Nonholonomic geometry of
Gibbs-Duhem symplectic metric structure

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Abstract: We prove that a six-dimensional nonholonomic thermodynamic system has a symplectic metric structure, characterized by rational components of the geometric objects. To build a significant Riemannian metric, the authors use a coframe with thermodynamical meaning, including the Gibbs-Duhem form. Based on this metric, the paper studies the geometry of a Gibbs-Duhem-Vranceanu-Riemann nonholonomic space and gives the physical interpretation of induced geometrical objects: coefficients of bilinear covariants, coefficients of Ricci (with three and four indexes), tangent vectors to geodesics etc. The original results include also the submanifold of bilinear covariants coefficients, the submanifold of Ricci rotation coefficients, and the submanifold of Ricci coefficients with four indexes.

Key–Words: Thermo-Chemical system, adapted frames, nonholonomic space, symplectic metric structure, geodesics, Ricci coefficients.

1 Nonholonomic theory of Gibbs-Duhem symplectic structure

In thermodynamics (see, [13]), the Gibbs-Duhem-Pfaff equation $SdT - VdP + N \mu = 0$ describes the relationship between changes in chemical potential for components in a thermodynamical system, in the 6-dimensional space $R^6 = \{(S, T, V, P, N, \mu)\}$, where $S$ - entropy, $T$ - temperature, $V$ - volume, $P$ - pressure, $N$ - the number of moles and $\mu$ - chemical potential.

In this paper we shall use the Vrânceanu theory ([11], [12]), combined with Udrisâte theory ([9], [10]), and some ideas in the papers [14], [15], to offer a nonholonomic study of Gibbs-Duhem symplectic structure $(R^6, \omega)$, $R^6 = \{(S, T, V, P, N, \mu)\}$, $\omega = SdT - VdP + N \mu$, where we preserve the names for the independent variables, but neither is restricted to positive values as in thermodynamics. The thermodynamic symplectic model created by us may be transferred into economics via a well-known dictionary created by the second author.

The geometrical modeling requires the coordinates
\[ x^1 = S, \quad x^2 = T, \quad x^3 = V, \quad x^4 = P, \quad x^5 = N, \quad x^6 = \mu. \]

Then the differential 1-form $\omega$ rewrites
\[ \omega^6 = \lambda_6^i dx^i, \quad i = 1, 6, \]
where
\[ \lambda_1^6 = \lambda_3^6 = \lambda_5^6 = 0, \quad \lambda_2^6 = x^1, \quad \lambda_4^6 = -x^3, \quad \lambda_6^6 = x^5. \]

To create an adapted moving coframe, to the Gibbs-Duhem 1-form $\omega^6$ we add another five linearly independent Pfaff forms
\[ \omega^1 = x^2 dx^1 \text{ (completely integrable form)}, \]
\[ \omega^2 = x^4 dx^3 \text{ (completely integrable form)}, \]
\[ \omega^3 = -x^1 dx^2 + x^3 dx^4 + x^6 dx^5, \]
\[ \omega^4 = -x^1 dx^2 - x^4 dx^3 + x^6 dx^5, \]
\[ \omega^5 = x^2 dx^1 + x^3 dx^4 + x^6 dx^5 \]
which can be written as
\[ \omega^a = \lambda^a_i dx^i, \quad a = 1, 5. \]
The reason for selecting the forms (2)+(3) is the thermodynamic significance of the attached geometry. These Pfaff forms are in fact actions (elementary thermo-chemical mechanical works) that deforms the initial Euclidean Geometry of the space. Mathematically, they determine a moving coframe (a system of congruences), with the moments \( \lambda^a_i \) ([11], [12]). Along the curves we can use the Vrăncăeanu notations \( ds^a = \omega^a, a = \overline{1,6} \). The congruences (2)+(3) are not orthogonal in the Euclidean space \( (\mathbb{R}^6, \delta_{ab}) \). The physical meaning of their restrictions is

- \( ds^1 = TdS = dQ \), where \( Q \) is the heat of the system;
- \( ds^2 = PdV = dW \), where \( W \) is the PV-work of the system;
- \( ds^3 = -SdT + VdP + \mu dN = dG \), where \( G \) is the Gibbs energy;
- \( ds^4 = -SdT - PdV + \mu dN = dA \), where \( A \) is the Helmholtz energy;
- \( ds^5 = TdS + VdP + \mu dN = dH \), where \( H \) is the enthalpy;
- \( ds^6 = SdT - VdP + N d\mu \) (Gibbs-Duhem-Pfaff).

