A characterization of projective special linear group $L_3(8)$ by nse

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Abstract: Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. In Khatami et al and Liu's works, $L_3(2)$ and $L_3(4)$ are unique determined by nse(G). In this paper, we prove that if G is a group such that nse(G)=nse$(L_3(8))$, then $G \cong L_3(8)$.

Key–Words: Element order, Projective special linear group, Thompson's problem, Number of elements of the same order, Simple group, Element order.

1 Introduction

We introduce some notations which is needed. Let $a.b$ denote the products of an integer a by an integer b. $L_n(q)$ denotes the projective special linear group of degree n over finite fields of order q. $U_n(q)$ denotes the projective special unitary group of degree n over finite fields of order q. Let r be a prime and G be a group. Then P_r denotes the Sylow r-subgroups of G and n_r or $n_r(G)$ denotes the number of Sylow r-subgroups of G. The notations are standard (see [1] and [15]).

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [16]).

Thompson's Problem. Let $T(G) = \{(n, s_n) | n \in \omega(G)$ and $s_n \in \text{nse}(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

If the groups G and H have the same order type, then $|G| = |H|$ and nse(G)=nse(H). The following results are gotten.

Result 1: Let G be a group and M some simple K_i-group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and nse(G)=nse(M) (see [12, 13]).

Result 2: The group A_{12} is characterizable by order and nse (see [7]).

Result 3: All sporadic simple groups are characterizable by nse and order (see [5]).

Result 4: $L_2(2^m)$ with $2^m + 1$ prime or $2^m - 1$ prime, is characterized by nse and order (see [14]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the **Thompson's Problem**, in other words, it remains only nse(G), whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set nse(G) (see [6, 15]). The author has proved that the groups $L_3(4)$, $U_3(5)$ and $L_5(2)$ are characterizable by nse (see [8, 9, 10]). In this paper, it is shown that the group $L_3(8)$ also can be characterized by nse.

2 Introduction

Lemma 1 *[2] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m || L_m(G)|$.*

Lemma 2 *[11] Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 3 *[15] Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 4 *[3, Theorem 9.3.1] Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be*

ISBN: 978-960-474-351-3

the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:

(1) $q_i^{\beta_i} \equiv 1$ (mod p_j) for some p_j.

(2) The order of some chief factor of G is divided by $q_i^{\beta_i}$.

To prove $G \cong L_3(8)$, we need the structure of simple K_4-groups.

Lemma 5 *[17] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9 or A_{10}.

(2) M_{11}, M_{12} or J_2.

(3) One of the following:

(a) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, and v is a prime greater than 3.

(b) $L_2(2^m)$, where $2^m - 1 = u$, $2^m + 1 = 3t^b$ with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$.

(4) One of the following 28 simple groups: $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ or ${}^2F_4(2)'$.

Lemma 6 *Let G be a simple K_4-group and $\{73\} \subseteq \pi(G) \subseteq \{2, 3, 7, 73\}$. Then $G \cong L_3(8)$, or $U_3(9)$.*

Proof. From Lemma 5(1)(2), order consideration rules out this case.

So we consider Lemma 5(3). We will deal with this with the following cases.

- Case 1. $G \cong L_2(r)$, where $r \in \{3, 7, 73\}$.
  - * Let $r = 3$, then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(r^2 - 1)| = 3$.
  - * Let $r = 7$, then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(r^2 - 1)| = 3$.
  - * Let $r = 73$, then $|\pi(r^2 - 1)| = 3$. Hence $G \cong L_2(73)$, but $37 \mid |G|$, a contradiction.
- Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 7, 73\}$.
  - * Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $\mathbf{N}$, a contradiction.
  - * Let $u = 7$, then $m = 3$, and $2^3 + 1 = 3t^b$. Thus $t = 3$ and $b = 1$. But $t > 3$, a contradiction.
  - * Let $u = 73$, then $2^m - 1 = 19$. But the equation has no solution in $\mathbf{N}$.
- Case 3. $G \cong L_2(3^m)$.

  We will consider the case by the following two cases.
  - * Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.
    We can suppose that $t \in \{3, 7, 73\}$
    Let $t = 3, 73$, the equation $3^m + 1 = 4t$ has no solution. So we rule out the case.
    Let $t = 7$, then $m = 3$ and so $3^3 - 1 = 2 \cdot 11$, which means $11 \mid |G|$, a contradiction.
  - * Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.
    We can suppose that $u \in \{3, 7, 19\}$
    Let $u = 3, 7, 73$, then the equation $3^m - 1 = 2u$ has no solution in $\mathbf{N}$, a contradiction.

In review of Lemma 5(4), $G \cong L_3(8)$, or $U_3(9)$. This completes the proof of the Lemma.

3 Main theorem and its proof

Let G be a group such that nse(G)=nse($L_3(8)$), and s_n be the number of elements of order n. By Lemma 3 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have

$$\{ \phi\ (m) \mid s_m; \quad m \mid \sum_{d \mid m} s_d \qquad (1)$$

Theorem 7 *Let G be a group with nse(G)=nse($L_3(8)$)={1, 4599, 257544, 261632, 784896, 1569792, 1709952, 1766016, 4709376, 5419008}, where $L_3(8)$ is the projective special linear group of degree 3 over field of order 8. Then G$\cong L_3(8)$.*

Proof. We prove the theorem by first proving that $\pi(G) \subseteq \{2, 3, 7, 73\}$, second showing that $|G| = |L_3(8)|$, and so $G \cong L_3(8)$.

ISBN: 978-960-474-351-3

By (1), $\pi(G) \subseteq \{2, 3, 5, 7, 19, 73, 784897, 1569793\}$. If $m > 2$, then $\phi(m)$ is even, then $s_2 = 4599, 2 \in \pi(G)$.

In the following, we prove that $784897 \notin \pi(G)$. If $784897 \in \pi(G)$, then by (1), $s_{784897} = 784896$. If $\phi(2.784897) \mid s_{2.784896}$, then $s_{2.784897}$=784896, 1569792, or 4709376. On the other hand, $2.784897 \mid 1 + s_2 + s_{784897} + s_{2.784897}$(=1574392, 2359288, 5498872), a contradiction. Therefore $2.784897 \notin \omega(G)$. Now we consider Sylow 784897-subgroup P_{784897} of G acts fixed point freely on the set of elements of order 2, then $|P_{784897}| \mid s_2$, a contradiction. Similarly we can prove that the prime 1569793 does not belong to $\pi(G)$. Hence we have $\pi(G) \subseteq \{2, 3, 5, 7, 19, 73\}$. Furthermore, by (1) s_3=261632, 4709276 or 5419008, s_5=257544, or 5419008, s_7=1709952, s_{19}=257544, or 5419008, and s_{73}=5419008.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1} \mid s_{2^a}$ and so $0 \le a \le 13$.

If $3^a \in \omega(G)$, then $1 \le a \le 4$.

If $2^a.3^b \in \omega(G)$, then $1 \le a \le 12$ and $1 \le b \le 4$.

If $7^a \in \omega(G)$, then $1 \le a \le 3$.

If $2^a.7^b \in \omega(G)$, then $1 \le a \le 12$ and $1 \le b \le 3$.

If $73^a \in \omega(G)$, then $1 \le a \le 2$. Since $s_{73^2} \notin$nse(G), then $a = 1$.

If $5^a \in \omega(G)$, then $a = 1$.

If $19^a \in \omega(G)$, then $a = 1$.

If $7.19 \in \omega(G)$, then $s_{7.19} \notin$nse(G). Therefore $7.19 \notin \omega(G)$.

If $2^a.19 \in \omega(G)$, then $1 \le a \le 12$

If $3^a.19 \in \omega(G)$, then $1 \le a \le 2$.

In the following, we first prove that $73 \in \pi(G)$, then consider the proper subset of $\{2, 3, 7, 73\}$ and finally the set $\{2, 3, 7, 73\}$.

As $\exp(P_2) = 2, \cdots, 2^{13}$, by Lemma 1, $|P_2| \mid 1 + s_2 + \cdots + s_{13}$ and so $|P_2| \mid 2^{13}$.

If $3 \in \pi(G)$, then as $\exp(P_3) = 3, 3^2, 3^3, 3^4$, by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} + s_{3^4}$ and so $|P_3| \mid 3^6$.

If $5 \in \pi(G)$, then by Lemma 1, $|P_5| \mid 1 + s_5$ and so $|P_5| = 5$.

If $7 \in \pi(G)$, then $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3}$ and so $|P_7| \mid 7^3$

If $19 \in \pi(G)$, then $|P_{19}| \mid 1 + s_{19}$ and so $|P_{19}|$=19.

To remove the primes 5 and 19, we must show that $73 \in \pi(G)$.

Assume that $73 \notin \pi(G)$.

If $3, 5, 7, 19 \notin \pi(G)$, then G is a 2-group and so $16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m$, where $k_1, k_2, \cdots, k_8, m$ are nonnegative integers and $0 \le \sum_{i=1}^{8} k_i \le 4$. Since $16482816 \le |G| = 2^m \le 16482816 + 4.5419008$, then $m = 24, 25$, a contradiction since m is at most 13.

Let $19 \in \pi(G)$. Then as $\exp(P_{19}) = 19$ and by Lemma 1, $|P_{19}| \mid 1 + s_{19}$. If $s_{19} = 257544$, then $|P_{19}| = 19$. Since $n_{19} = s_{19}/\phi(19)$, then $73 \in \pi(G)$, a contradiction. If $s_{19} = 5419008$, then $|P_{19}| \nmid 1+s_{19}$ and so $|P_{19}| = 19$. Since $n_{19} = s_{19}/\phi(19)$=301056. Since $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 19\}$, then we assume that $16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m.3^n.5^p.7^q.19$, where $k_1, k_2, \cdots, k_8, m$ are nonnegative integers and $0 \le \sum_{i=1}^{8} k_i \le 259$. Since 16482816$\le |G| \le 16482816 +$ 5419008.259, $1 \le m \le 13$, $0 \le n \le 6$, $0 \le p \le 1$ and $0 \le q \le 3$, then there is some solutions (m, n, p, q) such that the conditions. We only give an example for these case (the other cases can be ruled out as this case). For example, $|G| = 2^{10}.3^3.5.7.19$, then the number of Sylow 19-subgroups of G is 1, 20, 96, 210, 1008, 1920, 9216, 20160, 96768, and so the number of order 19 of G is 18, 360, 1728, 3780, 18144, 34560, 165888, 362880, 1741824, but these do not belong to nse(G), a contradiction.

Let $7 \in \pi(G)$. We know that $\exp(P_7) = 7, 7^2, 7^3, 7^3$.

If $\exp(P_7) = 7$, then by Lemma 1, $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^2$. If $|P_7| = 7$, then $n_7 = s_7/\phi(7)$ and so $73 \in \pi(G)$, a contradiction. If $|P_7| = 7^2$, then since $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 19\}$ and the above arguments, we can assume that $16482816 + 257544k_1 + 261632k_2 + 784896k_3 + 1569792k_4 + 1709952k_5 + 1766016k_6 + 4709376k_7 + 5419008k_8 = 2^m.3^n.5^p.7^2$, where $k_1, k_2, ..., k_8, m$ are nonnegative integers and $0 \le \sum_{i=1}^{8} k_i \le 230$. Since 16482816$\le$ $|G| \le$16482816+5419008.230, $1 \le m \le 13$, $0 \le n \le 6$, $0 \le p \le 1$ and $0 \le q \le 3$, then set $p = 1, (m, n)$=(12, 3), (13, 3), (13, 2), (10, 4), (11, 4), (12, 4), (13, 4), (9, 5), (10, 5), (11, 5), (12, 5), (13, 5), (7, 6), (8, 6), (9, 6), (10, 6), (11, 6), (12, 6) and set $p = 0, (m, n)$ =(9, 6), (10, 6), (11, 6), (12, 6), (13, 6), (11, 5), (12, 5), (13, 5), (13, 4). We only give an example for these cases (the other cases can be ruled out as this case). For example, $|G| = 2^{12}.3^3.5.7^2$, then the number of Sylow 5-subgroups of G is 1, 6, 16, 21, 36, 56, 96, 126, 196, 216, 256, 336, 441, 576, 756, 896, 1176, 1536, 2016, 2646, 3136, 3456, 4096, 5376, 7056, 9216, 12096, 14336, 18816, 32256, 42336, 50176, 55296, 86016, 112896, 193536, 301056, 677376, 1806336, and so the number of elements of order 5 of G is 4, 24, 64, 84, 144, 224, 384, 504, 784, 864, 1024, 1344, 1764, 2304, 3024, 3584, 4704, 6144, 8064, 10584, 12544, 13824, 16384, 21504, 28224, 36864, 48384, 57344, 75264, 129024, 169344, 200704, 221184, 344064, 451584, 774144, 1204224, 2709504, 1445068, but none of which belongs to nse(G), a contradiction.

