Abstract: In this paper we consider two concepts of exponential dichotomy (uniform and nonuniform) for nonautonomous variational difference equations in Banach spaces. Characterizations of these concepts are given. In addition, we prove a result about the robustness of the uniform exponential dichotomy property.

Key Words: Variational difference equations, discrete skew-evolution semiflows, exponential dichotomy
lution semiflow $\varphi$, then the pair $S = (\Phi, \varphi)$ is called a discrete skew-evolution semiflow on $X$.

Given a sequence $(A_m)_{m \in \mathbb{N}}$ with $A_m : X \to B(V)$ and a discrete evolution semiflow $\varphi : \Delta \times X \to X$, we consider the problem of existence of a sequence $(v_m)_{m \in \mathbb{N}}$ with $v_m : \mathbb{N} \times X \to V$ such that

$$v_{m+1}(n, x) = A_m(\varphi(m, n, x))v_m(n, x)$$

for all $(m, n, x) \in \Delta \times X$. We shall denote this problem with $(A, \varphi)$ and we say that $(A, \varphi)$ is a variational (nonautonomous) discrete-time system.

For $(m, n) \in \Delta$ we define the application $\Phi_m^n : X \to B(V)$ by

$$\Phi_m^n(x)v =
\begin{cases}
A_{m-1}((\varphi(m-1, n, x)) \ldots v, & \text{if } m > n \\
A_{n+1}((\varphi(n+1, n, x)) A_n(v), & \text{if } m = n.
\end{cases}$$

Remark 3 The mapping

$$\Phi : \Delta \times X \to B(V), \Phi(m, n, x)v = \Phi_m^n(x)v$$
is a discrete evolution cocycle over discrete evolution semiflow $\varphi$.

Definition 4 A continuous mapping $P : X \to B(V)$ which satisfies the relation $P(x)P(x) = P(x)$ for all $x \in X$, is called projector on $V$.

Remark 5 If $P$ is a projector on $V$, then the mapping

$$Q : X \to B(V), Q(x) = I - P(x)$$
is also a projector on $V$, called the complementary projector of $P$.

Definition 6 A projector $P$ on $V$ is said to be compatible with a system $(A, \varphi)$ if following relation hold:

$$P(\varphi(m, n, x)) \Phi_m^n(x) = \Phi_m^n(x)P(x)$$

for all $(m, n, x) \in \Delta \times X$, where $\Phi$ is the discrete evolution cocycle generated by the system $(A, \varphi)$.

Remark 7 If the projector $P$ is compatible with a system $(A, \varphi)$, then its complementary projector $Q$ is also compatible with the system $(A, \varphi)$.

3 Exponential dichotomy

Let $P : X \to B(V)$ a projector compatible with the system $(A, \varphi)$.

Definition 8 The system $(A, \varphi)$ is said to be:

(i) P-exponentially dichotomic (and denote P-e.d.) if there are three constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:

$$e^{\alpha(m-n)}(\|P_m^n(x)P(x)v\| + \|Q(x)v\|) \leq N(e^{\beta n} \|P(x)v\| + e^{\beta m} \|P_m^n(x)Q(x)v\|)$$

for all $(m, n, v) \in \Delta \times X \times V$.

(ii) P-uniformly exponentially dichotomic (and denote P-u.e.d.) if there are the constants $N \geq 1$ and $\alpha > 0$ such that:

$$e^{\alpha(m-n)}(\|P_m^n(x)P(x)v\| + \|Q(x)v\|) \leq N(\|P(x)v\| + \|P_m^n(x)Q(x)v\|)$$

for all $(m, n, x, v) \in \Delta \times X \times V$.

Remark 9 It is easy to see that $(A, \varphi)$ is:

(i) P-exponentially dichotomic if and only if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ with

$$e^{\alpha(m-n)}(\|P_m^n(x)P(x)v\| + \|P_m^n(x)Q(x)v\|) \leq N(e^{\beta n} \|P(x)v\| + e^{\beta m} \|P_m^n(x)Q(x)v\|)$$

for all $(m, n, p, x, v) \in T \times X \times V$.

(ii) P-uniformly exponentially dichotomic if and only if there are $N \geq 1$ and $\alpha > 0$ with

$$e^{\alpha(m-n)}(\|P_m^n(x)P(x)v\| + \|P_m^n(x)Q(x)v\|) \leq N(\|P_m^n(x)P(x)v\| + \|P_m^n(x)Q(x)v\|)$$

for all $(m, n, p, x, v) \in T \times X \times V$.

An example of variational discrete-time system with P-uniform exponential dichotomy property is given in

Example 10 Let $\mathcal{C} = \mathcal{C}(R, R)$ be the metric space of all continuous functions $x : R \to R$, with the topology of uniform convergence on compact subsets of $R$. $\mathcal{C}$ is metrizable relative to the metric $d(x_1, x_2) = \sup_{t \in R} |x_1(t) - x_2(t)|$.

Let $f : R_+ \to (0, \infty)$ be a decreasing function with the property that there exists $\lim_{t \to \infty} f(t) = l > 0$. We denote by $X$ the closure in $\mathcal{C}$ of the set $\{f_t, t \in R_+\}$, where $f_t(s) = f(t + s)$ for all $s \in R_+$. 

38
$R_+$. The mapping $\varphi : \Delta \times X \rightarrow X$ defined by
\[ \varphi(m, n, x) = x_{m-n} \] is a discrete evolution semiflow.

Let us consider the Banach space $V = \mathbb{R}^2$ and let
\[ A : X \rightarrow \mathcal{B}(\mathbb{R}^2) \] defined by
\[ A(x)(v_1, v_2) = \left( \frac{x(\tau)}{x(\tau + 1)} v_1, \frac{cx(\tau + 1)}{x(\tau)} v_2 \right) \]
for all $(x, v_1, v_2) \in X \times \mathbb{R}^2$, where $c > 1$.

Then
\[ \Phi_n(x)(v_1, v_2) = \left( \frac{e^{-c(m-n)x(\tau)}}{x(m-n + \tau)} v_1, \frac{e^{c(m-n)x(\tau)}}{x(\tau)} v_2 \right) \]
for all $(m, n, x, v_1, v_2) \in \Delta \times X \times \mathbb{R}^2$.

If we consider $P : X \rightarrow \mathcal{B}(\mathbb{R}^2)$ by
\[ P(x)(v_1, v_2) = (v_1, 0) \]
for all $(x, v_1, v_2) \in X \times \mathbb{R}^2$, then it results that
\[ |\Phi_n(x)P(x)(v_1, v_2)| + |Q(x)(v_1, v_2)| = \]
\[ = e^{-(m-n)x(\tau)} \left( |v_1| + \frac{e^{c(m-n)x(\tau)}}{x(\tau)} |v_2| \right) \leq \frac{e^{-(m-n)x(\tau)}}{1} \left( |v_1| + \frac{e^{c(m-n)x(\tau)}}{x(\tau)} |v_2| \right) =\]
\[ = N \left| x(\tau) \right| \left( |v_1| + \frac{e^{m-nx(\tau)}}{x(\tau)} |v_2| \right) =\]
\[ = N e^{-\alpha(m-n)} (|P(x)(v_1, v_2)| + |\Phi_n(x)Q(x)(v_1, v_2)|) \]
for all $(m, n, x, v_1, v_2) \in \Delta \times X \times \mathbb{R}^2$, where $N = \frac{x(0)}{\tau}$ and $\alpha = \ln c$.

Remark 11 It is obvious that
\[ P - u.e.d. \Rightarrow P - e.d. \]

The following example shows that the converse implication is not valid.

Example 12 Let $(X, d)$ be the metric space, $V$ the Banach space and $\varphi$ the evolution semiflow given as in Example 10.

We define the sequence of mappings $A_m : X \rightarrow \mathcal{B}(\mathbb{R}^2)$ by
\[ A_m(x)(v_1, v_2) = \left( \frac{u(m)}{u(m + 1)} e^{-\frac{1}{2} x(\tau)d\tau} v_1, \frac{u(m + 1)}{u(m)} e^{\frac{1}{2} x(\tau)d\tau} v_2 \right) \]
for all $(m, x, v_1, v_2) \in N \times X \times \mathbb{R}^2$, where the sequence $u : N \rightarrow \mathbb{R}$ is given by $u(m) = e^{m(1-\cos \frac{m\pi}{d})}$.

Then
\[ \Phi_m(x)(v_1, v_2) = \left( \frac{u(n)}{u(m)} e^{-\frac{1}{2} x(\tau)d\tau} v_1, \frac{u(m)}{u(n)} e^{\frac{1}{2} x(\tau)d\tau} v_2 \right) \]
for all $(m, n, x, v_1, v_2) \in \Delta \times X \times \mathbb{R}^2$.

We consider $P : X \rightarrow \mathcal{B}(\mathbb{R}^2)$ by
\[ P(x)(v_1, v_2) = (v_1, 0) \]
for all $(x, v_1, v_2) \in X \times \mathbb{R}^2$, and it results that
\[ |\Phi_m(x)| + |v_2| = \]
\[ = e^{m(1-\cos \frac{m\pi}{d})} - m\pi(1-\cos \frac{m\pi}{2}) \]
\[ - \int_0^{m-n} x(\tau)d\tau \]
\[ \leq e^{-\frac{1}{2} x(\tau)d\tau} \bigg( |v_1| + |\Phi_m(x)| \bigg) \leq \]
\[ \leq e^{-\frac{1}{2} x(\tau)d\tau} \bigg( |v_1| + |\Phi_m(x)| \bigg) \leq \]
for all $(m, n, x, v_1, v_2) \in \Delta \times X \times \mathbb{R}^2$, which prove that $(A, \varphi)$ is $P$-e.d.

Assume that the system $(A, \varphi)$ is $P$-u.e.d. According to Definition 8, there exist $N \geq 1$ and $\nu > 0$ such that
\[ m\pi(1 - \cos \frac{m\pi}{2} ) - m\pi(1 - \cos \frac{n\pi}{2} ) + \]
\[ + \int_0^{m-n} x(\tau)d\tau \leq \ln N - \nu(m-n) \]
for all $(m, n, x) \in \Delta \times X$. If we consider $n = 4k$ and $m = 4k + 2, k \in N$ we have that
\[ 2\nu + \int_0^{m-n} x(\tau)d\tau \leq \ln N \]
which, for $k \rightarrow \infty$, leads to a contradiction. This proves that $(A, \varphi)$ is not $P$-u.e.d.

Theorem 13 The system $(A, \varphi)$ is $P$-exponentially dichotomic if and only if there exist $c \geq 0, d > 0$ and $D \geq 1$ such that:
\[ \sum_{k=n}^{\infty} e^{d(k-n)} \| \Phi_k^p(x)P(x)v \| + \]
\[ + \sum_{k=n}^{m} e^{d(m-k)} \| \Phi_k^p(x)Q(x)v \| \leq \]
\[ \leq D(e^{cm} \| \Phi_k^p(x)P(x)v \| + e^{cm} \| \Phi_k^p(x)Q(x)v \|) \]
for all $(m, n, x, v) \in T \times X \times V$. 

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Proof. Necessity. From Remark 9, for \( d \in (0, \alpha) \) it results that
\[
\sum_{k=n}^{\infty} e^{d(k-n)} \| \Phi_k^p(x)P(x)v \| + \sum_{k=n}^{m} e^{d(m-k)} \| \Phi_k^p(x)Q(x)v \| \leq
\]
\[
\leq Ne^{-\alpha} \| \Phi_n^p(x)P(x)v \| \sum_{k=n}^{\infty} e(d-\alpha)(k-n) + Ne^{-\alpha} \| \Phi_n^p(x)Q(x)v \| \sum_{k=n}^{m} e(d-\alpha)(m-k) \leq
\]
\[
\leq De^{cn} \| \Phi_n^p(x)P(x)v \| + e^{cm} \| \Phi_n^p(x)Q(x)v \|)
\]
for all \((m, n, p, x, v) \in T \times X \times V\), where \( D = \frac{Ne^{-\alpha}D}{e^{cn-d}} \) and \( c = \beta \).