To prevent contradictory discussions, we accept the dimensionless expression of the 1-forms.

On the region where the matrix of moments \( (\lambda^a_i) \), \( a = \overline{1,6}, i = \overline{1,6} \) is nondegenerate, we introduce its inverse matrix \( (\mu^a_i) \). Explicitly,

\[
(\lambda^a_i) = \begin{pmatrix}
  x^2 & 0 & 0 & 0 & 0 & 0 \\
  0 & x^4 & 0 & 0 & 0 & 0 \\
  0 & -x^1 & 0 & x^3 & x^6 & 0 \\
  0 & -x^1 & -x^4 & 0 & x^6 & 0 \\
  x^2 & 0 & 0 & 0 & x^6 & 0 \\
  0 & 0 & 0 & 0 & -x^6 & 0 & x^5
\end{pmatrix},
\]

\[
(\mu^a_i) = \begin{pmatrix}
  \frac{1}{x^2} & 0 & 0 & 0 & 0 & 0 \\
  -\frac{1}{x^1} & 0 & 0 & 0 & 0 & 0 \\
  0 & \frac{1}{x^4} & 0 & 0 & 0 & 0 \\
  0 & -\frac{1}{x^1} & \frac{1}{x^2} & 0 & 0 & 0 \\
  -\frac{1}{x^2} & -\frac{1}{x^1} & -\frac{1}{x^4} & \frac{1}{x^7} & 0 & 0 \\
  \frac{1}{x^6} & \frac{1}{x^7} & \frac{1}{x^7} & -\frac{1}{x^7} & \frac{1}{x^7} & -\frac{1}{x^7}
\end{pmatrix}.
\]

on the subset \( D \subset \mathbb{R}^6 \) characterized by

\[
\det (\lambda^a_i) = -x^1 x^2 x^3 x^4 x^5 x^6 \neq 0.
\]

The columns of the matrix \( (\mu^a_i) \) represent six linearly independent vector fields (moving frame).

Also we remark that

\[
\omega^a = \lambda^a_i dx^i, a, i = \overline{1,6},
\]

is equivalent to

\[
dx^i = \mu^a_i \omega^a, \ i, a = \overline{1,6}.
\]

Particularly, for \( \omega^6 = 0 \), we get

\[
dx^i = \mu^a_i \omega^a, \ i = \overline{1,6}, a = \overline{1,5}.
\]

Thus, the GDP equation rewrites

\[
\lambda^a_i \mu^a_i \omega^a = 0, \ i = \overline{1,6}, a = \overline{1,5}.
\]

The moving coframe \( \omega^a, a = \overline{1,6} \), on \( (D \subset \mathbb{R}^6, \delta_{ab}) \) determines a Riemannian manifold \( (D \subset \mathbb{R}^6, g_{ij}, \omega^a) \) (with a positive definite metric), in which the system of congruences \( \omega^a \) is orthogonal. The Riemannian metric \( g_{ij} \) is given by the square of arc element ([12], p. 260)

\[
ds^2 = \delta_{ab} ds^a ds^b = g_{ij} dx^i dx^j, a, b = \overline{1,6}; \ i, j = \overline{1,6},
\]
or by the matrix

\[
(\mu^a_i) = (\delta_{ab} \lambda^b_j \lambda^a_i) =
\begin{pmatrix}
  2x^2 & 0 & 0 & 0 & x^2 x^3 & x^2 x^6 & 0 \\
  0 & 3x^1 & x^1 x^3 & -2x^1 x^3 & -2x^1 x^6 & x^1 x^5 & 0 \\
  0 & x^1 x^4 & 2x^1 x^2 & 0 & -x^4 x^6 & 0 & 0 \\
  x^2 x^3 & -2x^1 x^3 & 0 & 3x^3 x^2 & 2x^3 x^6 & -x^3 x^5 & 0 \\
  x^2 x^6 & -2x^1 x^6 & -x^4 x^6 & 2x^3 x^6 & 3x^6 x^2 & 0 & 0 \\
  0 & x^1 x^5 & 0 & -x^3 x^5 & 0 & x^5 & 0
\end{pmatrix}
\]

The triple \( (D \subset \mathbb{R}^6, g_{ij}, \omega^a) \) is called Gibbs-Duhem-Vrăncăeanu-Riemann nonholonomic space (GDVRNS).

Coming back to physical sense of our variables, we find

**Proposition 1** The components \( g_{ij} \) of the Vrăncăeanu-Riemann metric are thermo-chemical energies.

2. The bilinear covariant coefficients of GDVRNS

For a moving coframe \( \omega^a = \lambda^a_i dx^i, a = \overline{1,6} \) (orthonormal congruences), the components of the **bilinear covariant** are

\[
w^a_{bc} = (\frac{\partial \lambda^a_i}{\partial x^b} - \frac{\partial \lambda^a_i}{\partial x^c}) \mu^a_i \mu^a_i.
\]

They verify the relation \( w^a_{bc} = -w^a_{cb} \) (skew-symmetric in the indexes \( b \) and \( c \)). Consequently the maximum dimension of the subspace (essential parameters)

\[
W = \{ w^a_{bc} \in \mathbb{R}^{216} | a, b, c = 1, ..., 6; b < c \}
\]

is 90.

Let us find a meaning for the bilinear covariant on the manifold (GDVRNS) \( (D \subset \mathbb{R}^6, g_{ij}, \omega^a) \).
Theorem 2  (i) The components of the bilinear covariant are thermo-chemical potentials determined by the energies $T, S, PV$ and $N\mu$.

(ii) The hypersurface $w_{bc}^a = w_{bc}^a(x), a, b, c = 1, \ldots, 6; b < c$, of $W$ can be reparameterized as a hyper-plane of dimension $3$.

Proof. Using Maple Facilities (see, "with(DifferentialGeometry): with(Tensor)"), we compute the essential components of the bilinear covariant:

\[
\begin{align*}
& w_{13}^1 = -\frac{1}{x^1 x^2}, \quad w_{15}^1 = \frac{1}{x^1 x^2}, \quad w_{23}^2 = \frac{1}{x^3 x^4}, \\
& w_{24}^2 = -\frac{1}{x^3 x^4}, \quad w_{13}^3 = -\frac{1}{x^1 x^2} - \frac{1}{x^3 x^4}, \\
& w_{15}^3 = \frac{1}{x^1 x^2}, \quad w_{16}^3 = -\frac{1}{x^1 x^2}, \quad w_{23}^3 = -\frac{1}{x^3 x^4} + \frac{1}{x^5 x^6}, \\
& w_{24}^3 = -\frac{1}{x^3 x^4} + \frac{1}{x^5 x^6}, \quad w_{24}^4 = \frac{1}{x^3 x^4} - \frac{1}{x^5 x^6}, \quad w_{26}^5 = \frac{1}{x^7 x^8}, \\
& w_{34}^3 = -\frac{1}{x^5 x^6}, \quad w_{35}^3 = -\frac{1}{x^5 x^6}, \quad w_{36}^3 = -\frac{1}{x^5 x^6}, \\
& w_{46}^3 = \frac{1}{x^5 x^6}, \quad w_{36}^4 = \frac{1}{x^5 x^6}, \quad w_{46}^4 = \frac{1}{x^5 x^6}, \\
& w_{46}^5 = \frac{1}{x^3 x^4}, \quad w_{46}^6 = -\frac{1}{x^3 x^4} - \frac{1}{x^5 x^6}, \\
& w_{15}^5 = \frac{1}{x^1 x^2}, \quad w_{15}^6 = -\frac{1}{x^5 x^6}, \quad w_{16}^6 = -\frac{1}{x^5 x^6}, \\
& w_{24}^5 = \frac{1}{x^5 x^6}, \quad w_{24}^6 = -\frac{1}{x^5 x^6}, \quad w_{26}^5 = -\frac{1}{x^5 x^6}, \\
& w_{34}^5 = -\frac{1}{x^5 x^6}, \quad w_{35}^5 = \frac{1}{x^5 x^6}, \quad w_{36}^5 = -\frac{1}{x^5 x^6}, \\
& w_{46}^5 = \frac{1}{x^3 x^4}, \quad w_{46}^6 = -\frac{1}{x^3 x^4}, \quad w_{56}^5 = -\frac{1}{x^3 x^4}.
\end{align*}
\]

The rank of the Jacobi matrix associated to the reparameterization is $3$.

3 The Ricci rotation coefficients of GDVRNS

The Riemannian metric $g_{ij}$ determines the Levi-Civita connection. The associated components of $\Gamma^j_{ik}$ on the considered system of orthonormal congruences, are the Ricci rotation coefficients of these congruences ([12], p. 267):

\[
\gamma_{bc}^a = \frac{1}{2} (w_{bc}^a + w_{ca}^b + w_{ab}^c),
\]

where

\[
w_{bc}^a = \left(\frac{\partial \lambda^a_i}{\partial x^j} - \frac{\partial \lambda^b_i}{\partial x^j}\right) \mu^a_b \rho^\mu_c
\]

are the components of the bilinear covariants.

The Ricci rotation coefficients verify the relation

\[
\gamma_{bc}^a - \gamma_{cb}^a - \gamma_{bc}^a = 0
\]

which means that the Levi-Civita connection closes the infinitesimal parallellograms. More than that,

\[
\gamma_{bc}^a + \gamma_{ac}^b = 0 \Rightarrow \gamma_{ac}^a = 0, \quad \gamma_{bc}^a = -\gamma_{ac}^b, \quad (a \neq b),
\]

i.e., the Ricci rotation coefficients are skew-symmetric in the indexes $a$ and $b$. Consequently the maximum dimension of the subspace (essential parameters)

\[
\Gamma = \{ \gamma_{bc}^a \in R^{216} \mid a, b, c = 1, \ldots, 6; a < b\}
\]

is $90$.

Le us find a meaning for the Ricci rotation coefficients on the manifold (GDVRNS) $(D \subset R^6, \mu_{ij}, \omega^a)$.

Theorem 3  (i) The components of the Ricci rotation coefficients are thermo-chemical potentials determined by the energies $T, S, PV$ and $N\mu$.

(ii) The hypersurface $\gamma_{bc}^a = \gamma_{bc}^a(x), a, b, c = 1, \ldots, 6; a < b$, of $\Gamma$ can be reparameterized as a hyper-plane of dimension $3$.

Proof. Using Maple Facilities (see, "with(DifferentialGeometry): with(Tensor)"), we compute the essential Ricci rotation coefficients:

\[
\begin{align*}
\gamma_{31}^1 &= \frac{1}{x^1 x^2}, \quad \gamma_{33}^1 = \frac{x^1 x^2 + x^5 x^6}{x^1 x^2 x^5 x^6}, \quad \gamma_{34}^1 = \frac{x^1 x^2}{2 x^1 x^2 x^5 x^6}, \\
\gamma_{12}^3 &= \frac{1}{x^1 x^2}, \quad \gamma_{33}^3 = \frac{x^1 x^2 + x^5 x^6}{x^1 x^2 x^5 x^6}, \quad \gamma_{34}^3 = \frac{x^1 x^2}{2 x^1 x^2 x^5 x^6}.
\end{align*}
\]
\[ \gamma_{35}^1 = \frac{1}{2x^5x_0}, \quad \gamma_{36}^1 = - \frac{1}{2x^4x_0}, \quad \gamma_{43}^1 = x^1x^2 + x^5x_0, \]
\[ \gamma_{45}^1 = - \frac{1}{2x^4x^2}, \quad \gamma_{46}^1 = \frac{1}{2x^4x_0}, \quad \gamma_{51}^1 = - \frac{1}{2x^4x_0}, \]
\[ \gamma_{53}^1 = \frac{1}{2x^4x_0}, \quad \gamma_{64}^1 = - \frac{1}{2x^4x^2}, \quad \gamma_{65}^1 = - \frac{1}{x^4x^2}, \]
\[ \gamma_{66}^1 = \frac{1}{x^3x^2}, \quad \gamma_{67}^1 = - \frac{1}{x^3x_0}, \quad \gamma_{73}^1 = - \frac{1}{2x^3x_0}, \]
\[ \gamma_{75}^1 = 1/(x^3x^2), \quad \gamma_{76}^1 = 1/(x^3x_0), \quad \gamma_{77}^1 = 1/x^3x_0, \]

The parametric equations of Ricci rotation coefficients depend linearly on three parameters (for connections whose components are polynomials, see also [3] and [8])

\[ u = \frac{1}{x^3x^2}, \quad v = \frac{1}{x^3x_0}, \quad w = \frac{1}{x^3x_0}, \]

The rank of the Jacobi matrix associated to the reparameterization is 3.

4 The geodesics of GDVRNS

Generally, it is known that on a Riemannian manifold, the autoparallel curves of the Levi-Civita connection are geodesics, and conversely. Taking s as the arc of autoparallel curves \( s = \sqrt{c^2 + t_0} \), c being a positive constant, the equations of geodesics can be written as follows ([12], p. 274):

\[ \frac{dx^i}{ds} = \mu^i_a u^a, \quad \frac{du^a}{ds} = \gamma^a_b u^b u^c, \]

where \( u^a = \frac{ds}{ds^a} \) are the sinusines of the angles which the tangent to the curve makes with the five orthogonal congruences, and \( \gamma^i_a \) are the previous Ricci coefficients of rotations (signomials of variables \( \frac{1}{x^3x^2}, \frac{1}{x^3x_0}, \frac{1}{x^3x_0} \)).

The tangent vectors to geodesics are rational functions. The equation of geodesics can be written explicitly via “with(DifferentialGeometry); with(Tensor);”

5 The Ricci coefficients with four indexes of GDVRNS

The components of the curvature tensor field on the previous orthonormal congruences are the coefficients of Ricci with four indexes,

\[ \gamma^a_{bcd} = \frac{\partial \gamma^a_{bc}}{\partial s^d} - \frac{\partial \gamma^a_{bd}}{\partial s^c} + \gamma^a_{bc} \gamma^b_{cd} - \gamma^a_{bd} \gamma^b_{cd}, \quad (6) \]

where \( \frac{\partial}{\partial s^a} = \mu^i_a \frac{\partial}{\partial x^i} \), and \( \gamma^a_{bcd} \) are the Ricci rotation coefficients (5). The Ricci coefficients with four indexes satisfy identically

\[ \gamma^a_{bcd} + \gamma^a_{bdc} = 0, \quad \gamma^a_{bcd} + \gamma^a_{bde} = 0, \]

and consequently \( \gamma^a_{bdc} = \gamma^a_{bde} \).

Consequently the maximum dimension of the subspace (essential parameters)

\[ \text{Riem} = \{ \gamma^a_{bcd} \in \mathbb{R}^{1296} \mid c, d = 1, \ldots, 6 \} \]

is 105.

Le us find a meaning for the Ricci rotation coefficients with four indexes on the manifold (GDVRNS) \( D \subset \mathbb{R}^6, g_{ij}, \omega^a \).
Theorem 4 (i) The components of the Ricci coefficients with four indexes are thermo-chemical potentials determined by the energies $TS, PV, N\mu$ and their product $TSPV, TSN\mu$ respectively $PV\mu$.

(ii) The hypersurface $\gamma^a_{bcd} = \gamma^a_{bcd}(x)$, for $a, b, c = 1, ..., 6$; $a < b$, of Riem can be reparameterized as a hypersurface of dimension 3.

Proof. Using Maple Facilities (see, "with(DifferentialGeometry); with(Tensor);")

(i) We renounce to the list of components, being too long.

(ii) The parametric equations $\gamma^a_{bcd} = \gamma^a_{bcd}(x)$ depend quadratically on three parameters

$$u = \frac{1}{x^1 x^2}, \quad v = \frac{1}{x^3 x^4}, \quad w = \frac{1}{x^5 x^6}.$$ 

The rank of the Jacob matrix associated to the reparameterization is 3. Consequently, the submanifold of Ricci coefficients with four indexes in $H$ is a Steiner hypersurface of dimension 3 (for Steiner manifolds, see [7]).

Remark 5 This is an important example of Ricci coefficients with four indexes which are square functions in three parameters $u = \frac{1}{x^1 x^2}, \quad v = \frac{1}{x^3 x^4}, \quad w = \frac{1}{x^5 x^6}$ (see [3], [8]).

6 The Ricci tensor and scalar curvature of GDVRNS

The Ricci tensor is ([12], p. 294) is the contraction $\gamma_{bd} = \gamma^a_{bcd}$. Consequently the maximum dimension of the subspace (essential parameters)

$$\text{Ric} = \{ \gamma_{bd} \in \mathbb{R}^{36} | b, d = 1, ..., 6; b \leq d \}$$

is 21.

Le us find a meaning for the Ricci tensor components on the manifold (GDVRNS) $(D \subset \mathbb{R}^6, g_{ij}, \omega^a)$.

Theorem 6 (i) The components of the Ricci tensor are thermo-chemical potentials determined by the energies $TS, PV, N\mu$ and their products $TSPV, TSN\mu$ respectively $PV\mu$.

(ii) The hypersurface $\gamma^a_{bd} = \gamma^a_{bd}(x)$, for $a, b, c = 1, ..., 6$; $a < b$, of Ric can be reparameterized as a hypersurface of dimension 3.

Proof Using the signomials $\gamma^a_{bcd}$, and the Maple Facilities (see, "with(DifferentialGeometry); with(Tensor);"), we compute the essential components of Ricci tensor.

$$\gamma_{11} = \frac{1}{x^1 x^2 x^3} - \frac{3}{x^1 x^2 x^4} - \frac{4}{x^1 x^2 x^5} - \frac{4}{x^1 x^2 x^6},$$

$$\gamma_{12} = -\frac{2}{x^1 x^2 x^3} + \frac{4}{x^1 x^2 x^4} - \frac{2}{x^1 x^2 x^5} + \frac{2}{x^1 x^2 x^6},$$

$$\gamma_{13} = \frac{1}{x^1 x^2 x^3 x^4} - \frac{1}{x^1 x^2 x^4 x^5} - \frac{1}{x^1 x^2 x^5 x^6},$$

$$\gamma_{14} = \frac{1}{x^1 x^2 x^3 x^5} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6},$$

$$\gamma_{15} = \frac{1}{x^1 x^2 x^3 x^5} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6},$$

$$\gamma_{16} = \frac{1}{x^1 x^2 x^3 x^5} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6},$$

$$\gamma_{22} = \frac{1}{x^1 x^2 x^3 x^4} + \frac{3}{x^1 x^2 x^4 x^5} - \frac{4}{x^1 x^2 x^5 x^6} + \frac{4}{x^1 x^2 x^5 x^6},$$

$$\gamma_{23} = \frac{1}{x^1 x^2 x^3 x^4} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{2}{x^1 x^2 x^5 x^6},$$

$$\gamma_{24} = \frac{1}{x^1 x^2 x^3 x^4} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{5}{x^1 x^2 x^5 x^6},$$

$$\gamma_{25} = \frac{1}{x^1 x^2 x^3 x^4} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{5}{x^1 x^2 x^5 x^6},$$

$$\gamma_{26} = \frac{1}{x^1 x^2 x^3 x^4} - \frac{1}{x^1 x^2 x^4 x^6} + \frac{5}{x^1 x^2 x^5 x^6},$$

$$\gamma_{33} = \frac{1}{x^1 x^2 x^3 x^4} - \frac{2}{x^1 x^2 x^4 x^5} + \frac{2}{x^1 x^2 x^5 x^6},$$

$$\gamma_{34} = \frac{1}{x^1 x^2 x^3 x^4} + \frac{2}{x^1 x^2 x^4 x^5} + \frac{4}{x^1 x^2 x^5 x^6},$$

$$\gamma_{35} = \frac{1}{x^1 x^2 x^3 x^4} + \frac{1}{x^1 x^2 x^4 x^5} + \frac{1}{x^1 x^2 x^5 x^6},$$

$$\gamma_{36} = \frac{1}{x^1 x^2 x^3 x^4} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{5}{x^1 x^2 x^5 x^6},$$

$$\gamma_{44} = \frac{2}{x^1 x^2 x^3 x^5} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{1}{x^1 x^2 x^5 x^6},$$

$$\gamma_{45} = \frac{2}{x^1 x^2 x^3 x^5} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{1}{x^1 x^2 x^5 x^6},$$

$$\gamma_{46} = \frac{2}{x^1 x^2 x^3 x^5} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{1}{x^1 x^2 x^5 x^6},$$

$$\gamma_{55} = \frac{1}{x^1 x^2 x^3 x^5} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{3}{x^1 x^2 x^5 x^6},$$

$$\gamma_{56} = \frac{1}{x^1 x^2 x^3 x^5} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{3}{x^1 x^2 x^5 x^6},$$

$$\gamma_{66} = \frac{1}{x^1 x^2 x^3 x^5} + \frac{1}{x^1 x^2 x^4 x^6} + \frac{3}{x^1 x^2 x^5 x^6}.$$

(ii) This is an important example of Ricci tensor coefficients which are square functions in three parameters $u = \frac{1}{x^1 x^2}, \quad v = \frac{1}{x^3 x^4}, \quad w = \frac{1}{x^5 x^6}$ (see [3], [8]).

The scalar curvature $c = \delta^{bd} \gamma_{bd}$ is given by

$$c = -\frac{1}{x^1 x^2 x^3 x^4 x^5 x^6} + 4x^1 x^2 x^3 x^4 x^5 x^6 - 2x^1 x^2 x^3 x^4 x^5 x^6 + 11x^1 x^2 x^3 x^4 x^5 x^6 + 4x^1 x^2 x^3 x^4 x^5 x^6 + 2x^1 x^2 x^3 x^4 x^5 x^6 - 2x^1 x^2 x^3 x^4 x^5 x^6.$$
This is an important example of scalar curvature which is a square function in three parameters $u = \frac{1}{x_1^2 + x_2^2}$, $v = \frac{1}{x_3^2 + x_4^2}$, $w = \frac{1}{x_5^2 + x_6^2}$ (see [3], [8]).

From thermodynamical point of view, it can be observed that the scalar curvature depends only the energies $TS$, $PV$, $N\mu$ and their product $TSPV$, $TSN\mu$ respectively $PVN\mu$. On the other hand, the scalar curvature $c = -\frac{4}{P^2V^2} + \frac{2}{PVN\mu} - \frac{17}{N^2\mu^2} - \frac{4}{T^2S^2} - \frac{2}{TSN\mu} + \frac{2}{TSPV}$ reflects the saddle shape (behavior) in $R^6 = \{u = \frac{1}{T^3}, v = \frac{1}{P^3}, \mu = \frac{1}{N^3}, c\}$ via the submersion $\pi : R^6 \rightarrow R^3$, $\pi(T, S, P, V, N, \mu) = (u, v, w), \ u = \frac{1}{T^3}, \ v = \frac{1}{P^3}, \ w = \frac{1}{N^3}$.

7 Conclusions

This paper contains the application, to the Gibbs-Duhem-Pfaff differential 1-form, of the congruence theory of Vranceanu, combined with the Udrişte’s theory concerning even-dimensional nonholonomic equations. The physical meanings of the used congruences are underlined and a Riemannian metric is built. The proofs that the coefficients of bilinear covariants and the coefficients of Ricci (with three and four indexes) are signomials and the tangent vectors to geodesics are rational functions are also given.

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