Deterministic and stochastic dynamic model with delays of the financial crises contagions

AURA LOREDANA CIUDARIU
"Politehnica" University of Timişoara
Department of Mathematics
Bvd.Vasile Pârvan 2, Timişoara
ROMANIA
loredana.ciurdariu@mat.upt.ro

MIHAELA NEAMŢU
West University of Timişoara
Faculty of Economics and Business Administration
16 J.L. Pestalozzi Str., 300115 Timisoara
Romania
mihaela.neamtu@feaa.uvt.ro

Abstract: A nonlinear dynamic model is set up to describe the international financial crises contagion. The model is given by deterministic and stochastic dynamic model with delays. In the deterministic model we set the condition for the existence of the delay parameter value for which the model displays a Hopf bifurcation. For the stochastic system, we identify the differential equation for the square mean value. The existence of the Hopf bifurcation is analyzed. The last part of this paper includes numerical simulation and conclusions.

Key–Words: Financial crises contagion, stock return rate, nonlinear dynamics model, delay, stochastic differential equations.

1 Introduction

Since 1990s, the international financial crises have frequently broken out, and their contagions also greatly have increased.

In this paper, we investigate the financial crises contagions between two financial markets by analyzing their mutual impact changes within a short time during the crises. We try to introduce the differential dynamic methods to set up a non-linear model of the financial crisis contagion between two markets. We use the ordinary differential equation with delay and stochastic differential equation with delay to describe the mutual impact state between two countries or two financial markets within a short time during the crisis, then to discuss the infections state or path.

The paper is organized as follows. Section 2 focuses on setting to nonlinear contagion model by using the differential equation with delay and stochastic differential equation with delay. Section 3 analyzes the model described by the differential equation with delay, the Hopf bifurcation is found for delay as well as the normal form. Section 3 analyzes the square mean values of the stochastic disturbed linearized variables of deterministic system. Section 4 presents the numerical simulations of the dynamics of the determinist and stochastic system. The last section summarizes our studies.

2 The determinist and stochastic model

The financial crisis contagion firstly affects the financial security of a country. It may lead to the great volatility of the financial asset prices (such as stocks, bonds, currencies, real estate, etc) deteriorating the operating conditions of the financial institutions, cause the capital flight, decline the foreign exchange reserves, and increase the foreign debt.

We constructed a nonlinear dynamic model, as follows:

$$\begin{aligned} \dot{x}_1(t) &= a - x_1(t)x_2(t-\tau_2)^2 \\ \dot{x}_2(t) &= x_2(t)(-b + x_1(t-\tau_1)x_2(t)) \end{aligned} \tag{1}$$

where a, b are positive constants, a is the increasing rate of the average stock returns of A country under the normal situation, and b is the decreasing rate of the stock returns of B country. The variables $x_1(t)$, $x_2(t)$ are respectively the stock return rates of A and B country for $t \in \mathbf{R}_+$, and $x_1(t-\tau_1),\ x_2(t-\tau_2)$, for $t-\tau_1, t-\tau_2$ respectively, where $\tau_1 > 0,\ \tau_2 > 0$, are delays. For $\tau_1 = 0,\ \tau_2 = 0$, (1) describes the model from Chen [3].

From (1), it results that the equilibrium point is given by (x_{10}, x_{20}), where

$$x_{10} = \frac{b^2}{a}, \qquad x_{20} = \frac{a}{b}. \tag{2}$$

As follows, we shall analyze the model (1) in the case $b^3 < a^2$. In Chen [3] for $\tau_1 = 0,\ \tau_2 = 0$, the cases $a^2 < b^3$ and $a^2 = b^3$ are investigated.

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Let $(\Omega,\ \mathcal{F}_0,\ \mathcal{P}),\ t \geq 0$ be a given probability space Kloeden [4], and $w(t) \in \mathbf{R}$ be a scalar Wiener process defined on Ω having independent stationary Gauss increments with $w(0) = 0,\ E(w(t)-w(s)) = 0$, and $E(w(t)w(s)) = \min(t,s)$. The symbol E denotes the mathematical expectation. The sample trajectories of $w(t)$ are continuous, nowhere differentiable, and have infinite variation on any finite time interval [4].

What we are interested in knowing is the effect of the noise perturbation on the equilibrium point (x_{10}, x_{20}). The stochastic perturbation of (1) given by the form of a stochastic differential equations with delays is:

$$\begin{aligned} dx_1(t) = &\ a - x_1(t)x_2(t-\tau_2)^2 \\ &+\sigma(x_1(t)-x_{10})dw(t) \\ dx_2(t) = &\ x_2(t)(-b + x_1(t-\tau_1)x_2(t)) \\ &+\sigma(x_2(t)-x_{20})dw(t) \end{aligned} \tag{3}$$

where $\sigma > 0$ and $w(t)$ is a scalar Wiener process and we denote by $x_1(t) = x_1(t,\omega),\ x_2(t) = x_2(t,\omega)$ the components of a stochastic process on the probability space [4], [6], [7].

3 Hopf bifurcation analysis for deterministic model (1)

The system (1) has only one positive equilibrium point (x_{10}, x_{20}), where x_{10} and x_{20} are given by (2)

By carrying out the translation $u_1(t) = x_1(t) - x_{10},\ u_2(t) = x_2(t) - x_{20}$ from (1) we got the system:

$$\begin{aligned} \dot{u}_1(t) = &\ -\frac{a^2}{b^2}u_1(t) - 2bu_2(t-\tau_2) - \frac{b^2}{a}u_2^2(t-\tau_2) \\ &-\frac{2a}{b}u_1(t)u_2(t-\tau_2) - u_1(t)u_2^2(t-\tau_2) \\ \dot{u}_2(t) = &\ \frac{a^2}{b^2}u_1(t-\tau_1) + bu_2(t) + \frac{b^2}{a}u_2^2(t) \\ &+\frac{2a}{b}u_1(t-\tau_1)u_2(t) + u_1(t-\tau_1)u_2^2(t). \end{aligned} \tag{4}$$

The linearized system of (4) in (0,0) is given by:

$$\dot{y}(t) = Ay(t) + B_1 y(t-\tau_1) + B_2 y(t-\tau_2) \tag{5}$$

where $y(t) = (y_1(t), y_2(t))^\top$ and

$$A = \begin{pmatrix} -\frac{a^2}{b^2} & 0 \\ 0 & b \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & 0 \\ \frac{a^2}{b^2} & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & -2b \\ 0 & 0 \end{pmatrix}. \tag{6}$$

The characteristic function for (5) is given by

$$h(\lambda,\tau) = \lambda^2 + p\lambda - q + 2qe^{-\lambda\tau}. \tag{7}$$

where $\tau = \tau_1 + \tau_2$, and

$$p = \frac{a^2}{b^2} - b, \qquad q = \frac{a^2}{b}. \tag{8}$$

From (6) and (8) result:

Proposition 1 *If $b^3 < a^2$, and $\tau_1 = 0$, then the equation $h(\lambda, 0) = 0$ has the roots with a negative real part. The equilibrium point (x_{10}, x_{20}) is local asymptotically stabile.*

In what follows we discuss the stability of the delay model about the equilibrium point and the Hopf bifurcation.

Regarding the existence of the Hopf bifurcation, a critical time delay τ_0 must exist such that

$H_1:\quad \lambda_{1,2}(\tau_0) = \pm i\omega_0\ (\omega_0 > 0)$, are the roots of the equation $h(\lambda,\tau_0) = 0$, and all other eigenvalue have negative real part at $\tau = \tau_0$.

$H_2:\quad Re(\frac{d\lambda_{1,2}(\tau)}{d\tau})|_{\lambda=\lambda_1} \neq 0.$

For the existence of the condition H_1, we assume that there is a pair of imaginary roots for $h(\lambda,\tau) = 0$,. Let $\lambda = i\omega\ (\omega > 0)$ be one of its roots. Substituting in $h(\lambda,\tau) = 0$ we obtain:

$$-\omega^2 + p\,\omega\, i - q + 2q(\cos(\omega\tau) - i\sin(\omega\tau)) = 0. \tag{9}$$

From (9), ω satisfies equation:

$$\omega^4 + (p^2 + 2q)\omega^2 - 3q^2 = 0. \tag{10}$$

Because $(p^2+2q)^2 + 12q^2 > 0$, and $-3q^2 < 0$, the equation (10) has a positive root.

Let ω_0 be a positive root of (10), then the critical value of delay τ is given by:

$$\tau_0 = \frac{1}{\omega_0} arctg \frac{p\omega_0}{\omega_0^2 + q^2} \tag{11}$$

Let $\lambda = \lambda(\tau)$ be a solution of the equation $h(\lambda(t),\tau) = 0$. Differentiating with respect to τ, we have

$$\frac{d\lambda(\tau)}{d\tau} = \frac{2q\lambda(\tau)e^{-\lambda(\tau)\tau}}{2\lambda(\tau) + p - 2q\tau e^{-\lambda(\tau)\tau}} \tag{12}$$

From (12), we have:

$$\frac{d\lambda(\tau)}{d\tau}|_{\lambda=i\omega_0,\tau=\tau_0} = \frac{A_1 + iA_2}{B_1 + iB_2} \tag{13}$$

where

$$\begin{aligned} &A_1 = 2q\omega_0\sin(\omega_0\tau_0), A_2 = 2q\omega_0\cos(\omega_0\tau_0) \\ &B_1 = p - 2qr_0\cos(\omega_0\tau_0), B_2 = 2\omega_0 + 2q\tau_0\sin(\omega_0\tau_0) \end{aligned} \tag{14}$$

We denote by

$$\begin{array}{ll} M & = Re(\frac{d\lambda(\tau)}{d\tau}|_{\lambda=i\omega_0,\tau=\tau_0}) = \frac{A_1B_1+A_2B_2}{B_1^2+B_2^2} \\ N & = Im(\frac{d\lambda(\tau)}{d\tau}|_{\lambda=i\omega_0,\tau=\tau_0}) = \frac{A_2B_1-A_1B_2}{B_1^2+B_2^2} \end{array} \quad (15)$$

From (15) and (14) we get:

$$\begin{array}{l} M = \dfrac{\omega_0^2(p^2+4q+2\omega_0^2)}{p^2+4\omega_0^2+4q^2\tau_0^2+\tau_0 p(2\omega_0^2-q)} \\ N = \dfrac{p\omega_0(\omega_0^2+q)-2\omega_0^3 p-4q^2\omega_0\tau_0}{p^2+4\omega_0^2+4q^2\tau_0^2+\tau_0 p(2\omega_0^2-q)} \end{array} \quad (16)$$

From (16) $M \neq 0$. Then the Hopf bifurcation exists for the system (1).

Let τ_0 be given by (11) $\tau_2 \in (0, \frac{\tau_0}{2})$, and $\tau_1^0 = \tau_0 - \tau_2$. In the study of the Hopf bifurcation problem, first we transform the system (4) with $\tau_1 = \tau_1^0 + \mu$, where $\mu \in \mathbf{R}$, we regard μ as the bifurcation parameters.

For $\phi \in C([-\tau_2, 0], \mathbf{C}^2)$ we define a linear operator

$$L_\mu\phi = A\phi(0) + B_1\phi(-\tau_1) + B_2\phi(-\tau_2) \quad (17)$$

where $A,\ B_1,\ B_2$ are given by (6) and,

$$F(\mu,\phi) = \begin{pmatrix} U \\ V \end{pmatrix} \quad (18)$$

where

$$\begin{array}{l} U = -\dfrac{b^2}{a}\phi_2^2(-\tau_2) - \dfrac{2a}{b}\phi_1(0)\phi_2(-\tau_2) - \phi_1(0)\phi_2^2(-\tau_2) \\ V = \dfrac{b^2}{a}\phi_2^2(0) + \dfrac{2a}{b}\phi_1(-\tau_1^0)\phi_2(0) + \phi_1(-\tau_1^0)\phi_2^2(0) \end{array}$$

By the Riesz representation theorem, there is a matrix whose components are bounded variations functions, $\eta(\theta,\mu)$ with $\theta \in [-\tau_1^0, 0]$, such that

$$L_\mu\phi = \int_{-\tau_1^0}^0 d\eta(\theta,\mu)\phi(\theta), \theta \in [-\tau_1^0, 0]. \quad (19)$$

We can choose

$$\eta(\theta,\mu) = \begin{cases} A, & \theta = 0 \\ B_1\delta(\theta+\tau_1^0), & \theta \in [-\tau_1^0, -\tau_2) \\ B_2\delta(\theta+\tau_2), & \theta \in [-\tau_2, 0). \end{cases}$$

For $\phi \in C^1([-\tau_1^0, 0], \mathbf{R}^2)$, we define

$$\mathcal{A}(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau_1^0, 0) \\ \int_{-\tau_1^0}^0 d\eta(t,\mu)\phi(t), & \theta = 0 \end{cases} \quad (20)$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta =\in [-\tau_1^0, 0) \\ F(\mu,\phi), & \theta = 0. \end{cases} \quad (21)$$

Then, we can rewrite (4) in the following vector form:

$$\dot{u_t} = \mathcal{A}(\mu)u_t + \mathcal{R}(\mu)u_t \quad (22)$$

where $u = (u_1, u_2)^\top,\ u_t = u(t+\theta),\ \theta \in [-\tau_1^0, 0]$.

For $\psi \in C^1([0, \tau_1^0], \mathbf{R}^2)$ the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is defined as

$$\mathcal{A}^*\psi(s) = \begin{cases} \frac{d\phi(s)}{ds}, & s \in (0, \tau_1^0) \\ \int_{-\tau_1^0}^0 d\eta^\top(t,0)\psi(-t), & s = 0. \end{cases} \quad (23)$$

For $\phi \in C([-\tau_1^0, 0], \mathbf{C}^2)$ and $\psi \in C([0, \tau_1^0], \mathbf{C}^2)$ define the bilinear form:

$$\begin{array}{rl} <\psi,\phi>= & \bar{\psi}^\top(0)\phi(0) \\ & -\int_{-\tau_1^0}^0 \int_{\xi=0}^\theta \bar{\psi}^\top(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \end{array} \quad (24)$$

where $\eta(\theta) = \eta(\theta, 0)$.

Considering $\lambda_1 = i\omega_0, \quad \lambda_2 = -i\omega_0$, where ω_0 is a positive root of (10), from (20), the eigenvalue of $\mathcal{A}(0)$ corresponding to λ_1 is given by $v = (v_1, v_2)$, where

$$v_1 = \lambda_1 - b, \quad v_2 = \frac{a^2}{b^2}e^{-\lambda_1\tau_1^0} \quad (25)$$

and

$$v(\theta) = ve^{\lambda_1\theta}, \quad \theta \in [-\tau_1^0, 0] \quad (26)$$

is the eigenvector of $\mathcal{A}$.

From (23), we obtain that the eigenvector of $\mathcal{A}^*(0)$ corresponding to λ_2 is given by $v^* = (v_1^*, v_2^*)$, where

$$\begin{array}{rl} v_1^* = & \dfrac{((\lambda_2+\frac{a^2}{b^2})\tau_1^0 e^{\lambda_1\tau_1^0} + \frac{a^2}{b^2}e^{\lambda_2\tau_1^0})\bar{v_1}}{\frac{a^2}{b^2}e^{\lambda_2\tau_1^0}} \\ & + \dfrac{(\lambda_2+\frac{a^2}{b^2} - 2\frac{\varsigma a^2}{b}e^{\lambda_2\tau_1^0+\lambda_1\tau_2})\bar{v_2}}{\frac{a^2}{b^2}e^{\lambda_2\tau_1^0}} \\ v_2^* = & \dfrac{1 - w_1(\bar{v}_1 - 2b\bar{v}_2\tau_2 e^{\lambda_1\tau_2})}{\bar{v}_2 + \tau_1^0\bar{v}_1 e^{\lambda_1\tau_1^0}} \end{array} \quad (27)$$

and

$$v^*(s) = v^*e^{\lambda_1 s}, \quad s \in [0, \infty). \quad (28)$$

We can verify that

$$<v^*, v>= 1, \quad <v^*, \bar{v}>= 0.$$

Using the approach of Hassard et al [2] we next compute the coordinates to describe the center manifold Ω_0 at $\mu = 0$. Let $\mu_t = u(t+\theta),\ \theta \in [-\tau_1^0, 0)$, be the solution of (4), when $\mu = 0$. We define

$$z(t) =< v^*, u_t >, w(t,\theta) = u_t(\theta) - 2Re\{z(t)v(\theta)\}. \quad (29)$$