Sufficiency. We observe that from hypothesis it results
\[
e^{d(m-n)} \| \Phi_m^p(x)P(x)v \| + e^{d(m-n)} \| \Phi_n^p(x)Q(x)v \| \leq
\]
\[
\leq \sum_{k=n}^{\infty} e^{d(k-n)} \| \Phi_k^p(x)P(x)v \| + \sum_{k=n}^{m} e^{d(m-k)} \| \Phi_k^p(x)Q(x)v \| \leq
\]
\[
\leq D(e^{cn} \| \Phi_n^p(x)P(x)v \| + e^{cm} \| \Phi_n^p(x)Q(x)v \|)
\]
for all \((m, n, p, x, v) \in T \times X \times V\), which proove that \((A, \varphi)\) is P-e.s.

The next result is specific systems with P-uniform exponential dichotomy property and can not be generalized to the case of P-exponential dichotomy. For example, the system considered in Example 12 not satisfy the following lemma.

Lemma 14 The system \((A, \varphi)\) is P-uniformly exponentially dichotomic if and only if there exists a decreasing sequence of real numbers \((a_n)_n \) with \( a_n \to 0 \) such that
\[
\| \Phi_m^p(x)P(x)v \| + \| \Phi_m^p(x)Q(x)v \| \leq a_{m-n}(\| \Phi_n^p(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \|)
\]
for all \((m, n, p, x, v) \in T \times X \times V\).

Proof. Necessity. From Remark 9 it results that there exist the constants \( N \geq 1 \) and \( \alpha > 0 \) such that
\[
\| \Phi_m^p(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \| \leq
\]
\[
\leq Ne^{-\alpha(m-n)}(\| \Phi_n^p(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \|)
\]
for all \((m, n, p, x, v) \in T \times X \times V\).

If we consider \( a_n = Ne^{-\alpha n} \), then
\[
\| \Phi_m^p(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \| \leq a_{m-n}(\| \Phi_n^p(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \|)
\]
for all \((m, n, p, x, v) \in T \times X \times V\).

Sufficiency. If \( a_n \to 0 \) then there exists \( k \in \mathbb{N}^\ast \) with \( a_k < 1 \). Also, for any \((m, n) \in \Delta \) there exist \( p \in \mathbb{N} \) and \( r \in [0, k) \) such that \( m = n + pk + r \).

From hypothesis and Remark 9 we obtain
\[
\| \Phi_m^p(x)P(x)v \| \leq a_0 \| \Phi_n^{+pk}(x)P(x)v \| \leq
\]
\[
\leq a_0a_k \| \Phi_n^{+pk}(x)Q(x)v \| \leq a_0 \| Q(x)v \| \leq \| Q(x)v \| \leq a_k \| \Phi_n^{+2k}(x)Q(x)v \| \leq a_k \| \Phi_n^{+k}(x)Q(x)v \| \leq a_k \| \Phi_n^p(x)Q(x)v \| \leq a_k \| \Phi_n^p(x)Q(x)v \|
\]
which implies that
\[
\| \Phi_m^p(x)P(x)v \| + \| Q(x)v \| \leq a_0 \| \Phi_n^{+pk}(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \|
\]
\[
= \| \Phi_n^p(x)P(x)v \| + \| \Phi_n^p(x)Q(x)v \|
\]
\[
= a_0e^{\alpha k} \| \Phi_n^p(x)Q(x)v \|
\]
for all \((m, n, x, v) \in \Delta \times X \times V\), where \( N = 1 + a_0e^{\alpha k} \) and \( \alpha = -\ln a_k \).
Corollary 15 For every system \((A, \varphi)\) the following assertions are equivalent:

(i) \((A, \varphi)\) is \(P\)-uniformly exponentially dichotomic;
(ii) there exist \(d > 0\) and \(D \geq 1\) such that:

\[
\sum_{k=n}^{\infty} e^{d(k-n)} \left\| \Phi^n_k(x)P(x)v \right\| +
\sum_{k=n}^{m} e^{d(m-k)} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq D \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, p, x, v) \in T \times X \times V\);

(iii) there exists \(D \geq 1\) such that:

\[
\sum_{k=n}^{\infty} \left\| \Phi^n_k(x)P(x)v \right\| + \sum_{k=n}^{m} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq D \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, p, x, v) \in T \times X \times V\).

Proof. (i) \(\Rightarrow\) (ii) From Remark 9, for \(d \in (0, \alpha)\) we obtain that:

\[
\sum_{k=n}^{\infty} e^{d(k-n)} \left\| \Phi^n_k(x)P(x)v \right\| +
\sum_{k=n}^{m} e^{d(m-k)} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq N \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, p, x, v) \in T \times X \times V, \) where \(D = \frac{Ne^{-\alpha}}{e^{-\alpha} - d} \).

(ii) \(\Rightarrow\) (iii) It is obvious.

(iii) \(\Rightarrow\) (i) We observe that (iii) implies that:

\[
(m - n + 1) \left( \left\| \Phi^n_m(x)P(x)v \right\| + \left\| \Phi^n_m(x)Q(x)v \right\| \right) =
\sum_{k=n}^{m} \left( \left\| \Phi^n_m(x)P(x)v \right\| + \left\| \Phi^n_m(x)Q(x)v \right\| \right) 
\leq D \left( \sum_{k=n}^{\infty} \left\| \Phi^n_k(x)P(x)v \right\| + \sum_{k=n}^{m} \left\| \Phi^n_k(x)Q(x)v \right\| \right) 
\leq D^2 \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, p, x, v) \in T \times X \times V\). By Lemma 14 it results that \((A, \varphi)\) is P-u.e.d.

A sufficient condition for P-exponential dichotomy is given by

Theorem 16 If there are \(b > c \geq 0\) and \(B \geq 1\) such that:

\[
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)P(x)v \right\| +
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq D \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, x, v) \in X \times \Delta \times X \times V\), then the system \((A, \varphi)\) is P-exponentially dichotomic.

Proof. From hypothesis it follows that:

\[
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)P(x)v \right\| +
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq D \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, x, v) \in X \times \Delta \times X \times V\) which leads to the conclusion that the system \((A, \varphi)\) is P-exponentially dichotomic.

For uniform case, we get the following characterization.

Corollary 17 If there are \(b > 0\) and \(B \geq 1\) such that:

\[
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)P(x)v \right\| +
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq D \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, x, v) \in X \times \Delta \times X \times V\), then the system \((A, \varphi)\) is P-uniformly exponentially dichotomic.

Proof. According to the hypothesis we have that:

\[
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)P(x)v \right\| +
\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi^n_k(x)Q(x)v \right\| 
\leq D \left( \left\| \Phi^n_k(x)P(x)v \right\| + \left\| \Phi^n_k(x)Q(x)v \right\| \right)
\]

for all \((m, n, x, v) \in X \times \Delta \times X \times V\). Therefore the system \((A, \varphi)\) is P-uniformly exponentially dichotomic.

Remark 18 Theorems 13 and 16 are generalizations of the results obtained by R. Datko [4], respectively E. Barbashin [11].
4 The robustness of the uniform exponential dichotomy

Given a discrete variational system \((A, \varphi)\) with the uniform exponential dichotomy property, we ask ourselves how much we can disturb it such that the property of uniform exponential dichotomy still holds. We call this problem robustness and we study in this section.

Let \((A, \varphi)\) be a discrete variational system and \((A_\lambda, \varphi)\), \(\lambda \in \mathbb{R}_+\) the perturbed system given by the relationship

\[ v_{m+1}(n, x) = (A_m (\varphi(m, n, x)) + \lambda I) v_m(n, x) \]

for all \((m, n, x) \in \Delta \times X\).

The discrete evolution cocycle associated to perturbed system \((A_\lambda, \varphi)\) is

\[ \Phi : \Delta \times X \to B(V), \]

\[ \Phi^m_n(x) = (A_{m-1} (\varphi(m - 1, n, x)) + \lambda I) \cdot \cdot \cdot \cdot (A_{n+1} (\varphi(n + 1, n, x) + \lambda I)) (A_n(x) + \lambda I). \]

Let \(P : X \to B(V)\) be a projector compatible with the system \((A, \varphi)\).

**Theorem 19** Assume that the system \((A, \varphi)\) is \(P\)-uniformly exponentially dichotomic such that \(1 \leq N < e^\alpha\), where \(N\) and \(\alpha\) are the constants given in Definition 8. Then for all \(\lambda \in \mathbb{R}_+\) with \(0 < \lambda < 1 - Ne^{-\alpha}\), the perturbed system \((A_\lambda, \varphi)\) is \(P\)-uniformly exponentially dichotomic.

**Proof.** From hypothesis we have that there exist the constants \(N \geq 1\) and \(\alpha > 0\) such that

\[ e^{\alpha(m-n)}(\|\Phi^m_n(x)P(x)v\| + \|Q(x)v\|) \leq \]

\[ \leq N(\|P(x)v\| + \|\Phi^m_n(x)Q(x)v\|) \]

for all \((m, n, x, v) \in \Delta \times X \times V\).

Then, we obtain

\[ \|\Phi^m_n(x)P(x)v\| = \]

\[ = \|\Phi^m_{m-1}(\varphi(m - 1, n, x))\Phi^m_{m-2}(x)P(x)v\| \leq \]

\[ \leq \|\Phi^m_{m-1}(\varphi(m - 1, n, x)) + \lambda I\|\|\Phi^m_{m-1}(x)P(x)v\| \leq \]

\[ \leq (Ne^{-\alpha} + \lambda)^m \|P(x)v\| \leq \]

\[ \leq (Ne^{-\alpha} + \lambda)^{m-n} \|P(x)v\| \]

and

\[ \|\Phi^m_n(x)Q(x)v\| = \]

\[ = \|\Phi^m_{m-1}(\varphi(m - 1, n, x))\Phi^m_{m-1}(x)Q(x)v\| \geq \]

\[ \geq \|\Phi^m_{m-1}(\varphi(m - 1, n, x))\Phi^m_{m-1}(x)Q(x)v\| \geq \]

\[ \geq \frac{e^\alpha}{N} \|\Phi^m_{m-1}(x)Q(x)v\| = \]

\[ = \frac{e^\alpha}{N} \|\Phi^m_{m-2}(\varphi(m - 2, n, x))\Phi^m_{m-2}(x)Q(x)v\| \geq \]

\[ \geq \left(\frac{e^\alpha}{N}\right)^2 \|\Phi^m_{m-2}(x)Q(x)v\| \geq \]

\[ \geq \ldots \geq \left(\frac{e^\alpha}{N}\right)^{m-n} \|Q(x)v\| \]

so

\[ \|Q(x)v\| \leq (Ne^{-\alpha})^{m-n} \|\Phi^m_n(x)Q(x)v\| \]

which implies that

\[ \|\Phi^m_n(x)P(x)v\| + \|Q(x)v\| \leq \]

\[ \leq (Ne^{-\alpha} + \lambda)^{m-n} \left(\|P(x)v\| + \|\Phi^m_n(x)Q(x)v\|\right) \]

for all \((m, n, x, v) \in \Delta \times X \times V\). If we consider \(a_n = (Ne^{-\alpha} + \lambda)^n\), then from Lemma 14 it results that the perturbed system \((A_\lambda, \varphi)\) is \(P\)-uniformly exponentially dichotomic.

**References:**


