Extreme Value Theory and Bootstrap Approach for Robust Value at Risk Estimation under Basel II

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Abstract: - Banking legislature allows banks to use internal models to estimate risk metrics such as Value at Risk (VaR). These metrics indirectly determine regulatory capital needed to be held for different kinds of risks. The article aims to make a comparison of classic methods (historical simulation and bootstrap approach) allowed in Basel II framework to Extreme Value Theory. Influence of selected approach to resulting VaR is stressed. The problem is demonstrated on positively skewed data with extreme values that reflect nature of many financial processes. Our findings suggest alternative methods assess higher estimates of VaR than the accepted one and are more accurate in terms of estimator variance.

Key-Words: Value at risk, bootstrap sampling, extreme value theory, extremes identification, economic capital, regulatory capital

1. Introduction

Banks and insurance institutions hold decisive position in financial system as well as in overall macroeconomic environment. The industry faces both internal (operational) and external (market and credit) risks. To make the systems stable and safe, banks have to respect strict regulations.

Almost 24-year development of banking legislative under the Bank for International Settlement (starting with Basel I) has evolved into general framework grounded in a three-pillar system – capital framework; risk management and supervision pillar and market discipline pillar. Because of the nature of such highly stochastic environment, contemporary regulation philosophy is closely tied to advanced mathematical and statistical procedures. One of the most widely used (and criticized) approach used in required capital determination within Basel II [4] legislative is the Value at Risk (VaR) method.

Critique of the VaR methodology is twofold. The first argues the methodological properties itself and following misleading interpretation. The second highlights technical problems such as parameter estimation and distribution approximation. This paper focuses mainly on the VaR methodology from the latter perspective. We suggest Extreme Value Theory concept to be more appropriate method for modelling skewed and non-normal tailed distribution of losses in a sense of safety (overestimating risk rather than underestimating) than methods allowed under Basel II.

First section introduces regulatory framework and economic capital and its connection to VaR methodology. A description of historical simulation, Monte Carlo simulation and EVT approach follows. Finally, comparable results are presented.

2. Methods

2.1. Banking regulation

It can be distinguished between two kinds of capital. Regulatory capital is the minimal amount of risk capital to be held to meet regulatory rules and application guidance. Economic capital is a result of shareholder’s trade-off between solvency and profitability. Although both of them measure very similar-meaning variable, final figures very often differs. According to Saita [14], performance indicators such a RAROC are usually based on economic capital while regulatory is omitted.

Several approaches for economic capital stipulation were introduced (i.e., CreditMetrics or Moody's KMV). Basel II motivates bank’s management to measure different sources of risks by implementing these internal models to be more accurate in the risk-evaluation processes. If banks use internal-rating-based approach these metrics are tightened with regulatory capital. Own risk measures are
transformed into risk weights specified by the Basel Committee. [2]

In 1999, Basel Committee for Banking Supervision released the New Basel Capital accord. According to this legislature (also known as Basel II, [4]) VaR value can be used (after certain requirements are met) to determine capital adequacy matching to unexpected credit losses. Exposures to credit risk \( E^* \) are then calculated as

\[
E^* = \max \left\{ 0, \left[ \sum E_i - \sum C_i \right] + \text{VaR} \right\},
\]

where \( E_i \) is current value of exposure \( i \) and \( C_i \) the received collateral value.

2.2. Value at Risk

VaR represents overall portfolio’s absolute risk measure, generally defined as (when the losses are with positive sign)

\[
\alpha = P(x \leq \text{VaR}) = \int_{\text{VaR}}^\infty f(x) \, dx = F(\text{VaR})
\]

where \( f(x) \) is the probability density function of a variable (i.e., losses over some time period from portfolio’s value changes) and \( F(x) \) its corresponding cumulative distributive function. Value higher than \( x \) will occur only with probability \( \alpha \).

Although numerous articles were written on VaR (for a historical review see [7]), some topics remain uncovered. In probability theory, two fundamental approaches coexist each of which tackles the problem in a different way. While the majority prefers the classical frequentist approach, the Bayesian one is more promising in incorporating genuine expert information [5] and handling non-linear systems [15]

Crucial point in the VaR estimation process lies in identifying appropriate distribution which generates the data. To create it, several approaches were designed. Starting from the simplest historical

\[1 \text{ Credit risk is considered for demonstration purposes only, although Basel (2006) allows using VaR for market and operational risks, as well.} \]

simulation form through mean-variance to simulation models, all of them suffer from the main idea of VaR – estimating overall distribution which may underestimate the real risk hidden in the tail(s).