If $\exp(P_7) = 7^2$, then by Lemma 1, $|P_7| \mid 1+s_7+s_{7^2}$ for s_{7^2}=257544 or 5419008, and so $|P_7| \mid 7^2$. If s_{7^2}=257544, then 73 $\in \pi(G)$ since $n_7 = s_{7^2}/\phi(7^2)$, a contradiction. If s_{7^2}=5419008, then n_7=129024. By Sylow's theorem, $n_7 = 7k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$.

If $\exp(P_7) = 7^3$, then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3}$ for s_{7^3}=257544, or 5419008, and so $|P_7| \mid 7^3$. If s_{7^3}=257544, then 73 $\in \pi(G)$ since $n_7 = s_{7^3}/\phi(7^3)$, a contradiction. If s_{7^3}=5419008, then n_7=18432. By Sylow's theorem, $n_7 = 7k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$.

Let $5 \in \pi(G)$. As $\exp(P_5)$=5, by Lemma 1, $|P_5| \mid 1 + s_5$ for s_5=257544 or 5419008, and so $|P_5| = 5$. If s_5=257544, then 73 $\in \pi(G)$ since $n_5 = s_5/\phi(5)$, a contradiction. If s_5=5419008, then n_5=1354752. By Sylow's theorem, $n_5 = 5k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$.

Let $3 \in \pi(G)$. We know that $\exp(P_3) = 3, 3^2, 3^3, 3^4$.

If $\exp(P_3)$=3, then by Lemma 1, $|P_3| \mid 1 + s_3$ for s_3=261632, 4709376 or 5419008, and so $|P_3| = 3$. If s_3=261632, 4709376, then $n_3 = s_3/\phi(3)$ and so 73 $\in \pi(G)$, a contradiction. If s_3=5419008, then n_3=2709504. By Sylow's theorem, $n_3 = 3k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$.

If $\exp(P_3) = 3^2$, then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2}$ for s_{3^2}=261632, 4709376 or 5419008, and so $|P_3| \mid 3^2$. If s_{3^2}=261632, 4709376, then $n_{3^2} = s_3/\phi(3^2)$ and so 73 $\in \pi(G)$, a contradiction. If s_{3^2}=5419008, then n_3=903168. By Sylow's theorem, $n_3 = 3k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$.

If $\exp(P_3) = 3^3$, then by Lemma 1, $|P_3| \mid 1+s_3+s_{3^2}+s_{3^3}$ for s_{3^3}=257544, 1766016,4709376, 257544, 1766016,4709376 or 5419008, and so $|P_3| \mid 3^4$. Let $|P_3| = 3^3$. If s_{3^3}=257544, 1766016,4709376, then $n_3 = s_{3^3}/\phi(3^3)$ and so 73 $\in \pi(G)$, a contradiction. If s_{3^3}=5419008, then n_3=301056. By Sylow's theorem, $n_3 = 3k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$. Let $|P_3| = 3^4$. Then $16482816+257544k_1+261632k_2+784896k_3+1569792k_4+1709952k_5+1766016k_6+4709376k_7+5419008k_8 = 2^m.3^4$, where $k_1, k_2, \cdots, k_8, m$ are non-negative integers and $0 \le \sum_{i=1}^{8} k_i \le 56$. Since $16482816 \le |G| \le 16482816 + 5419008.56$, then $m = 18, ..., 21$, a contradiction.

If $\exp(P_3) = 3^4$, then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} + s_{3^4}$ and so $|P_3| \mid 3^6$. Let $|P_3| = 3^4$. Since s_{3^4}=1766016 or 4709376, then $n_3 = s_{3^4}/\phi(3^4)$ and so 73$\in \pi(G)$, a contradiction. If $|P_3| > 3^4$, then by Lemma 2, $s_{3^4} = 3^4 t$ for some non-negative integer t. But the equation has no solution in $\mathbf{N}$ since $s_{7^4} \in$nse(G).

Therefore 73 $\in \pi(G)$.

In the following, we prove that the primes 5 and 19 do not belong to $\pi(G)$.

Let $5 \in \pi(G)$. If $5.73 \in \omega(G)$, then by (1), $s_{5.73} \notin$nse(G). It follows that the Sylow 5-subgroup of G acts fixed point freely on the set of elements of order 73 and so $|P_5| \mid s_{73}$, a contradiction. Therefore $5 \notin \pi(G)$. Similarly $19 \notin \pi(G)$.

Therefore we have that $\{2, 73\} \subseteq \pi(G) \subseteq \{2, 3, 7, 73\}$

Case a. $\pi(G) = \{2, 73\}$.

As $\exp(P_{73})$=73, then by Lemma 1, $|P_{73}| \mid 1+s_{73}$ and so $|P_{73}| = 73$. Since $n_{73} = s_{73}/\phi(73)$, then $7 \in \pi(G)$, a contradiction.

Case b. $\pi(G) = \{2, 3, 73\}$.

The proof is the same as Case a

Case c. $\pi(G) = \{2, 7, 73\}$.

We know that $\exp(P_7) = 7, 7^2, 7^3, 7^3$.

If $\exp(P_7) = 7$, then by Lemma 1, $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^2$. If $|P_7| = 7$, then $n_7 = s_7/\phi(7)$ and so $3 \in \pi(G)$, a contradiction. If $|P_7| = 7^2$, then $16482816+257544k_1+261632k_2+784896k_3+1569792k_4+1709952k_5+1766016k_6+4709376k_7+5419008k_8 = 2^m.7^2.73$, where $k_1, k_2, \cdots, k_8, m$ are non-negative integers and $0 \le \sum_{i=1}^{8} k_i \le 17$. Since $16482816 \le |G| \le 16482816 + 5419008.17$, then $m = 13$ and $|G| = 2^{13}.7^2.73$, thus the number of Sylow 73-subgroups of G is 1, 512, 28672, and so the number of order 73 of G is 72, 36864, 2064384, but none of which belongs to nse(G), a contradiction.

If $\exp(P_7) = 7^2$, then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2}$ for s_{7^2}=257544 or 5419008, and so $|P_7| \mid 7^2$. If s_{7^2}=257544, then $3 \in \pi(G)$ since $n_7 = s_{7^2}/\phi(7^2)$, a contradiction. If s_{7^2}=5419008, then n_7=129024. By Sylow's theorem, $n_7 = 7k + 1$ for some non-negative integer k, but the equation has no solution in $\mathbf{N}$.

If $\exp(P_7) = 7^3$, then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3}$ and so $|P_7| \mid 7^3$. Since s_{7^3}=257544, 5419008, then $3 \in \pi(G)$ since $n_7 = s_{7^3}/\phi(7^3)$, a contradiction.

Case d. $\pi(G) = \{2, 3, 7, 73\}$.

In the following, we first show that $|G| = 2^m.3^n$. and second prove that $G \cong L_3(8)$.

Step 1. $|G| = 2^m.3^n.7^2.73$, where $m = 9, 10, 11, 12$ and $n = 2, 3$.

We have known that $|P_{73}|$=73.

If $2.73 \in \omega(G)$, set P and Q are Sylow 73-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{2.73} = \phi(2.73).n_{73}.k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_{73})$. Since $n_{73} = s_{73}/\phi(73)$=5419008/72, 5419008 $\mid s_{2.73}$ and so $s_{2.73}$=5419008. But by Lemma 1, $2.73 \mid 1 + s_2 + s_{73} + s_{73}$, a contradiction. Therefore $2.73 \notin \omega(G)$, it follows that the Sylow 2-subgroup of G acts fixed point freely on the set of elements of order 73, $|P_2| \mid s_{73}$ and so $|P_2| \mid 2^{12}$.

If $3.73 \in \omega(G)$, then by (1), $3.73 \mid 1 + s_3 + s_{73} + s_{3.73}$. Therefore $3.73 \notin \omega(G)$, it follows that the Sylow 3-subgroup of G acts fixed point freely on the set of order 73 and so $|P_3| \mid s_{73}$. Hence $|P_3| \mid 3^3$. Similarly $7.73 \notin \omega(G)$ and $|P_7| \mid 7^2$.

Therefore we can assume that $|G| = 2^m.3^n.7^p.73$. Since $16482816 = 2^9.3^2.7^2.73 \leq |G| = 2^m.3^n.7^p.73$, then $|G| = 2^m.3^n.7^2.73$, where $m = 9, 10, 11, 12$ and $n = 2, 3$.

Step 2. $G \cong L_3(8)$

First prove that there is no group such that $|G| = 2^{10}.3^3.7^2.73$ and nse(G)=nse$(L_3(8))$, similarly we rule out the other cases except for $|G| = 2^9.3^2.7^2.73$. Then get the desired result by [13].

Let $|G| = 2^{10}.3^2.7^2.73$ and nse(G)=nse$(L_3(8))$.

Let G be soluble. Since $n_{73} = s_{73}/\phi(73) = 512.3.49$, then by Lemma 4, $3 \equiv 1 \pmod{73}$, a contradiction. So G is insoluble.

Therefore G has a normal series $1 \leq K \leq L \leq G$ such that L/K is isomorphic to a simple K_i-group with $i = 3, 4$ as 5329 does not divide the order of G.

If L/K is isomorphic to a simple K_3-group, from [4], $L/K \cong L_2(7)$, $L_2(8)$, $U_3(3)$. From [1], $n_7(L/K) = n_7(L_2(7))$=8, and so by [13] $n_7(G) = 8t$ and $7 \not| t$. Hence the number of elements of order 7 in G is: $s_7 = 8t \cdot 6 = 48t$ for some non-negative integer t. Since $s_7 \in$nse(G), then $s_7 = 1709952$ and so $t = 35624$. Therefore $8.61.73 \mid |K| \mid 2^7.3.7.73$, which is a contradiction. For the groups $L_2(8)$ and $U_3(3)$, similarly we can rule out.

Hence G is isomorphic to a simple K_4-group, then by Lemma 6, $L/K \cong L_3(8)$, $U_3(9)$. Order consideration rules out the group $U_3(9)$. Hence $L/K \cong L_3(8)$.

Let $\bar{G} = G/K$ and $\bar{L} = L/K$. Then

$L_3(8) \leq \bar{L} \cong \bar{L}C_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leq \bar{G}/C_{\bar{G}}(\bar{L}) = N_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leq Aut(\bar{L})$

Set $M = \{xK \mid xK \in C_{\overline{G}}(\overline{L})\}$, then $G/M \cong \overline{G}/C_{\overline{G}}(\overline{L})$ and so $L_3(8) \leq G/M \leq$Aut$(L_3(8))$. Therefore $G/M \cong L_3(8)$, $G/M \cong 2.L_3(8)$, $G/M \cong 3.L_3(8)$ or $G/M \cong 6.L_3(8)$.

If $G/M \cong L_3(8)$, then order consideration $|M| = 3$ and $M = Z(G)$. So there exists an element of order 3.73, which is a contradiction.

If $G/M \cong 2.L_3(8)$, then $G \cong 2.L_3(8)$. By Sylow's theorem, the number of the Sylow 73-subgroups of G is 1, 147, 512, 75264, and so the number of elements of order 73 of G is 72, 10548, 37376, 5419008, but none of which belongs to nse(G), a contradiction.

If $G/M \cong 3.L_3(8)$ or $G/M \cong 6.L_3(8)$, then order consideration rules out these cases.

Similarly we can rule out the cases "$|G| = 2^m.3^2.7^2.73$ with $m = 11, 12$" and "$|G| = 2^m.3^3.7^2.73$ with $m = 9, 10, 11, 12$".

Therefore $|G| = 2^9.3^2.7^2.73 = |L_3(8)|$ and by assumption, nse(G)=nse$(L_3(8))$, then by [13], $G \cong L_3(8)$.

This completes the proof of the theorem.

4 Conclusion

The following theorem is the main theorem

Theorem 8 *Let G be a group such that nse(G)=nse$(L_3(8))$. Then $G \cong L3(8)$*

As a corollary, we have the following corollary of Theorem 8.

Corollary 9 *Let G be a group such that nse(G)=nse$(L_3(8))$ and $|G| = |L_3(8)|$. Then $G \cong L3(8)$*

Proof. See [13].

Acknowledgments The object is supported by the Department of Education of Sichuan Province (Grant No: 12ZB085, 12ZB291 and 13ZA0119). The authors are very grateful for the helpful suggestions of the referee.

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