A Study of Centerless Grinding

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Abstract: Our goal is to perform a study on the dynamics of the piece in the centerless grinding process. The model constructed in this paper is a high non-linear one, resulted from the geometry of the centerless grinding. For this model we perform the development into Taylor series to obtain the depth of cut as function of the displacement of the piece, as a third degree polynomial. In this form, we get three potential positions of equilibrium for the piece. To avoid two such equilibrium positions (which are unstable) and lead to a mathematical model that uses the Heaviside step function, we determine the geometric condition for the existence and uniqueness of the equilibrium position. We also prove that this unique position of equilibrium is a simply stable one. This condition is one and the same to the condition of existence for the harmonic development of the solution around the simply stable equilibrium position. We determine this solution till the third order harmonic and we present the condition to avoid the secular terms in the solution. In this way we give a theoretical explanation for the triangular shape of the obtained piece by centerless grinding. The study assumed that the cutting force is a linear expression in depth of cut and the piece does not loose the contact to both cutting disc and driven disc.

Key-Words: - Grinding, harmonic, equilibrium, stability, triangular

1 Introduction
The cutting of the cylindrical work-pieces by centerless grinding is an often met procedure in machines construction. Its simplicity and the fact that it does not require a high specialized work-hand make this procedure a very advantageous on from the economical point of view. In addition, the procedure is also self-corrected, i.e. it does not require a high precision for the initial positioning of the work-piece, the cutting precision being not influenced by this initial positioning.

Fig. 1. Centerless grinding.

The procedure is described in Figure 1. The cylindrical work-piece of centre \( O \) and radius \( R \) is supported on the support \( S \) which is an inclined plan of angle \( \beta < 5^\circ \), and is acted in a rotational motion by the training abrasive disk of fixed centre \( O_1 \). The grinding is performed by the abrasive disc of centre \( O_2 \).

The training disc has the angular velocity \( \omega_1 \), while the grinding disc has the angular velocity \( \omega_2 \).

2 Mathematical model
Let us denote by \( O \) the equilibrium position of the work-piece and by \( O' \) a new position of it, resulted from the vibrations during the grinding process (Fig. 2). We denote by \( x \) the distance \( OO' \), and by \( \alpha \) the angle made by the straight line \( OO_2 \) with the direction of displacement of the center of work-piece, \( OO' \).

Fig. 2. Mathematical model.
From the triangle $O_2OO'$, applying the theorem of cosine, one obtains the relation
\begin{equation}
D_1^2 = D^2 + x^2 - 2Dx \cos \alpha,
\end{equation}
where $D$ and $D_1$ are the measurements of the segments $O_2O$ and $O_2O'$, respectively.

Relation (1) leads to
\begin{equation}
D_1 = \sqrt{D^2 + x^2 - 2Dx \cos \alpha} = \sqrt{1 + a_1x + a_2x^2},
\end{equation}
in which
\begin{equation}
a_1 = -\frac{2 \cos \alpha}{D}, \quad a_2 = \frac{1}{D^2}.
\end{equation}

On the other hand,
\begin{equation}
\frac{d}{dx} \left[ (1 + a_1x + a_2x^2)^{\frac{1}{2}} \right] = \frac{1}{2} \left( 1 + a_1x + a_2x^2 \right)^{\frac{3}{2}} (a_1 + 2a_2x).
\end{equation}

It results the differential equation
\begin{equation}
m\ddot{x} = F,
\end{equation}
wherefrom
\begin{equation}
\ddot{x} = d_1x + d_2x^2 + d_3x^3,
\end{equation}
with
\begin{equation}
d_1 = \frac{b_1}{mCD}, \quad i = \frac{1}{\alpha},
\end{equation}
where $b_1 = \frac{a_1}{2} = -\frac{\cos \alpha}{D}$, \hspace{1cm} (11)
\begin{equation}
b_2 = a_2 - \frac{3a_1^2}{4} = 1 - \frac{3 \cos^2 \alpha}{D^2},
\end{equation}
\begin{equation}
b_3 = \frac{15a_1^3}{8} - 9a_1a_2 = \frac{\cos \alpha(18 - 15 \cos^2 \alpha)}{D^3}.
\end{equation}

3 The equation of motion

We will assume that the cutting force is proportional to the depth of cut $w$,
\begin{equation}
F_{cut} = C_i w,
\end{equation}
where $C_i$ is a constant that depends on the working conditions.

It follows that the force that acts on the direction $OO'$, of displacement of the work-piece, may be expressed by
\begin{equation}
F = C(D - D_1),
\end{equation}
where $C$ is a constant which depends on $C_i$ and the geometry of the machine tool.

Keeping into account the relation (10), the expression (15), can be put in the form
\begin{equation}
F_{cut} = \frac{b_1}{mCD} \left( 1 + b_1x + b_2x^2 + b_3x^3 \right),
\end{equation}
so that results the equation of motion
\begin{equation}
m\ddot{x} = F,
\end{equation}
wherefrom
\begin{equation}
\ddot{x} = d_1x + d_2x^2 + d_3x^3,
\end{equation}
with
\begin{equation}
d_1 = \frac{b_1}{mCD}, \quad i = \frac{1}{\alpha},
\end{equation}
where $b_1 = \frac{a_1}{2} = -\frac{\cos \alpha}{D}$, \hspace{1cm} (11)
\begin{equation}
b_2 = a_2 - \frac{3a_1^2}{4} = 1 - \frac{3 \cos^2 \alpha}{D^2},
\end{equation}
\begin{equation}
b_3 = \frac{15a_1^3}{8} - 9a_1a_2 = \frac{\cos \alpha(18 - 15 \cos^2 \alpha)}{D^3}.
\end{equation}

We make now the change of variable
\begin{equation}
x = y - \frac{d_2}{3d_3}, \quad \ddot{x} = \ddot{y};
\end{equation}
therefore
\begin{equation}
d_1x + d_2x^2 + d_3x^3 =
\end{equation}
\begin{equation}
\frac{2d_3^3}{27d_2^3} - \frac{d_3d_2^2}{3d_3} + \left( d_1 - \frac{d_3^2}{3d_3} \right) y + d_3y^3.
\end{equation}
It results the differential equation
\begin{equation}
\ddot{y} = \frac{2d_3^3}{27d_2^3} - \frac{d_3d_2^2}{3d_3} + \left( d_1 - \frac{d_3^2}{3d_3} \right) y + d_3y^3.
\end{equation}
A particular solution of this equation is
\begin{equation}
y_p = \left( \frac{2d_3^3}{27d_2^3} - \frac{d_3d_2^2}{3d_3} \right) t^3 + \frac{t^2}{2},
\end{equation}
where $t$ is the variable time.

But
and choosing the angle \( \alpha \) from the condition
\[ 1 - 3 \cos^2 \alpha = 0, \cos \alpha = \frac{1}{\sqrt{3}}, \alpha = 60.817^\circ, \] (25)
the particular solution (23) becomes the null solution.

### 4 The study of the homogeneous differential equation

Further on, we will analyze the homogeneous differential equation obtained from the expression (22), i.e.
\[
y = \left(d_1 - \frac{d_2^2}{3d_3}\right)y + d_3y^3
\] (26)
or
\[
y = e_1y + e_3y^3,
\] (27)
with
\[
e_1 = d_1 - \frac{d_2^2}{3d_3}, \quad e_3 = d_3.
\] (28)

We have
\[
e_1 = \frac{1}{mCD}\left(b_1 - \frac{b_2^2}{3b_3}\right) = - \frac{1}{mCD}\left[\cos \alpha + \frac{(1 - 3 \cos^2 \alpha)}{\cos \alpha(18 - 15 \cos^2 \alpha)}\right];
\] (29)
hence we may consider
\[
e_1 = -\omega^2
\] (30)
and the equation (27) becomes
\[
y + \omega^2y = e_3y^3.
\] (31)

We will the multiple scales of time method and we will write
\[T_n = e^{\kappa t}, \quad n = 0,1,2,\ldots,\] (32)
so that
\[
\frac{d}{dr} = \frac{dT_0}{dr} \frac{\partial}{\partial T_0} + \frac{dT_1}{dr} \frac{\partial}{\partial T_1} + \frac{dT_2}{dr} \frac{\partial}{\partial T_2} + \ldots = \frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} + \frac{\partial}{\partial T_2} + \ldots,
\] (33)
\[
\frac{d^2}{dr^2} = \frac{d}{dr}\left(\frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} + \frac{\partial}{\partial T_2} + \ldots\right) = \frac{\partial^2}{\partial T_0^2} + \frac{\partial^2}{\partial T_1^2} + \frac{\partial^2}{\partial T_2^2} + \ldots
\] (34)
We are searching the solution in the form
\[
y(t,\varepsilon) = e_{y_1}(T_0, T_1, T_2, \ldots) + e^2y_2(T_0, T_1, T_2, \ldots) + e^3y_3(T_0, T_1, T_2, \ldots) + \ldots,
\] (35)
where \( \varepsilon \) is a small parameter.

Replacing in the equation (31) and equating the coefficients of the same powers of \( \varepsilon \), we obtain
\[
\frac{\partial^2y_1}{\partial T_0^2} + \omega^2y_1 = 0,
\] (36)
\[
\frac{\partial^2y_2}{\partial T_0^2} + 2\frac{\partial^2y_1}{\partial T_0^2} - 2\frac{\partial^2y_1}{\partial T_0^2} + \omega^2y_3 = 0,
\] (37)
\[
\frac{\partial^2y_3}{\partial T_0^2} + \omega^2y_3 = -2\frac{\partial^2y_2}{\partial T_0^2} + \omega^2y_3 = 0.
\] (38)

The equation (36) leads us to the complex solution
\[
y_1 = A(T_1, T_2)e^{i\omega T_0} + \bar{A}(T_1, T_2)e^{-i\omega T_0},
\] (39)
in which \( A \) is a complex function depending on \( T_1 \) and \( T_2 \), while \( \bar{A} \) is its complex conjugate.

The equation (37) offers now
\[
\frac{\partial A}{\partial T_1} = 0,
\] (40)
that is, the function \( A \) does not depend on \( T_1 \); hence \( A = A(T_2) \).

The solution of the equation (40) is now
\[
y_2 = 0.
\] (41)
The equation (38) becomes
\[
\frac{\partial^2y_3}{\partial T_0^2} + \omega^2y_3 = -2\frac{\partial^2y_2}{\partial T_0^2} + \omega^2y_3 = 0.
\] (42)

The secular terms disappear in \( y_3 \) if
\[
2i\omega \frac{\partial A}{\partial T_2} - 3e_3A^2\bar{A} = 0.
\] (43)

Writing now
\[ A = \frac{1}{2} \alpha e^{i \beta}, \]  
where \( \alpha \) and \( \beta \) are real functions in \( T_2 \), and

\[ \omega \frac{d \alpha}{dT_2} = 0, \quad \omega \alpha \frac{d \beta}{dT_2} + \frac{3}{8} \varepsilon_3 \alpha^3 = 0. \]  

The first relation (46) shows that \( \alpha \) is a constant, while the second relation (46) leads to

\[ \beta = -\frac{3}{8} \varepsilon_3 \alpha^2 T_2 + \beta_0, \]  
where \( \beta_0 \) is a constant.

In conclusion,

\[ A = \frac{1}{2} \alpha e^{i \left(-\frac{3}{8} \varepsilon_3 \alpha^2 + \beta_0\right)} = \frac{1}{2} \alpha e^{i \left(-\frac{3}{8} \varepsilon_3 \alpha^2 + \beta_0\right)}. \]  

The equation (43) reads

\[ \frac{\partial^2 y_3}{\partial T_0^2} + \omega^2 y_3 = e_3 A \cos(3\omega T_0) \]  
and it has a harmonic solution in \( 3\omega T_0 \).

It results the solution of the equation (27) as

\[ y(t, \varepsilon) = y_0 + e^{i \varepsilon} y_3 \]  
and it contains terms in \( \cos(\omega t) \) and \( \cos(3\omega t) \); hence we deduce that the solution of the equation (18) contains terms in \( \cos(\omega t) \) and \( \cos(3\omega t) \).

Moreover, the harmonic \( \cos(3\omega t) \) leads to the triangular form of the work-piece.

### 5 The equilibrium positions

We will study the equation (27) instead of the equation (18), these equations being equivalent and having the same number of real roots.

The equilibrium positions are the solutions of the equations

\[ e_1 y + e_3 y^3 = 0, \]  
wherefrom

\[ y_1 = 0 \]  
and

\[ y_{2,3} = \pm \sqrt{-\frac{e_1}{e_3}}. \]  

We saw that \( e_1 < 0 \) for \( \cos \alpha > 0 \) (according to the relation (30)). Keeping into account the expression of \( e_3 \),

\[ e_3 = d_3 = \frac{b_3}{mCD} = \frac{\cos \alpha (18 - 15 \cos^2 \alpha)}{mCD^4}, \]  
it results that \( e_3 > 0 \) and therefore the roots \( y_{2,3} \) given by the expression (53) always exist for the equation (51).

The solution (52) of the equation (51) is unique if and only if \( e_i, e_1 > 0 \), which leads to

\[ - \frac{1}{mCD^2} \left[ \cos \alpha + \frac{(1 - 3 \cos^2 \alpha)^2}{\cos \alpha (18 - 15 \cos^2 \alpha)} \right] \times \cos \alpha (18 - 15 \cos^2 \alpha) \]  
wherefrom

\[ \cos^2 \alpha (18 - 15 \cos^2 \alpha) + (1 - 3 \cos^2 \alpha)^2 < 0, \]  
which is absurd; there will always exist three equilibrium positions.

The equation (27) may be put in the form of the following system

\[ \ddot{\xi}_1 = \xi_2, \quad \ddot{\xi}_2 = e_1 \xi_1 + e_3 \xi_3^3, \]  
where \( \xi_1 = y, \quad \xi_2 = y, \) and it leads to the characteristic equation

\[ -\lambda - \frac{1}{\xi_1 + 3e_3 \xi_3^2 - \lambda} = 0, \]  
wherefrom

\[ \lambda^2 - (e_1 + 3e_3 \xi_3^2) = 0, \]  
with the solutions

\[ \lambda_{1,2} = \pm \sqrt{e_1 + 3e_3 \xi_3^2}, \]  
\( \xi_3 \) marking an equilibrium position.

For \( e_1 = 0, \) the characteristic equation becomes

\[ \lambda^2 - e_3 = 0 \]  
and since \( e_3 < 0, \) result pure imaginary roots

\[ \lambda_{1,2} = \pm \sqrt{-e_3}; \]  

hence the equilibrium is simply stable.

For \( y_{2,3} = \pm \sqrt{-\frac{e_1}{e_3}} \) the characteristic equation reads

\[ \lambda^2 + 2e_1 = 0 \]  
and it has the real roots

\[ \lambda_3 = \sqrt{-2e_1} > 0, \quad \lambda_4 = -\sqrt{-2e_1} < 0. \]  

Since \( \lambda_3 > 0 \) it results that these equilibrium positions are unstable.

In addition, the relation (53) defines the limits between which the work-piece oscillates.

### 6 Conclusions

In our paper we described a model with one degree of freedom for the centerless grinding. The model is highly nonlinear because of the geometry of the cutting process. Using the development in a Taylor series for one component of the grinding force, a nonlinear non-homogeneous ordinary differential
equation, the nonlinearity being defined by a third degree polynomial. With the aid of the normal form theory we eliminated the second degree term and using the multiple scales of time method the solution around the stable equilibrium position is described. This solution contains a harmonic which causes the triangular form of the work-piece. The particular solution of the differential equation can be eliminated by a specific choose of the geometry of grinding. We also proved the existence of three equilibrium positions for the geometry described in Figure 1. The process is stable and self corrector because the zero equilibrium solution is stable, while the other two equilibrium positions (one negative and one positive) are unstable and these last two equilibrium positions define the zone where the work-piece oscillates.

References: