghe Marinescu [10].

We suppose that the conditions of theorem 2 are fulfilled and therefore the integral equation (1) has a unique solution in the sphere $\overline{B}(f;r)\subset C[a,b]$.

We denote this solution by $x^*\in \overline{B}(f;r)\subset C[a,b]$ and it can be obtained by the successive approximations method, starting at any element $x_0\in \overline{B}(f;r)\subset C[a,b]$. In addition, if x_m is the m-th successive approximation, then the estimation (2) is true.

Therefore, for the determination of x^* we apply the successive approximations method.

To get a better result, it is considered an equidistant division Δ of the interval $[a,b]$ through the points $a=t_0<t_1<\ldots<t_n=b$.

Now, we have the sequence of successive approximations:

$$x_0(t_k)=f(t_k),\quad x_0\in \overline{B}(f;r)\subset C[a,b]$$

$$x_1(t_k)=\int_a^b K(t_k,s)\cdot h(s,x_0(s),x_0(a),x_0(b))ds+f(t_k)$$

$$x_2(t_k)=\int_a^b K(t_k,s)\cdot h(s,x_1(s),x_1(a),x_1(b))ds+f(t_k)$$

$$\cdots\cdots\cdots\cdots \tag{4}$$

$$x_{m+1}(t_k)=\int_a^b K(t_k,s)\cdot h(s,x_m(s),x_m(a),x_m(b))ds+f(t_k)$$

.. . .

Next, in the following two sections we present two methods for approximating the solution of integral equation (1), obtained by applying the successive approximations method and using also, the trapezoids formula and the rectangles formula for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence.

3 Approximation of the solution using the trapezoids formula

To that effect we suppose that $K\in C^2([a,b]\times[a,b])$, $h\in C^2([a,b]\times J^3)$, $J\subset \mathbf{R}$ is a closed interval and $f\in C^2[a,b]$.

We will approximate the terms of the successive approximations sequence using the trapezoids formula:

$$\int_a^b g(t)dt=\frac{b-a}{2n}\left[g(a)+2\sum_{i=1}^{n-1}g(t_i)+g(b)\right]+R^T(g) \tag{5}$$

where $g\in C^2[a,b]$, $a=t_0<t_1<\ldots<t_n=b$ is an equidistant division of the interval $[a,b]$, and $R^T(g)=\sum_{i=1}^{n}R_i^T(g)$ is the rest of formula (5), having the estimation:

$$\left|R^T(g)\right|\le M^T\,\frac{(b-a)^3}{12n^2}. \tag{6}$$

For the calculation of the integrals that appear in the terms of the successive approximations sequence, the trapezoids formula (5)+(6) is going to be applied.

In the general case for $x_m(t_k)$ we have:

$$x_m(t_k)=\frac{b-a}{2n}\left[K(t_k,a)\cdot h(a,x_{m-1}(a),x_{m-1}(a),x_{m-1}(b))+\right.$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i)\cdot h(t_i,x_{m-1}(t_i),x_{m-1}(a),x_{m-1}(b))+$$

$$\left.+K(t_k,b)\cdot h(b,x_{m-1}(b),x_{m-1}(a),x_{m-1}(b))\right]+$$

ISBN: 978-960-474-351-3

$$+ f(t_k) + R^T_{m,k}\ , \quad k = \overline{0,n}\ ,\ m \in N \qquad (7)$$

with the estimation of the rest

$$\left|R^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2} \cdot$$

$$\cdot \max_{s\in[a,b]} \left| \left[K(t_k, s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))\right]''_s \right| . \quad (8)$$

Since $K\in C^2([a,b]\times[a,b])$ and $h\in C^2([a,b]\times J^3)$, it results that $K\cdot h\in C^2([a,b]\times[a,b]\times J^3)$ and there exists the derivative of the function $K\cdot h$ from the expression of $R^T_{m,k}$, and therefore it has to be calculated. So, we have:

$$\frac{d(K\cdot h)}{ds} = \frac{\partial K}{\partial s} h + K\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial s}\right)$$

$$\frac{d^2(K\cdot h)}{ds^2} = \frac{\partial^2 K}{\partial s^2}\cdot h + 2\frac{\partial K}{\partial s}\cdot\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial s}\right) +$$

$$+ K\left(\frac{\partial^2 h}{\partial s^2} + 2\frac{\partial^2 h}{\partial x_{m-1}\partial s}\cdot\frac{\partial x_{m-1}}{\partial s} + \frac{\partial^2 h}{\partial x^2_{m-1}}\cdot\left(\frac{\partial x_{m-1}}{\partial s}\right)^2 + \right.$$

$$\left. + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial^2 x_{m-1}}{\partial s^2}\right)$$

and therefore

$$\left[K(t_k, s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))\right]''_s =$$

$$= \frac{\partial^2 K}{\partial s^2}\cdot h + 2\frac{\partial K}{\partial s}\cdot\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}} x'_{m-1}(s)\right) +$$

$$+ K\cdot\left(\frac{\partial^2 h}{\partial s^2} + \frac{\partial^2 h}{\partial x_{m-1}\partial s} x'_{m-1}(s) + \frac{\partial^2 h}{\partial x^2_{m-1}}\left(x'_{m-1}(s)\right)^2 + \right.$$

$$\left. + \frac{\partial h}{\partial x_{m-1}} x''_{m-1}(s)\right)$$

and

$$x_{m-1}(t) = \int_a^b K(t,s)\cdot h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds +$$

$$+ f(t)$$

$$x'_{m-1}(t) = \int_a^b \frac{\partial K(t,s)}{\partial t} h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds +$$

$$+ f'(t)$$

$$x''_{m-1}(t) = \int_a^b \frac{\partial^2 K(t,s)}{\partial t^2} h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds +$$

$$+ f''(t)\ .$$

If we take into account the expression of the derivatives of $x_{m-1}(t)$ and we denote

$$M^T_1 = \max_{\substack{|\alpha|\le 2 \\ t,s\in[a,b]}} \left|\frac{\partial^\alpha K(t,s)}{\partial t^{\alpha_1}\cdot\partial s^{\alpha_2}}\right| ,$$

$$M^T_2 = \max_{\substack{|\alpha|\le 2 \\ s\in[a,b]}} \left|\frac{\partial^\alpha h(s,u,v,w)}{\partial s^{\alpha_1}\cdot\partial u^{\alpha_2}}\right| ,$$

$$M^T_3 = \max_{\substack{\alpha\le 2 \\ t\in[a,b]}} \left|f^{(\alpha)}(t)\right| ,$$

then we obtain the following estimations for $x_{m-1}(t)$ and its derivatives:

$$\left|x_{m-1}(t)\right| \le M^T_1 M^T_2 (b-a) + M^T_3\ ,$$

$$\left|x'_{m-1}(t)\right| \le M^T_1 M^T_2 (b-a) + M^T_3\ ,$$

$$\left|x''_{m-1}(t)\right| \le M^T_1 M^T_2 (b-a) + M^T_3\ ,$$

while for the derivative of function $K\cdot h$ we have the estimation:

$$\left|\left[K(t_k, s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))\right]''_s\right| \le$$

$$\le 4M^T_1 M^T_2 + 5M^T_1 M^T_2\left[M^T_1 M^T_2 (b-a) + M^T_3\right] +$$

$$+ M^T_1 M^T_2\left[M^T_1 M^T_2 (b-a) + M^T_3\right]^2 = M^T_0\ .$$

It is obvious that M^T_0 doesn't depend on m and k, therefore the estimation of the rest $R^T_{m,k}$ is:

$$\left|R^T_{m,k}\right| \le M^T_0 \cdot \frac{(b-a)^3}{12n^2}\ , \qquad (9)$$

where $M^T_0 = M^T_0(K, D^\alpha K, h, D^\alpha h, f, D^\alpha f), |\alpha| \le 2$, and thus, we obtain a formula for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence.

Using the method of successive approximations and the formula (7) with the estimation of the rest resulted from (9), we suggest further on an algorithm, in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence. Thus, we have:

$x_0(t_k) = f(t_k)$

$$x_1(t_k) = \int_a^b K(t_k, s)\cdot h(s, f(s), f(a), f(b))ds + f(t_k) =$$

$$= \frac{b-a}{2n}\left[K(t_k,a)\cdot h(a, f(a), f(a), f(b)) + \right.$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, f(t_i), f(a), f(b)) +$$