2.3. Standard approaches

Under Basel II three approaches to VaR estimation for regulatory capital purposes are allowed:

1. Variance–Covariance methods

Based on the portfolio volatility, descriptive statistics (such a central moments) are derived to be used as parameters in parametric probability model. The simplest model refer to normal distribution but more proficient methods were introduced to handle non-normal distributions as well (variance is not even needed to be computed). Although many computationally extensive methods (kernel estimators) were introduced and are available in statistical packages such R, SPSS, Matlab, we restrict our paper to mathematical estimator using Cornish-Fisher Expansion (CF). This expansion starts with assumption about underlying distribution \( z \approx N(0,1) \) but ends with transformed non-normal \( z_c \) distribution

\[
z_c = z + \frac{2}{6} \frac{z^2 - 1}{S} + \frac{2}{24} \frac{z^3 - 3z}{K} - \frac{3z^2 - 5z}{36}
\]

\( S \) represents skewness and \( K \) kurtosis of variable \( z \).

2. Historical simulation

Historical distribution of returns updates new data by previous patterns in changes. This approach presumes existence of some underlying repeating structure such as time-invariant probabilistic model. Only the corresponding quantile is computed. This procedure can be used under Basel II only if data sample is sufficiently large and proved over specified time horizon.

3. Monte Carlo simulation

Monte Carlo procedures (including bootstrap) are sampling procedures which draw random samples from the initial sample to estimate value of predefined quantity. We propose bootstrap method to be suitable for deriving VaR because of its properties (sampling with replacement, computationally efficient, no strict assumptions). After bootstrapped values are known, graphical analysis follows, providing useful information about
quantity variation and sensitivity with respect to initial settings.

2.4. Extreme Value Theory

Observed data and their approximation by any classical distribution may underestimate the risk. Although the largest part of a distribution can fit perfectly, tail values may cause real damage. Financial data tend to have “fat” tails and thus need to be treated carefully. Extreme Value Theory deals with this extreme part of distribution as with another property of the phenomena being investigated.

If any observation in a dataset would be considered as a sample from its random variable, Fisher-Tippet theorem [1] can be used to estimate underlying distribution. This theorem states that sequence of $iid^2$ random variables and their extreme values can converge only to one of three distributions – Fréchet, Weibull and Gumbel. According to Beirlant [1] a general limit distribution combining the previously mentioned exists.

Two principal models are recognized from the point of determining extreme values. The simpler one, Block maxima, estimates the extremes in fixed time period or within logically justified blocks of variables. Limit distribution here is Generalised Extreme Value distribution (GEV) written as follows:

$$GEV_{\xi,\mu,\sigma} = \begin{cases} e^{-\left(1+\frac{\xi x-\mu}{\sigma}\right)^{\frac{1}{\xi}}} & \text{if } \xi \neq 0 \\ e^{-\frac{x-\mu}{\sigma}} & \text{if } \xi = 0 \end{cases}$$

(4)

where $\xi$ is a tail index, $\mu$ location and $\sigma$ scale parameter. Larger $\xi$ produces fatter distribution. When $\xi > 0$, distribution becomes Fréchet. With $\xi = 0$, GEV becomes Gumbell and $\xi < 0$ produces Weibull distribution. [9,11]

Computationally-intensive peak over threshold (POT) method is the second approach. Extreme values are those exceeding a threshold value. Limit distribution used in POT method is Generalised Pareto Distribution (GPD).

$$GPD_{\xi}(x) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-1/\xi} & \xi \neq 0 \\ 1 - e^{\left(-x/\beta\right)} & \xi = 0 \end{cases}$$

(5)

When $\xi > 0$, GPD becomes ordinary Pareto distribution, also known as distribution of large losses in actuarial statistics. In case of $\xi = 0$, GPD is identical to exponential distribution while with $\xi < 0$, GPD is known as Pareto type II distribution. [9,11]

Threshold value can be estimated by informed guess but another two mathematical approaches were developed: first one is based on semiparametric statistics named Hill estimator. [6] The second one is Mean Excess Function $M(t)$ which computes mean value of values exceeding threshold.

$$M(t) = E\left(X - t \mid X > u\right)$$

(6)

Within the normal part of distribution and moving to extremes (to the larger numbers) mean value should rise steadily because extreme values does not play such an important part. Fluctuating of M function suggest extreme part break.

3. Results

Assume a portfolio consisting of $X$ entities. Histogram (Fig.1) provides information about distribution of losses (with positive sign) in the last 200 days and estimated probability density function.

3.1. Bootstrapped estimate

To set Value at Risk according to the standard methods, we proceed with estimating density function of the gathered data through rescaled histogram. This allows to compute historical VaR (HS). After the shape of distribution is estimated using Cornish-Fisher, kernel estimation component risk contribution with no weighting preferences (authors Epperlein and Smillie, implementation in [12]) is provided as well.

<table>
<thead>
<tr>
<th>Historical simulation</th>
<th>Cornish-Fisher</th>
<th>Normal distribution</th>
<th>Kernel estimator</th>
</tr>
</thead>
</table>

Table 1 – Classic methods (Own processing, [12])
Using normal distribution for estimating VaR at 5% level showed the lowest value. Difference between HS, kernel and CF estimates is only 0.3 in favour of HS (from the point of conservativeness).

<table>
<thead>
<tr>
<th></th>
<th>approximation</th>
<th></th>
<th>approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>41.59</td>
<td>41.27</td>
<td>39.14</td>
<td>41.59</td>
</tr>
</tbody>
</table>

If the previous approach fails, sampling methods known as Monte Carlo may be used instead. In the paper, original data were replicated using bootstrapped sampling procedure of 1000 replicates. To make data replicable, appropriate distribution and parameterisation have to be set. Gamma distribution was used and parameters were estimated using maximum likelihood estimator (MLE). After the distribution is estimated, data can be generated randomly from it.

Values constituting the confidence intervals and mean values are presented at Table 1.

### Table 2 – Bootstrap confidence intervals [13]

<table>
<thead>
<tr>
<th>VaR(α)</th>
<th>0.9</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower CI</td>
<td>31.835</td>
<td>35.271</td>
<td>41.700</td>
</tr>
<tr>
<td>Mean</td>
<td>35.052</td>
<td>40.825</td>
<td>46.025</td>
</tr>
<tr>
<td>Upper CI</td>
<td>39.392</td>
<td>42.687</td>
<td>56.957</td>
</tr>
</tbody>
</table>

### 3.1. EVT estimate

If our data is periodical variable, we would choose as extreme values those which are the highest on every sub-period. Our data do not follow any similar pattern, thus we will not be concerned about Block maxima approach.

Using the statistically naïve POT method and selecting the 90% largest value as a threshold, our cut-off value would be 35.39. Graphical analysis for appropriate threshold selection can be provided through mean-excess plot. When the data’s trend remains stable, no extreme values are present. After data become more jittered, they should be considered as inconsistent with the preceding data. Fig. 3 shows more dispersion when x exceeds 40.
After the threshold is set, the remaining data can be approximated using a distribution described in equation 4. Estimate was provided using Maximum Likelihood function.

Table 3 Parameter estimates (Own processing)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\xi$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>-0.1012147</td>
<td>6.6263301</td>
</tr>
<tr>
<td>Standard error</td>
<td>0.3068897</td>
<td>2.6515820</td>
</tr>
</tbody>
</table>

Estimated value for the shape parameter $\xi$ is small relative to its standard error. Because the parameter determines type of definition used for approximation, the information should be treated carefully. QQplot below supports our choice of distribution and parameterisation.

Fig 5 – QQ plot for GPD (Own processing, 11)

In Fig. 5 the estimate is denoted as a solid line. Vertical dashed line shows .95 quantile considered at VaR at $\alpha = 0.05$. Horizontal line defines confidence interval on a level scaled on the second $y$ axis.

Fig 6 – VaR(.95) Confidence interval [16]

At the predefined level the confidence interval is asymmetric which is to be expected due to underlying distribution’s skewness.

Table 4 Parameter estimates [13]

<table>
<thead>
<tr>
<th></th>
<th>Lower CI</th>
<th>Estimate</th>
<th>Upper CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>40.59341</td>
<td>41.89757</td>
<td>44.6070</td>
</tr>
</tbody>
</table>

4. Conclusion

In 1996, Basel Committee released an amendment to the Capital Accord which allowed banks to use VaR variance-covariance method, historical and Monte Carlo simulations. EVT was neither allowed nor mentioned. [4] This paper compared VaR quantified by all previously mentioned methods with following results.

The lowest VaR estimate was provided by approximation using normal distribution (39.14). Direct quantile estimate within historical simulation and kernel procedures follows (41.58). Other estimates were accompanied by confidence intervals in which the worst case scenario (upper CI value) was considered as VaR value. Using Monte Carlo (bootstrapped) method VaR is smaller (42.687, overall interval width = 7.416) than the EVT’s (44.607, width = 4.01).

These findings suggest that officially approved methods are inappropriate (when the loss distribution has similar positive skew shape as ours) in terms of identifying potential risk value. The size of VaR consequently influences the capital
requirement which allows banks to hold less capital reserves when standard methods are used.

In May 2012 consultative document [3], fundamental review of the trading book were published by the Bank for International Settlement. This document extends impact of Basel 2.5 rules. This European legislative primarily aims at banks’ trading books and suggests to completely reject VaR in favour of ES (expected shortfall) and other risk metrics. This step can be interpreted as conceptual framework abandonment which might be partly caused by inappropriate computing methods used.

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References: