Localization of Compact Invariant Sets of Nonlinear Systems

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Abstract: The method of finding domains in the state space of a nonlinear system which contain all compact invariant sets is presented. We consider continuous-time and discrete-time dynamical systems and control systems which may contain disturbances. The results are based on the fact that any continuous function reaches maximum and minimum on a compact set. This method provides estimates of the areas with the chaotic behavior of trajectories. As examples, the Lorenz system and the Hénon system are considered.

Key-Words: Localization, compact invariant sets, Lorenz system, Hénon system

1 Introduction

The study of compact invariant sets is one of the important topics in the qualitative theory of ordinary differential equations closely related to analysis of a long-time behavior of a nonlinear system. Many researchers have been interested in the idea of finding some geometrical bounds for objects in the phase space such as attractors, periodical orbits and chaotic dynamics of a nonlinear autonomous system

\[ \dot{x} = f(x), \quad f \in C^1(\mathbb{R}^n). \]  

(1)

Some results were obtained by Lyapunov-type functions [2, 16–18, 20] and by the semipermeable surfaces [3].

Another approach was proposed in [10, 11] for solving the localization problem of all periodical solutions of the system (1).

This method was developed in [1, 12–15, 19] in the case of compact invariant sets. Afterwards this localization method was extended to control systems and dynamical systems with disturbances [5, 8]. The case of discrete-time systems is considered in [5, 7, 9]. In all these cases the localization method is based on the fact that any continuous function reaches maximum and minimum on a compact set. In the case of chaotic systems it is important that the localization method does not use the numerical calculation of any trajectory.

In this article we develop the main ideas of the localization method.

2 Compact Invariant Sets

In this Section, when we talk about a localization we have in mind the following problem: find the set \( \Omega \subset \mathbb{R}^n \) (a localization set) that contains all compact invariant sets of the system (1).

For any \( x_0 \in \mathbb{R}^n \) by \( x(t, x_0), t \in \mathbb{R} \) we denote a solution of the system (1), with \( x(0, x_0) = x_0 \). A set \( G \subset \mathbb{R}^n \) is called invariant for (1) if for any \( x_0 \in G \) we have: \( x(t, x_0) \in G \) for all \( t \in \mathbb{R} \).

Important classes of compact invariant sets are equilibrium points, periodic orbits, heteroclinic orbits, homoclinic cycles and orbits.

Let \( Q \) be a subset in \( \mathbb{R}^n \). We define a maximal compact invariant set of the system (1) contained in \( Q \) as a compact invariant set in \( Q \) containing any compact invariant set of the system (1) contained in \( Q \). A maximal compact invariant set contained in \( Q \) may not exist.

2.1 Localization of Compact Invariant Sets

For a function \( \varphi \in C^1(Q) \), we introduce the set

\[ S_\varphi(Q) = \{ x \in Q : \ L_f \varphi = 0 \}, \]

where \( L_f \varphi \) is a Lie derivative of the function \( \varphi(x) \) with respect to the vector field \( f(x) \) of the system (1) and define

\[ \varphi_{\text{inf}}(Q) = \inf_{x \in S_\varphi(Q)} \varphi(x), \quad \varphi_{\text{sup}}(Q) = \sup_{x \in S_\varphi(Q)} \varphi(x). \]

Theorem 1 [12] For any \( \phi(x) \in C^\infty(\mathbb{R}^n) \) all compact invariant sets of the system (1) located in \( Q \) are contained in the set defined by the formula

\[ \Omega_{\varphi}(Q) = \{ x \in Q : \varphi_{\text{inf}}(Q) \leq \varphi(x) \leq \varphi_{\text{sup}}(Q) \} \]
as well. If \( \Omega_\varphi(Q) \) is a compact set then the system (1) has a maximal compact invariant set located in \( \Omega_\varphi(Q) \).

In Theorem 1 a smooth function defined on some subset \( Q \) of the state space is associated with a localization set. This localization set contains all compact invariant sets located in \( Q \) while this function itself is called localizing.

**Theorem 2** Let \( \varphi_m(x) \), \( m = 0, 1, 2, \ldots \) be a sequence of functions from \( C^1(\mathbb{R}^n) \). The localization sets

\[
\Omega_0 = \Omega_{\varphi_0}(\mathbb{R}^n), \quad \Omega_m = \Omega_{\varphi_m}(\Omega_{m-1}), \quad m > 0,
\]

contain any compact invariant set of the system (1) and

\[
\Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_m \supset \ldots .
\]

Proofs of these results may be realized in the same way like in [10, 11].

### 2.2 The Lorenz system

The Lorenz system

\[
\begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\
\dot{x}_2 &= rx_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - bx_3 
\end{align*}
\]

with positive parameters \( \sigma, r \) and \( b \) has a chaotic dynamics for the some values of parameters, for example if \( \sigma = 10, r = 28, b = \frac{8}{3} \).

To find compact localization set for the Lorenz system with these values of parameters we consider three functions

\[
\begin{align*}
p_1(x_1, x_2, x_3) &= x_1^2 - 2\sigma x_3, \\
p_2(x_1, x_2, x_3) &= \frac{x_2^2 + x_3^2}{2} - rx_3, \\
p_3(x_1, x_2, x_3) &= x_1^2 - \frac{2\sigma - M}{r}(x_2^2 + x_3^2) - Mx_3,
\end{align*}
\]

where \( M < 2\sigma \)

By using the localizing function \( p_1 \) we have that the set \( S := S_{p_1}(\mathbb{R}^3) \) is defined by equality

\[
x_3 = b^{-1}x_1^2.
\]

Hence

\[
p_1|_S = (1 - 2\sigma b^{-1})x_1^2,
\]

\( p_{1\inf}(\mathbb{R}^3) = -\infty, \quad p_{1\sup}(\mathbb{R}^3) = 0 \)

and we get the localization set

\[
\Omega_{p_1}(\mathbb{R}^3) = \{x_1^2 - 2\sigma x_3 \leq 0\}.
\]

For the localizing function \( p_2 \) we notice that the set \( S_{p_2}(\mathbb{R}^3) \) is defined by equality

\[
x_2^2 = -bx_3^2 + bx_3.
\]

Following the calculations, we find

\[
p_{2\inf}(\mathbb{R}^3) = -\frac{r^2}{2}, \quad p_{2\sup}(\mathbb{R}^3) = p_* := \frac{r^2(b - 2)^2}{8(b - 1)}
\]

and we came to the localization set

\[
\Omega_{p_2}(\mathbb{R}^3) = \left\{\frac{x_2^2 + x_3^2}{2} - rx_3 \leq p_*\right\}.
\]

Let us consider the localizing function \( p_3 \). For this function we get that \( p_{3\inf}(\mathbb{R}^3) = -\infty, \)

\[
p_{3\sup}(\mathbb{R}^3) = p_* := \frac{M^2(b - 2\sigma)^2r}{8\sigma(2\sigma - M)(\sigma - b)}
\]

and

\[
\Omega_{p_3}(\mathbb{R}^3) = \left\{x_1^2 - \frac{2\sigma - M}{r}(x_2^2 + x_3^2) - Mx_3 \leq p_*\right\}.
\]

Hence we found the \( M \)-parametric family of localization sets \( \omega_M = \Omega_{p_M}(\mathbb{R}^3) \). The intersection of the localization sets is a localization set. Therefore all compact invariant sets of the Lorenz system are contained in the sets

\[
\omega = \cap_{M < 2\sigma} \omega_M, \\
\Omega = \Omega_{p_1}(\mathbb{R}^3) \cap \Omega_{p_2}(\mathbb{R}^3) \cap \omega.
\]

The localization set \( \omega \) is an unbounded set (see Fig. 1) and the localization set \( \Omega \) is a compact set (see Fig. 2).

![Figure 1: The localizing set \( \omega \) for the compact invariant sets of the Lorenz system](image-url)
3 Robustly Invariant Compact Sets

In this Section, we consider a version of the localization method for localizing robustly invariant compact sets of a continuous-time dynamical system with disturbance, i.e., the construction of sets in the phase space of the system that include all the robustly invariant compact sets of the system.

Consider a dynamical system of the form

$$\dot{x} = f(x, w),$$

where \( f : X \times W \rightarrow X \) is a continuously differentiable mapping, \( X \subset \mathbb{R}^n \) is an open subset, \( W \subset \mathbb{R}^p \) and \( w \) is a disturbance.

The solution of the system (2) is a pair of functions \( (x(t), w(t)), x : [-T, T] \rightarrow \mathbb{R}^n, w : [-T, T] \rightarrow W \) satisfying the equation of the system. Here, \( w(t) \) and \( x(t) \) are assumed to be continuous and smooth (continuously differentiable) functions, respectively.

The trajectory of the system (2) is a curve in the phase space that is parametrically defined by the equation \( x = x(t) \), where \( x(t) \), together with some function \( w(t) \), is a solution of the system (2).

A set \( M \subset X \) is said to be robustly invariant for dynamical system (2) if, given an arbitrary point \( x_0 \in M \), any trajectory \( x(t) \) of the system (2) that satisfies the condition \( x(t_0) = x_0 \) is contained entirely in \( M \).

Let \( Q \) be a subset in \( X \subset \mathbb{R}^n \). For a function \( \varphi \in C^1(Q) \), we introduce the set

$$\Sigma_{\varphi}^-(Q) = \{ x \in Q : \varphi(F(x)) - \varphi(x) \geq 0 \},$$

and define

$$\varphi_{\inf}(Q) = \inf_{x \in \Sigma_{\varphi}^-(Q)} \varphi(x), \quad \varphi_{\sup}(Q) = \sup_{x \in \Sigma_{\varphi}^+(Q)} \varphi(x).$$

Theorem 3 Each robustly invariant compact set of the system (2) containing in \( Q \) is contained in the set

$$\Omega_{\varphi}(Q) = \{ x \in Q : \varphi_{\inf}(Q) \leq \varphi(x) \leq \varphi_{\sup}(Q) \}. $$

4 Compact Invariant Sets of discrete-time systems

In this Section, when we talk about a localization we have in mind the following problem: find the set \( \Omega \subset \mathbb{R}^n \) (a localization set) that contains all compact invariant (positively invariant or negatively invariant) sets of the system

$$x_{k+1} = F(x_k),$$

where \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous mapping.

A subset \( K \subset \mathbb{R}^n \) is said to be positively (negatively) invariant for the system (3) if \( F(K) \subset K \) \((F^{-1}(K) \subset K)\). A compact set of a discrete system is one that is both positively and negatively invariant.

The positively invariant sets of a discrete system include equilibria (stationary points), i.e., points satisfying the equation \( F(x) = x \); periodic orbits; and attractors. If the discrete system is reversible, i.e., the mapping \( F \) is a homeomorphism, these sets are invariant. However, if the system is not reversible, this is not true in the general case.

4.1 Localization of Compact Positively Invariant Sets

Let \( Q \) be a subset in \( \mathbb{R}^n \). For a function \( \varphi \in C(Q) \), we introduce the sets

$$\Sigma_{\varphi}^+(Q) = \{ x \in Q : \varphi(F(x)) - \varphi(x) \leq 0 \},$$

and define

$$\varphi^l_{\inf}(Q) = \inf_{x \in \Sigma_{\varphi}^+(Q)} \varphi(x), \quad \varphi^l_{\sup}(Q) = \sup_{x \in \Sigma_{\varphi}^+(Q)} \varphi(x).$$

Theorem 4 Each compact positively invariant set of the system (3) containing in \( Q \) is contained in the set

$$\Omega_{\varphi}^l(Q) = \{ x \in Q : \varphi^l_{\inf}(Q) \leq \varphi(x) \leq \varphi^l_{\sup}(Q) \}. $$

4.2 Localization of Compact Negatively Invariant Sets

Let

$$\varphi^r_{\inf}(Q) = \inf_{x \in F^{-1}(\Sigma_{\varphi}^-(Q))} \varphi(x),$$

$$\varphi^r_{\sup}(Q) = \sup_{x \in F^{-1}(\Sigma_{\varphi}^+(Q))} \varphi(x).$$
where \( \hat{F}(A) \) is a set of points whose complete preimage is contained in \( A \).

**Theorem 5** Each compact negatively invariant set of the system (3) containing in \( Q \) is contained in the set

\[
\Omega_\varphi^L(Q) = \{ x \in Q : \varphi_{\text{inf}}(Q) \leq \varphi(x) \leq \varphi_{\text{sup}}(Q) \}.
\]

If the dynamical system is reversible, then Theorem 5 is reduced to Theorem 4.

### 4.3 Localization of Compact Invariant Sets

Define

\[
\varphi_{\text{inf}}(Q) = \inf_{x \in \Sigma_{\varphi}(Q) \cap F(\Sigma_{\varphi}(Q))} \varphi(x),
\]

\[
\varphi_{\text{sup}}(Q) = \sup_{x \in \Sigma_{\varphi}(Q) \cap F(\Sigma_{\varphi}(Q))} \varphi(x).
\]

**Theorem 6** Any compact invariant set of the system (3) containing in \( Q \) is contained in the set

\[
\Omega_\varphi(Q) = \{ x \in Q : \varphi_{\text{inf}}(Q) \leq \varphi(x) \leq \varphi_{\text{sup}}(Q) \}.
\]

### 4.4 The Hénon system

Consider the system

\[
\begin{align*}
x_{n+1} &= a + by_n - x_n^2, \\
y_{n+1} &= x_n,
\end{align*}
\]

(4)

proposed by Hénon as a discrete-time analogue of the Lorenz system [4]. To localize the positive invariant compact sets we use the linear function \( \phi(x, y) = Cx + Dy \). Then we apply shifts of the obtained localizing sets along system trajectories. Finally, we take into account that the inverse system for (4) is the Hénon system with other parameters. The result is shown in fig. 3.

### 5 Robustly Invariant Compact Sets of Discrete-Time Systems

In this Section, we consider a version of the localization method for localizing robustly invariant compact sets of a discrete-time dynamical system with disturbance

\[
x_{k+1} = F(x_k, w_k),
\]

(5)

where \( F : \mathbb{R}^n \times W \rightarrow \mathbb{R}^n \) is a continuous mapping and \( w \in W \) are disturbances.

#### 5.1 Robustly Positively Invariant Compact Sets

A set \( M \subset \mathbb{R}^n \) is called positively invariant if \( F(x, w) \subset M \) for any \( x \in M \) and \( w \in W \). This definition can be restated as follows: a set \( M \) is positively invariant if \( F(M \times W) \subset M \). The definition implies that, if \( x_0 \in \mathbb{R}^n \) belongs to a positively invariant set \( M \), then, for any disturbances \( w_k \in W \), the trajectory \( x_k \) of the system determined by the relations \( x_{k+1} = F(x_k, w_k), k = 0, 1, \ldots \), lies entirely in \( M \).

Let \( Q \) be a subset in \( \mathbb{R}^n \). For a function \( \varphi \in C(Q) \), we introduce the sets

\[
\Sigma_{\varphi,w}^+(Q) = \{ x \in Q : \inf_{w \in W} \varphi(F(x, w)) - \varphi(x) \geq 0 \},
\]

\[
\Sigma_{\varphi,w}^-(Q) = \{ x \in Q : \inf_{w \in W} \varphi(F(x, w)) - \varphi(x) \leq 0 \},
\]

and define

\[
\varphi_{\text{inf}}^+(Q) = \inf_{x \in \Sigma_{\varphi,w}^+(Q)} \varphi(x),
\]

\[
\varphi_{\text{sup}}^+(Q) = \sup_{x \in \Sigma_{\varphi,w}^+(Q)} \varphi(x).
\]

**Theorem 7** Any positively invariant compact set of the system (5) that is contained in a set \( Q \) is also contained in the set

\[
\Omega_\varphi^+(Q) = \{ x \in Q : \varphi_{\text{inf}}^+(Q) \leq \varphi(x) \leq \varphi_{\text{sup}}^+(Q) \}.
\]

#### 5.2 Robustly Negatively Invariant Compact Sets

A set \( M \subset \mathbb{R}^n \) is called negatively invariant if, for any point \( \eta \in \mathbb{R}^n \) satisfying the condition \( F(\eta, w) \in M \) for some \( w \in W \), we have \( \eta \in M \). This means
Given an arbitrary set $Q \subset \mathbb{R}^n$, we introduce the notation
\[
\tilde{F}^{-1}(G) = \bigcup_{w \in W} F_w^{-1}(G).
\]

A necessary and sufficient condition for negative invariance can be stated as follows: a set $M \subset \mathbb{R}^n$ is negatively invariant for system (5) if and only if $\tilde{F}^{-1}(M) \subset M$.

Let $Q$ be a subset in $\mathbb{R}^n$. For a function $\varphi \in C(Q)$, we introduce the sets
\[
\Sigma_+^\varphi(Q) = \{ x \in Q : \sup_{\eta \in F^{-1}(x)} \varphi(\eta) - \varphi(x) \leq 0 \},
\]
\[
\Sigma_-^\varphi(Q) = \{ x \in Q : \inf_{\eta \in F^{-1}(x)} \varphi(\eta) - \varphi(x) \geq 0 \},
\]
and define
\[
\varphi_{\inf}^l(Q) = \inf_{x \in \Sigma_+^\varphi(Q)} \varphi(x), \quad \varphi_{\sup}^l(Q) = \sup_{x \in \Sigma_-^\varphi(Q)} \varphi(x).
\]

**Theorem 8** Any negatively invariant compact set of system (5) contained in $Q$ is also contained in the set
\[
\Omega^l_\varphi(Q) = \{ x \in Q : \varphi_{\inf}^l(Q) \leq \varphi(x) \leq \varphi_{\sup}^l(Q) \}.
\]

### 5.3 Robustly Invariant Compact Sets

A set $M \subset \mathbb{R}^n$ is called robustly invariant for system (5) if it is both positively and negatively robustly invariant for this system; i.e.,
\[
F(M \times W) \subset M, \quad \tilde{F}^{-1}(M) \subset M.
\]

Given a set $Q \subset \mathbb{R}^n$, define
\[
\varphi_{\inf}(Q) = \inf_{x \in \Sigma_+^\varphi(Q) \cap \Sigma_-^\varphi(Q)} \varphi(x),
\]
\[
\varphi_{\sup}(Q) = \sup_{x \in \Sigma_+^\varphi(Q) \cap \Sigma_-^\varphi(Q)} \varphi(x).
\]

**Theorem 9** Any negatively invariant compact set of system (5) contained in $Q$ is also contained in the set
\[
\Omega_\varphi(Q) = \{ x \in Q : \varphi_{\inf}(Q) \leq \varphi(x) \leq \varphi_{\sup}(Q) \}.
\]

### 6 Conclusion

The presented localization method is based on the use of a function. This method can be applied to various types of nonlinear dynamical systems with continuous and discrete time. In all cases the localizing set has the same form $a \leq \varphi(x) \leq b$ where bound values $a$ and $b$ are calculated for every type of systems by specific algorithms. The references show that the localization method is applicable in wide number of scientific areas.

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**References:**