$$\left. + K(t_k,b)\cdot h(b, f(b), f(a), f(b))\right] + f(t_k) + R_{1,k}^T =$$

$$= \tilde{x}_1(t_k) + R_{1,k}^T \quad , \quad k = \overline{0,n}$$

$$x_2(t_k) = \int_a^b K(t_k,s)\cdot h(s, x_1(s), x_1(a), x_1(b))ds + f(t_k) =$$

$$= \frac{b-a}{2n}\left[K(t_k,a)\cdot h(a, x_1(a), x_1(a), x_1(b)) + \right.$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, x_1(t_i), x_1(a), x_1(b)) +$$

$$\left. + K(t_k,b)\cdot h(b, x_1(b), x_1(a), x_1(b))\right] + f(t_k) + R_{2,k}^T =$$

$$= \frac{b-a}{2n}\left[K(t_k,a)\cdot \right.$$

$$\cdot h(a, \tilde{x}_1(a) + R_{1,0}^T, \tilde{x}_1(a) + R_{1,0}^T, \tilde{x}_1(b) + R_{1,0}^T) +$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot$$

$$\cdot h(t_i, \tilde{x}_1(t_i) + R_{1,i}^T, \tilde{x}_1(a) + R_{1,i}^T, \tilde{x}_1(b) + R_{1,i}^T) +$$

$$\left. + K(t_k,b)\cdot h(b, \tilde{x}_1(b) + R_{1,n}^T, \tilde{x}_1(a) + R_{1,n}^T, \tilde{x}_1(b) + R_{1,n}^T)\right] +$$

$$+ f(t_k) + R_{2,k}^T =$$

$$= \frac{b-a}{2n}\left[K(t_k,a)\cdot h(a, \tilde{x}_1(a), \tilde{x}_1(a), \tilde{x}_1(b)) + \right.$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, \tilde{x}_1(t_i), \tilde{x}_1(a), \tilde{x}_1(b)) +$$

$$\left. + K(t_k,b)\cdot h(b, \tilde{x}_1(b), \tilde{x}_1(a), \tilde{x}_1(b))\right] + f(t_k) + \tilde{R}_{2,k}^T =$$

$$= \tilde{x}_2(t_k) + \tilde{R}_{2,k}^T , \quad k = \overline{0,n} \quad ,$$

where

$$\left|\tilde{R}_{2,k}^T\right| \le \frac{b-a}{2n}\cdot M_K(\alpha+\beta+\gamma)\cdot$$

$$\cdot\left(\left|R_{1,0}^T\right| + 2\sum_{i=1}^{n-1}\left|R_{1,i}^T\right| + \left|R_{1,n}^T\right|\right) + \left|R_{2,k}^T\right| \le$$

$$\le M_K(\alpha+\beta+\gamma)(b-a)M_0^T\,\frac{(b-a)^3}{12n^2} + M_0^T\,\frac{(b-a)^3}{12n^2} =$$

$$= \frac{(b-a)^3}{12n^2}\cdot M_0^T\left[M_K(\alpha+\beta+\gamma)\cdot(b-a)+1\right] .$$

This reasoning continues for m = 3 and through induction we obtain:

$$x_m(t_k) = \frac{b-a}{2n}\left[K(t_k,a)\cdot h(a, \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)) + \right.$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, \tilde{x}_{m-1}(t_i), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)) +$$

$$\left. + K(t_k,b)\cdot h(b, \tilde{x}_{m-1}(b), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b))\right] +$$

$$+ f(t_k) + \tilde{R}_{m,k}^T = \tilde{x}_m(t_k) + \tilde{R}_{m,k}^T , \quad k = \overline{0,n} ,$$

and

$$\left|\tilde{R}_{m,k}^T\right| \le \frac{(b-a)^3}{12n^2}\cdot$$

$$\cdot M_0^t \cdot\left\{\left[M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]^{m-1} + \cdots + 1\right\},$$

$k = \overline{0,n}$.

Since the conditions of theorem 2 of existence and uniqueness of the solution of integral equation (1), are fulfilled, it results that $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a) < 1$, and we have the estimation:

$$\left|\tilde{R}_{m,k}^T\right| \le \frac{(b-a)^3}{12n^2\left[1 - M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]}\cdot M_0^T . \qquad (10)$$

Thus we have obtained a new sequence, $(\tilde{x}_m(t_k))_{m\in N}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in N}$ using an equidistant division of the interval [a,b], Δ: $a = t_0 < t_1 < ... < t_n = b$, with the following error in calculation:

$$\left|x_m(t_k) - \tilde{x}_m(t_k)\right| \le$$

$$\le \frac{(b-a)^3}{12n^2\left[1 - M_K(\alpha+\beta+\gamma)(b-a)\right]}M_0^T \qquad (11)$$

which, using the Chebyshev norm, becomes:

$$\left\|x_m - \tilde{x}_m\right\|_{C[a,b]} \le$$

$$\le \frac{(b-a)^3}{12n^2\left[1 - M_K(\alpha+\beta+\gamma)(b-a)\right]}M_0^T . \qquad (12)$$

Now, using the estimates (2) and (12) it is obtain the following result.

Theorem 3 *Suppose that the conditions of theorem 2 are fulfilled. In addition, we assume that the exact solution* x^* *of the integral equation* (1) *is approximated by the sequence* $(\tilde{x}_m(t_k))_{m\in N}$, $k = \overline{0,n}$, *on the nodes* t_k , $k = \overline{0,n}$, *of the equidistant division* Δ *of the interval* [a,b], *using the successive*

ISBN: 978-960-474-351-3

approximations method (4) *and the trapezoids formula* (5)+(6).

Under these conditions, the error of approximation is given by the following evaluation:

$$\|x^* - \tilde{x}_m\|_{C[a,b]} \le$$

$$\le \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\|x_1 - x_0\|_{C[a,b]} +$$

$$+\frac{(b-a)^3}{12n^2[1-M_K(\alpha+\beta+\gamma)(b-a)]}M_0^T . \qquad (13)$$

4 Approximation of the solution using the rectangles formula

To that effect we suppose that $K \in C^1([a,b]\times[a,b])$, $h \in C^1([a,b]\times J^3)$, $J \subset \mathbf{R}$ is a closed interval, $f \in C^1[a,b]$.

We will approximate the terms of the successive approximations sequence using the rectangles formula, considering the intermediary points of the division Δ of the interval $[a,b]$ on the left end of the partial intervals $\xi_I = t_i$, namely

$$\int_a^b g(t)dt = \frac{b-a}{n}[g(a) + \sum_{i=1}^{n-1} g(t_i)] + R^D(g) , \qquad (14)$$

or considering the intermediary points of the division of the interval $[a,b]$ on the right end of the partial intervals $\xi_I = t_{i+1}$, namely

$$\int_a^b g(t)dt = \frac{b-a}{n}\left[\sum_{i=1}^{n-1} g(t_i) + g(b)\right] + R^D(g) \qquad (14')$$

with a fine enough division of the interval $[a,b]$, Δ: $a = t_0 < t_1 < ... < t_n = b$, and we have assumed that $g \in C^1[a,b]$.

The rest of the formula $R^D(g) = \sum_{i=1}^{n} R_i^D(g)$ has the estimation:

$$\left|R^D(g)\right| \le M^D \frac{(b-a)^2}{n} . \qquad (15)$$

For the calculation of the integrals that appear in the terms of the successive approximations sequence, the rectangles formula (14)+(15) is going to be applied.

In the general case for $x_m(t_k)$ we have:

$$x_m(t_k) = \frac{b-a}{n}[K(t_k,a)\cdot h(a, x_{m-1}(a), x_{m-1}(a), x_{m-1}(b)) +$$

$$+\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, x_{m-1}(t_i), x_{m-1}(a), x_{m-1}(b))] +$$

$$+ f(t_k) + R_{m,k}^D ,\ k = \overline{0,n},\ m \in N \qquad (16)$$

with the estimation of the rest

$$\left|R_{m,k}^D\right| \le \frac{(b-a)^2}{n} \cdot$$

$$\cdot \max_{s\in[a,b]}\left|\frac{dK(t_k,s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))}{ds}\right| . \qquad (17)$$

Since $K \in C^1([a,b]\times[a,b])$ and $h \in C^1([a,b]\times J^3)$, it results that $K\cdot h \in C^1([a,b]\times[a,b]\times J^3)$ and there exists the derivative of the function $K\cdot h$ from the expression of $R_{m,k}^D$, and therefore it has to be calculated. So, we have:

$$\frac{d(K\cdot h)}{ds} = \frac{\partial K}{\partial s}h + K\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial s}\right),$$

and therefore

$$[K(t_k,s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))]_s' =$$

$$= \frac{\partial K}{\partial s}\cdot h + K\cdot\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}x_{m-1}'(s)\right)$$

and

$$x_{m-1}(t) = \int_a^b K(t,s)\cdot h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds +$$

$$+ f(t)$$

$$x'_{m-1}(t) = \int_a^b \frac{\partial K(t,s)}{\partial t} h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds +$$

$$+ f'(t) .$$

If we take into account the expression of the derivative of $x_{m-1}(t)$ and we denote

$$M_1^D = \max_{\substack{|\alpha|\le 1\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s)}{\partial t^{\alpha_1}\cdot\partial s^{\alpha_2}}\right| ,$$

$$M_2^D = \max_{\substack{|\alpha|\le 1\\ s\in[a,b]}}\left|\frac{\partial^\alpha h(s,u,v,w)}{\partial s^{\alpha_1}\cdot\partial u^{\alpha_2}}\right| ,$$

$$M_3^D = \max_{\substack{\alpha\le 1\\ t\in[a,b]}}\left|f^{(\alpha)}(t)\right| ,$$

then we obtain the following estimations for $x_{m-1}(t)$ and its derivative:

$$\left|x_{m-1}(t)\right| \le M_1^D M_2^D (b-a) + M_3^D ,$$

$$\left|x'_{m-1}(t)\right| \le M_1^D M_2^D (b-a) + M_3^D ,$$

while for the derivative of function $K\cdot h$ we have the estimation:

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$$\left|\left[K(t_k,s)\cdot h(s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))\right]_s^{'}\right|\le$$

$$\le M_1^D M_2^D\left[2+M_1^D M_2^D(b-a)+M_3^D\right]=M_0^D.$$

It is obvious that M_0^D doesn't depend on *m* and *k*, so the estimation of the rest is:

$$\left|R_{m,k}^D\right|\le M_0^D\cdot\frac{(b-a)^3}{12n^2},\qquad(18)$$

where $M_0^D=M_0^D(K,D^\alpha K,h,D^\alpha h,f,D^\alpha f)$, $|\alpha|\le 1$ and thus, we obtain a formula for the approximate, calculation of the integrals that appear in the terms of the successive approximations sequence.

Using the method of successive approximations and the formula (16) with the estimation of the rest resulted from (18), we suggest further on an algorithm in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and we obtain:

$x_0(t_k)=f(t_k)$

$$x_1(t_k)=\frac{b-a}{n}\left[K(t_k,a)\cdot h(a,f(a),f(a),f(b))+\right.$$

$$\left.+\sum_{i=1}^{n-1}K(t_k,t_i)\cdot h(t_i,f(t_i),f(a),f(b))\right]+f(t_k)+R_{1,k}^D=$$

$$=\tilde{x}_1(t_k)+R_{1,k}^D,\quad k=\overline{0,n}$$

$$x_2(t_k)=\int_a^b K(t_k,s)\cdot h(s,x_1(s),x_1(a),x_1(b))ds+f(t_k)=$$

$$=\frac{b-a}{n}\left[K(t_k,a)\cdot h(a,x_1(a),x_1(a),x_1(b))+\right.$$

$$\left.+\sum_{i=1}^{n-1}K(t_k,t_i)\cdot h(t_i,x_1(t_i),x_1(a),x_1(b))\right]+$$

$$+f(t_k)+R_{2,k}^D=\frac{b-a}{n}\cdot$$

$$\cdot\left[K(t_k,a)\cdot h(a,\tilde{x}_1(a)+R_{1,0}^D,\tilde{x}_1(a)+R_{1,0}^D,\tilde{x}_1(b)+R_{1,0}^D)+\right.$$

$$+\sum_{i=1}^{n-1}K(t_k,t_i)\cdot$$

$$\left.\cdot h(t_i,\tilde{x}_1(t_i)+R_{1,i}^D,\tilde{x}_1(a)+R_{1,i}^D,\tilde{x}_1(b)+R_{1,i}^D)\right]+$$

$$+f(t_k)+R_{2,k}^D=$$

$$=\frac{b-a}{n}\left[K(t_k,a)\cdot h(a,\tilde{x}_1(a),\tilde{x}_1(a),\tilde{x}_1(b))+\right.$$

$$\left.+\sum_{i=1}^{n-1}K(t_k,t_i)\cdot h(t_i,\tilde{x}_1(t_i),\tilde{x}_1(a),\tilde{x}_1(b))\right]+$$

$$+f(t_k)+\tilde{R}_{2,k}^D=$$

$$=\tilde{x}_2(t_k)+\tilde{R}_{2,k}^D,\quad k=\overline{0,n},$$

where

$$\left|\tilde{R}_{2,k}^D\right|\le\frac{b-a}{n}M_K(\alpha+\beta+\gamma)\left(\left|R_{1,0}^D\right|+\sum_{i=1}^{n-1}\left|R_{1,i}^D\right|\right)+\left|R_{2,k}^D\right|\le$$

$$\le(b-a)M_K(\alpha+\beta+\gamma)M_0^D\frac{(b-a)^2}{n}+M_0^D\frac{(b-a)^2}{n}=$$

$$=\frac{(b-a)^2}{n}\cdot M_0^D\left[M_K(\alpha+\beta+\gamma)(b-a)+1\right].$$

This reasoning continues for $m = 3$ and through induction we obtain

$$x_m(t_k)=\frac{b-a}{n}\left[K(t_k,a)\cdot h(a,\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b))+\right.$$

$$\left.+\sum_{i=1}^{n-1}K(t_k,t_i)\cdot h(t_i,\tilde{x}_{m-1}(t_i),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b))\right]+$$

$$+f(t_k)+\tilde{R}_{m,k}^D=\tilde{x}_m(t_k)+\tilde{R}_{m,k}^D,\quad k=\overline{0,n},$$

and

$$\left|\tilde{R}_{m,k}^D\right|\le\frac{(b-a)^2}{n}\cdot M_0^D\cdot$$

$$\cdot\left\{\left[M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]^{m-1}+\cdots+1\right\},$$

$k=\overline{0,n}$.

Since the conditions of theorem 2 of existence and uniqueness of the solution of integral equation (1), are fulfilled, it results that $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a)<1$, and we have the estimation:

$$\left|\tilde{R}_{m,k}^D\right|\le\frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D.\qquad(19)$$

Thus, we have obtained the sequence, $\left(\tilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $\left(x_m\right)_{m\in N}$ using an equidistant division of the interval [*a*,*b*], $\Delta: a=t_0<t_1<\ldots<t_n=b$, with the following error in calculation:

$$\left|x_m(t_k)-\tilde{x}_m(t_k)\right|\le$$

$$\le\frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D\qquad(20)$$

which, using the Chebyshev norm, becomes:

$$\left\|x_m-\tilde{x}_m\right\|_{C[a,b]}\le$$

ISBN: 978-960-474-351-3

$$\leq \frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D. \qquad (21)$$

Now, using the estimates (2) and (21) it is obtain the following result.

Theorem 4 *Suppose that the conditions of theorem* 2 *are fulfilled. In addition, we assume that the exact solution* x^* *of the integral equation* (1) *is approximated by the sequence* $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$ *on the nodes* t_k, $k=\overline{0,n}$, *of the equidistant division* Δ *of the interval* $[a,b]$, *using the successive approximations method* (4) *and the rectangles formula* (14)+(15).

Under these conditions, the error of approximation is given by the evaluation:

$$\|x^*-\tilde{x}_m\|_{C[a,b]}\leq$$

$$\leq \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\|x_1-x_0\|_{C[a,b]}+$$

$$+\frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D. \qquad (22)$$

5 Conclusion

Regarding to the two methods presented in this paper, we observe the following:

a) In both cases we used the method of successive approximations;

b) The terms of successive approximations sequence were approximated using two quadrature formulas: the trapezoids formula (5)+(6), and the rectangles formula (14)+(15), respectively.

c) In both cases we used an equidistant division, Δ, of the interval $[a,b]$ through the points $a=t_0<t_1<...<t_n=b$; also, in the both cases, we obtain a new sequence $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in N}$;

d) We obtained the following estimates of the error of approximation, on nodes, of the terms of successive approximations sequence:

- using the trapezoids formula:

$$\left|\tilde{R}_{m,k}^T\right|\leq \frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]}\cdot M_0^T$$

- using the rectangles formula:

$$\left|\tilde{R}_{m,k}^D\right|\leq \frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D$$

and both values M_0^T and M_0^D, from evaluation of rests, are independent of *m* and *k*;

e) Finally, the error of approximation of the exact solution of the integral equation (1) through the terms of the new string $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, is given by the relation (13) when we used the trapezoids formula and by the relation (22) when we used the rectangles formula.

f) Of the two estimates of remainder, from above, we deduce that the approximation error of the solution obtained by applying the successive approximations method is less if the trapezoidal formula is used, than if the rectangle formula is used.

Finally, it should be noted that all articles and books, respectively, from references constituted an important research material in preparation of this article.

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