A characterization of projective special linear group $L_3(8)$ by nse

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Abstract: Let $G$ be a group and $\omega(G)$ be the set of element orders of $G$. Let $k \in \omega(G)$ and $s_k$ be the number of elements of order $k$ in $G$. Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. In Khatami et al and Liu’s works, $L_3(2)$ and $L_3(4)$ are unique determined by nse. In this paper, we prove that if $G$ is a group such that $\text{nse}(G) = \text{nse}(L_3(8))$, then $G \cong L_3(8)$.

Key–Words: Element order, Projective special linear group, Thompson’s problem, Number of elements of the same order, Simple group, Element order.

1 Introduction
We introduce some notations which is needed. Let $a,b$ denote the products of an integer $a$ by an integer $b$. $L_n(q)$ denotes the projective special linear group of degree $n$ over finite fields of order $q$. $U_n(q)$ denotes the projective special unitary group of degree $n$ over finite fields of order $q$. Let $r$ be a prime and $G$ be a group. Then $P_r$ denotes the Sylow $r$-subgroups of $G$ and $n_r$ or $n_r(G)$ denotes the number of Sylow $r$-subgroups of $G$. The notations are standard (see [1] and [15]).

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [16]).

Thompson’s Problem. Let $T(G) = \{(n,s_n)|n \in \omega(G) \text{ and } s_n \in \text{nse}(G)\}$, where $s_n$ is the number of elements with order $n$. Suppose that $T(G)=T(H)$. If $G$ is a finite solvable group, is it true that $H$ is also necessarily solvable? If the groups $G$ and $H$ have the same order type, then $|G|=|H|$ and $\text{nse}(G) = \text{nse}(H)$. The following results are gotten.

Result 1: Let $G$ be a group and $M$ some simple $K_i$-group, $i = 3, 4$, then $G \cong M$ if and only if $|G|=|M|$ and $\text{nse}(G)=\text{nse}(M)$ (see [12, 13]).

Result 2: The group $A_{12}$ is characterizable by order and nse (see [7]).

Result 3: All sporadic simple groups are characterizable by nse and order (see [5]).

Result 4: $L_2(2^m)$ with $2^m+1$ prime or $2^m-1$ prime, is characterized by nse and order (see [14]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson’s Problem, in other words, it remains only nse(G), whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set $\text{nse}(G)$ (see [6, 15]). The author has proved that the groups $L_3(4), U_3(5)$ and $L_5(2)$ are characterizable by nse (see [8, 9, 10]). In this paper, it is shown that the group $L_3(8)$ also can be characterized by nse.

2 Introduction

Lemma 1 [2] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m || L_m(G)$.

Lemma 2 [11] Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^r m$ with $(p,m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

Lemma 3 [15] Let $G$ be a group containing more than two elements. If the maximal number $s$ of elements of the same order in $G$ is finite, then $G$ is finite and $|G| \leq s(s^2-1)$.

Lemma 4 [3, Theorem 9.3.1] Let $G$ be a finite solvable group and $|G| = mn$, where $m = p_1^{a_1} \cdots p_r^{a_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and $h_m$ be
Let $G$ be a group such that $\text{nse}(G)=\text{nse}(L_3(8))$, and $s_n$ be the number of elements of order $n$. By Lemma 3 we have that $G$ is finite. We note that $s_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have
\begin{equation}
\{ \phi(m) \mid s_m; \ m \mid \sum_{d|m} s_d \}
\end{equation}

**Theorem 7** Let $G$ be a group with $\text{nse}(G)=\text{nse}(L_3(8))\in\{1, 4599, 257544, 261632, 784896, 1569792, 1709952, 1766016, 4709376, 5419008\}$, where $L_3(8)$ is the projective special linear group of degree $3$ over field of order $8$. Then $G \cong L_3(8)$.

**Proof.** We prove the theorem by first proving that $\pi(G) \subseteq \{2, 3, 7, 73\}$, second showing that $|G| = |L_3(8)|$, and so $G \cong L_3(8)$. 

**Case 3.** $G \cong L_2(3^m)$.

We will consider the case by the following two cases.

* Subcase 3.1. $3^m+1 = 4t$ and $3^m-1 = 2u^2$.

We can suppose that $t \in \{3, 7, 73\}$

Let $t = 3, 7, 73$, the equation $3^m+1 = 4t$ has no solution. So we rule out the case.

Let $t = 7$, then $m = 3$ and so $3^3-1 = 2 \cdot 11$, which means $11 \mid |G|$, a contradiction.

* Subcase 3.2. $3^m+1 = 4t$ and $3^m-1 = 2u$.

We can suppose that $u \in \{3, 7, 19\}$

Let $u = 3, 7, 73$, then the equation $3^m-1 = 2u$ has no solution in $N$, a contradiction.

In review of Lemma 5(4), $G \cong L_3(8)$, or $U_3(9)$.

This completes the proof of the Lemma.

**3 Main theorem and its proof**

Let $G$ be a group such that $\text{nse}(G) = \text{nse}(L_3(8))$, and $s_n$ be the number of elements of order $n$. By Lemma 3 we have that $G$ is finite. We note that $s_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have
\begin{equation}
\{ \phi(m) \mid s_m; \ m \mid \sum_{d|m} s_d \}
\end{equation}
By (1), $\pi(G) \subseteq \{2, 3, 5, 7, 19, 73, 784897, 1569793\}$. If $m > 2$, then $\phi(m)$ is even, then $s_2 = 4599, 2 \in \pi(G)$.

In the following, we prove that $784897 \not\in \pi(G)$. If $784897 \in \pi(G)$, then by (1), $s_{784897} = 784896$. If $2 \not\in \pi(G)$, then $s_2 = 784896 \in \pi(G)$, and $4709376$. On the other hand, $2.784897 \not\in \pi(G)$, a contradiction. Therefore $2.784897 \notin \omega(G)$. Now we consider Sylow $2$-subgroup $P_{784897}$ of $G$ acts fixed point freely on the set of elements of order $2$, then $P_{784897} \not\in s_2$, a contradiction. Similarly we can prove that the prime $1569793$ does not belong to $\pi(G)$. Hence we have $\pi(G) \subseteq \{2, 3, 5, 7, 19, 73\}$. Furthermore, by (1) $s_3 = 261632, 4709276$ or $5419008, s_5 = 257544, 5419008, s_7 = 1709952, 257544, 5419008, s_{784897} = 5419008$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1} | s_2$ and so $0 \leq a \leq 13$. If $2^a \in \omega(G)$, then $1 \leq a \leq 4$. If $2^a \in \omega(G)$, then $1 \leq a \leq 12$ and $1 \leq b \leq 4$. If $2^a \in \omega(G)$, then $1 \leq a \leq 3$. If $2^a \in \omega(G)$, then $1 \leq a \leq 12$ and $1 \leq b \leq 3$. Since $s_{784897} \not\in \text{nse}(G)$, then $a = 1$. If $2^a \in \omega(G)$, then $a = 1$. If $3^a \in \omega(G)$, then $a = 1$. If $3^a \in \omega(G)$, then $a = 1$. If $7^a \in \omega(G)$, then $1 \leq a \leq 2$. If $7^a \in \omega(G)$, then $1 \leq a \leq 2$. If $7^a \in \omega(G)$, then $1 \leq a \leq 2$.

19. 7 \in \pi(G). We know that $\exp(P_2) = 7, 7^2, 7^3, 7^3$. If $7^a \in \omega(G)$, then $s_7.19 \not\in \text{nse}(G)$. Therefore $7.19 \in \omega(G)$. If $7^a \in \omega(G)$, then $1 \leq a \leq 12$. If $7^a \in \omega(G)$, then $1 \leq a \leq 12$. If in the following, we first prove that $73 \in \pi(G)$, then consider the proper subset of $\{2, 3, 7, 73\}$ and finally the set $\{2, 3, 7, 73\}$.

As $\exp(P_2) = 2, \ldots, 2^{13}$, by Lemma 1, $|P_2| = \prod_{i=1}^{13} p_i$, where $p_i$ are prime numbers and $p_i < 7$. If $3 \in \pi(G)$, then as $\exp(P_3) = 3, 3^2, 3^3, 3^4$, by Lemma 1, $|P_3| = 1 + s_3 + s_{32} + s_{33} + s_{34}$ and so $|P_3| = 3^9$. If $5 \in \pi(G)$, then by Lemma 1, $|P_5| = 1 + s_5$ and so $|P_5| = 5$. If $7 \in \pi(G)$, then $|P_7| = 1 + s_7 + s_{72} + s_{73}$ and so $|P_7| = 7^3$. If $19 \in \pi(G)$, then $|P_{19}| = 1 + s_{19}$ and so $|P_{19}| = 19$.

To remove the primes 5 and 19, we must show that $73 \in \pi(G)$.

Assume that $73 \not\in \pi(G)$. If $3, 5, 7, 19 \not\in \pi(G)$, then $G$ is a 2-group and so $16482816 \neq 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m$, where $k_1, k_2, \ldots, k_8, m$ are nonnegative integers and $0 \leq \sum_{i=1}^{8} k_i \leq 4$. Since $16482816 \leq |G| = 2^m \leq 16482816 + 4.5419008$, then $m = 24, 25$, a contradiction since $m$ is at most 13.

Let $19 \in \pi(G)$. Then as $\exp(P_{19}) = 19$ and by Lemma 1, $|P_{19}| = 1 + s_{19}$, and if $s_{19} = 257544$, then $|P_{19}| = 19$. Since $s_{19} = 5419008$, then $|P_{19}| = 1 + s_{19}$ and so $|P_{19}| = 19$. Since $s_{19} = 5419008$, then $|P_{19}| = 1 + s_{19}$, a contradiction. Since $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 19\}$, then we assume that $16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m.3^3.5^7.7^3$, where $k_1, k_2, \ldots, k_8, m$ are nonnegative integers and $0 \leq k_i \leq 5$. Since $16482816 \leq |G| \leq 16482816 + 5419008 = 257544$, then $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 19\}$, then we assume that $16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m.3^3.5^7.7^3$, where $k_1, k_2, \ldots, k_8, m$ are nonnegative integers and $0 \leq \sum_{i=1}^{8} k_i \leq 230$. Since $16482816 \leq |G| \leq 16482816 + 5419008.230, 1 \leq m \leq 13, 0 \leq n \leq 6, 0 \leq p \leq 1$ and $0 \leq q \leq 3$, then set $p = 1, (m, n) = (12, 3), (13, 3), (12, 2), (10, 4), (11, 4), (12, 4), (13, 4), (9, 5), (10, 5), (11, 5), (12, 5), (13, 5), (7, 6), (8, 6), (9, 6), (10, 6), (11, 6), (12, 6) and set $p = 0, (m, n) = (9, 6), (10, 6), (11, 6), (12, 6), (13, 6), (14, 5), (13, 5), (13, 4)$. We only give an example for these cases (the other cases can be ruled out as this case). For example, $|G| = 2^{10}.3^3.5.7.19$, then the number of Sylow 19-subgroups of $G$ is 1, 20, 96, 210, 1008, 1920, 9216, 20160, 96768, and so the number of order 19 of $G$ is 18, 360, 1728, 3780, 18144, 34560, 165888, 362880, 1741824, but these do not belong to nse(G), a contradiction.
Let 5 ∈ \(\pi(G)\). As \(\exp(P_3) = 5\), by Lemma 1, \(|P_3| = 1 + s_3\) for \(s_3 = 257544\) or \(5419008\), and so \(|P_3| = 5\). If \(s_3 = 257544\), then \(73 \in \pi(G)\) since \(n_3 = s_3/\phi(3)\), a contradiction. If \(s_3 = 5419008\), then \(n_3 = 18432\). By Sylow’s theorem, \(n_7 = 7k + 1\) for some non-negative integer \(k\), but the equation has no solution in \(N\).

Let 3 ∈ \(\pi(G)\). By Lemma 1, \(|P_2| = 3^2\), then \(|P_2| = 1 + s_2\) for \(s_2 = 261632\), \(4709376\) or 5419008, and so \(|P_2| \neq 3\). If \(s_2 = 261632\), \(4709376\), then \(n_2 = s_2/\phi(3)\) and so \(73 \in \pi(G)\), a contradiction. If \(s_2 = 5419008\), then \(n_2 = 2709504\). By Sylow’s theorem, \(n_2 = 3k + 1\) for some non-negative integer \(k\), but the equation has no solution in \(N\).

If \(\exp(P_3) = 3^3\), then by Lemma 1, \(|P_3| = 1 + s_3 + s_{32}\) for \(s_{32} = 261632\), \(4709376\) or 5419008, and so \(|P_3| = 3^3\). If \(s_{32} = 261632\), \(4709376\), then \(n_{32} = s_{32}/\phi(3)\) and so \(73 \in \pi(G)\), a contradiction. If \(s_{32} = 5419008\), then \(n_{32} = 903168\). By Sylow’s theorem, \(n_3 = 3k + 1\) for some non-negative integer \(k\), but the equation has no solution in \(N\).

If \(\exp(P_3) = 3^3\), then by Lemma 1, \(|P_3| = 1 + s_3 + s_{32} + s_{33}\) for \(s_{33} = 257544\), \(1766016\), \(4709376\), 257544, \(1766016\), \(4709376\) or 5419008, and so \(|P_3| = 3^3\). Let \(|P_3| = 3^3\). If \(s_{33} = 257544\), \(1766016\), \(4709376\), then \(n_{33} = s_{33}/\phi(3)\) and so \(73 \in \pi(G)\), a contradiction. If \(s_{33} = 5419008\), then \(n_{33} = 301056\). By Sylow’s theorem, \(n_3 = 3k + 1\) for some non-negative integer \(k\), but the equation has no solution in \(N\). Let \(|P_3| = 3^3\). Then 16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m \cdot 3^4, where \(k_1, k_2, \ldots, k_8, m\) are non-negative integers and \(0 \leq \sum_{i=1}^{8} k_i < 56\). Since 16482816 ≤ |\(G| ≤ 16482816 + 5419008.56, then \(m = 18, \ldots, 21\), a contradiction.

If \(\exp(P_3) = 3^4\), then by Lemma 1, \(|P_3| = 1 + s_3 + s_{32} + s_{33} + s_{34}\) and so \(|P_3| = 3^6\). Since \(s_{34} = 1766016\) or \(4709376\), then \(n_3 = s_{33}/\phi(3)\) and so \(73 \in \pi(G)\), a contradiction. If \(|P_3| > 3^4\), then by Lemma 2, \(s_{34} = 3^t\) for some non-negative integer \(t\). But the equation has no solution in \(N\) since \(s_{74} \notin \text{nse}(G)\).

Therefore \(73 \in \pi(G)\).

In the following, we prove that the primes 5 and 19 do not belong to \(\pi(G)\).

Let 5 ∈ \(\pi(G)\). If \(5.73 \in \omega(G)\), then by (1), \(s_{5,73} \notin \text{nse}(G)\). It follows that the Sylow 5-subgroup of \(G\) acts fixed point freely on the set of elements of order 73 and so \(|P_5| \neq 73\), a contradiction. Therefore 5 ∉ \(\pi(G)\). Similarly 19 ∉ \(\pi(G)\).

Therefore we have that \(\{2, 73\} \subseteq \pi(G) \subseteq \{2, 3, 7, 73\}\).

Case a. \(\pi(G) = \{2, 73\}\).

As \(\exp(P_3) = 7\), then by Lemma 1, \(|P_3| = 1 + s_3\) and so \(|P_3| = 73\). Since \(n_3 = s_3/\phi(73)\), then \(7 \in \pi(G)\), a contradiction.

Case b. \(\pi(G) = \{2, 3, 73\}\).

The proof is the same as Case a.

Case c. \(\pi(G) = \{2, 7, 73\}\).

We know that \(\exp(P_7) = 7, 7^2, 7^3, 7^3\).

If \(\exp(P_7) = 7\), then by Lemma 1, \(|P_7| = 1 + s_7\) and so \(|P_7| \neq 7\). If \(|P_7| = 7\), then \(n_7 = s_7/\phi(7)\) and so \(3 \in \pi(G)\), a contradiction. If \(|P_7| = 7^2\), then 16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m \cdot 7^2, 73, where \(k_1, k_2, \ldots, k_8, m\) are non-negative integers and \(0 \leq \sum_{i=1}^{8} k_i < 17\). Since 16482816 ≤ |\(G| ≤ 16482816 + 5419008.17, then \(m = 13\) and |\(G| = 2^3 \cdot 7^2 \cdot 73, thus the number of Sylow 73-subgroups of \(G\) is 1, 512, 28672, and so the number of order 73 of \(G\) is 72, 36864, 2064384, but none of which belongs to \(\text{nse}(G)\), a contradiction.

If \(\exp(P_7) = 7^2\), then by Lemma 1, \(|P_7| = 1 + s_7 + s_{72}\) for \(s_{72} = 257544\) or 5419008, and so \(|P_7| \neq 7^2\). If \(s_{72} = 257544\), then \(3 \in \pi(G)\) since \(n_7 = s_7/\phi(7^2)\), a contradiction. If \(s_{72} = 5419008\), then \(n_7 = 129024\). By Sylow’s theorem, \(n_7 = 7k + 1\) for some non-negative integer \(k\), but the equation has no solution in \(N\).

If \(\exp(P_7) = 7^3\), then by Lemma 1, \(|P_7| = 1 + s_7 + s_{72} + s_{73}\) and so \(|P_7| \neq 7^3\). Since \(s_{73} = 257544\) or 5419008, then \(n_7 = s_{72}/\phi(7^3)\) and so \(3 \in \pi(G)\), a contradiction. If \(|P_7| > 7^3\), then by Lemma 2, \(s_{73} = 3^t\) for some non-negative integer \(t\). But the equation has no solution in \(N\).

Case d. \(\pi(G) = \{2, 3, 7, 73\}\).

In the following, we first show that |\(G| = 2^m \cdot 3^n\), and second prove that \(G \cong L_3(8)\).

Step 1. \(|G| = 2^m \cdot 3^n \cdot 7^2 \cdot 73, where \(m = 9, 10, 11, 12\) and \(n = 2, 3\).
We have known that $|P_{73}|=73$.

If $2.73 \in \omega(G)$, set $P$ and $Q$ are Sylow 73-subgroups of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_G(Q)$ are also conjugate in $G$. Therefore we have $s_2.73 = \phi(2.73)N_{73}k$, where $k$ is the number of cyclic subgroups of order 2 in $C_G(P_{73})$. Since $s_{73} = s_2.73/\phi(73) = 5419008/72$, $5419008 \div s_{73}$ and so $s_{73} = 5419008$. But by Lemma 1, $2.73 = 1 + s_2 + s_7 + s_{73}$, a contradiction. Therefore $2.73 \notin \omega(G)$, it follows that the Sylow 2-subgroup of $G$ acts fixed point freely on the set of elements of order 73, $|P_2| \div s_{73}$ and so $|P_2| = 2^{12}$.

If $3.73 \in \omega(G)$, then by (1), $3.73 = 1 + s_3 + s_{73} + s_{73}$, therefore $3.73 \notin \omega(G)$, it follows that the Sylow 3-subgroup of $G$ acts fixed point freely on the set of order 73 and so $|P_3| \div s_{73}$. Hence $|P_3| = 3^3$. Similarly $7.3 \notin \omega(G)$ and $|P_7| \div 7^2$.

Therefore we can assume that $|G| = 2^{m}.3^{n}.7^p.73$. Since $16482816 = 2^9.3^2.7^2.73$ then $|G| = 2^m.3^n.7^p.73$, where $m = 9, 10, 11, 12$ and $n = 2, 3$.

Step 2. $G \cong L_3(8)$

First prove that there is no group such that $|G| = 2^{10}.3^2.7^2.73$ and nse($G$)$=nse(L_3(8))$, similarly we rule out the other cases except for $|G| = 2^9.3^2.7^2.73$. Then get the desired result by [13].

Let $|G| = 2^{10}.3^2.7^2.73$ and nse($G$)$=nse(L_3(8))$.

Let $G$ be soluble. Since $s_{73} = s_{73}/\phi(73) = 5123.49$, then by Lemma 4, 3 $\equiv 1$ (mod 73), a contradiction. So $G$ is insoluble.

Therefore $G$ has a normal series $1 \leq K \leq L \leq G$ such that $L/K$ is isomorphic to a simple $K_i$-group with $i = 3, 4$ as 5329 does not divide the order of $G$.

If $L/K$ is isomorphic to a simple $K_3$-group, from [4], $L/K \cong L_2(7)$, $L_2(8)$, $U_3(3)$. From [1], $n_7(L/K) = n_7(L_2(7))$, and so by [13] $n_7(G) = 8t$ and $7 \nmid t$. Hence the number of elements of order 7 in $G$ is: $s_7 = 8t \cdot 6 = 48t$ for some non-negative integer $t$. Since $s_7 \in \text{nse}(G)$, then $s_7 = 1709952$ and so $t = 35624$. Therefore $8.61.73 \div |K| \div 2^7.3.7.73$, which is a contradiction. For the groups $L_2(8)$ and $U_3(3)$, similarly we can rule out.

Hence $G$ is isomorphic to a simple $K_4$-group, then by Lemma 6, $L/K \cong L_3(3)$, $U_3(3)$. Order consideration rules out the group $U_3(3)$. Hence $L/K \cong L_3(3)$.

Let $G = G/K$ and $L = L_3/K$. Then $L_3(8) \leq L \cong L_3(3)/\text{Aut}(L)$ and consider condition rules out the group $U_3(3)$. Hence $L/K \cong L_3(3)$.

Let $G = G/K$ and $L = L_3/K$. Then $L_3(8) \leq L \cong L_3(3)/\text{Aut}(L) \cong L_3(3)/\text{Aut}(L) = N_G(L)/\text{Aut}(L)$.

Set $M = \{xK \mid xK \in C_G(L)\}$, then $G/M \cong L_3(3)$ and so $L_3(8) \leq G/M \leq \text{Aut}(L_3(8))$. Therefore $G/M \cong L_3(8)$, $G/M \equiv 2L_3(8)$, $G/M \cong 3L_3(8)$ or $G/M \equiv 6L_3(8)$.

If $G/M \cong L_3(8)$, then order consideration $|M| = 3$ and $M = Z(G)$. So there exists an element of order 3.73, which is a contradiction.

If $G/M \cong 2L_3(8)$, then $G \cong 2L_3(8)$. By Sylow’s theorem, the number of the Sylow 73-subgroups of $G$ is 1, 147, 512, 75264, and so the number of elements of order 73 of $G$ is 72, 10548, 37376, 5419008, but none of which belongs to nse($G$), a contradiction.

If $G/M \cong 3L_3(8)$ or $G/M \cong 6L_3(8)$, then order consideration rules out these cases. Similarly we can rule out the cases $|G| = 2^m.3^2.7^2.73$ with $m = 11, 12$ and $|G| = 2^m.3^2.7^2.73$ with $m = 9, 10, 11, 12$.

Therefore $|G| = 2^9.3^2.7^2.73 = |L_3(8)|$ and by assumption, nse($G$)$=nse(L_3(8))$, then by [13], $G \cong L_3(8)$.

This completes the proof of the theorem.

4 Conclusion

The following theorem is the main theorem

Theorem 8 Let $G$ be a group such that nse($G$)$=nse(L_3(8))$. Then $G \cong L_3(8)$

As a corollary, we have the following corollary of Theorem 8.

Corollary 9 Let $G$ be a group such that nse($G$)$=nse(L_3(8))$ and $|G| = |L_3(8)|$. Then $G \cong L_3(8)$

Proof. See [13].

Acknowledgments The object is supported by the Department of Education of Sichuan Province (Grant No: 12ZB085, 12ZB291 and 13ZA0119). The authors are very grateful for the helpful suggestions of the referee.

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