On the center manifold Ω_0, we have

$$w(t,\theta) = w(z(t), \bar{z}(t), \theta) \quad (30)$$

where

$$w(z,\bar{z},\theta)=w_{20}(\theta)\frac{z^2}{2}+w_{11}(\theta)z\bar{z}+w_{02}(\theta)\frac{\bar{z}^2}{2}+... \quad (31)$$

and $z, \bar{z}$ are the local coordinates of the center manifold Ω_0 in the direction of v^* and $\bar{v}^*$, respectively. Notice that for $\mu = 0$ and for the solution $u_t \in \Omega_0$ of (4), we have

$$\dot{z}(t) = \lambda_1 z(t) + g(z(t), \bar{z}(t)) \quad (32)$$

with

$$g(z, \bar{z}) = v^{*\top} \mathcal{R}(w(z, \bar{z}, \theta) + 2Re(z_1 v(\theta))). \quad (33)$$

We expand the function $g(z, \bar{z})$ on the center manifold Ω_0 in the powers of z and $\bar{z}$

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2}. \quad (34)$$

From (33), (32), and (34) using the method of [2] we get:

Proposition 2 *For the system (4), we have:*

$$g_{20} = \bar{v}_1^* F_{120} + \bar{v}_2^* F_{220}, g_{11} = \bar{v}_1^* F_{111} + \bar{v}_2^* F_{211},$$
$$g_{02} = \bar{v}_1^* F_{102} + \bar{v}_2^* F_{202}, g_{21} = \bar{v}_1^* F_{121} + \bar{v}_2^* F_{221}. \quad (35)$$

Where $\bar{v}_1^*$, $\bar{v}_2^*$ are given by (27) and

$$\begin{aligned}
&F_{120} = -\tfrac{2b^2}{a}v_2^2 e^{2\lambda_2\tau_2} - \tfrac{4a}{b}v_1v_2e^{\lambda_2\tau_2},\\
&F_{111} = -\tfrac{b^2}{a}v_2\bar{v}_2 - \tfrac{2a}{b}(v_1\bar{v}_2e^{\lambda_2\tau_2} + \bar{v}_1v_2e^{\lambda_2\tau_2})\\
&F_{102} = \bar{F}_{120}\\
&F_{121} = -\tfrac{2b^2}{a}(2v_2w_{211}(-\tau_2)e^{\lambda_2\tau_0} + \bar{v}_2w_{220}(-\tau_2)e^{\lambda_1\tau_2}) -\\
&-\tfrac{4a}{b}(v_1w_{211}(-\tau_2) + \bar{v}_2v_2w_{111}(0)e^{\lambda_2\tau_2}).\\
&F_{220} = \tfrac{2b^2}{a}v_2^2 + \tfrac{4a}{b}v_1v_2e^{\lambda_2\tau_1^0},\\
&F_{211} = \tfrac{b^2}{a}v_2\bar{v}_2 + \tfrac{2a}{b}(v_1\bar{v}_2e^{\lambda_2\tau_1^0} + \bar{v}_1v_2e^{\lambda_1\tau_1^0})\\
&F_{202} = \bar{F}_{220},\\
&F_{221} = \tfrac{2b^2}{a}(2v_2w_{211}(0) + \bar{v}_2w_{220}(0)) + \tfrac{4a}{b}(v_1w_{211}(0)e^{\lambda_2\tau_1^0}\\
&+\tfrac{1}{2}\bar{v}_1w_{220}(0)e^{\lambda_1\tau_1^0} + v_2w_{111}(-\tau_1) + \tfrac{1}{2}w_{120}(-\tau_1)\bar{v}_2)\\
&+4v_1v_2\bar{v}_2e^{\lambda_2\tau_1^0} + 2v_2^2\bar{v}_1e^{\lambda_1\tau_1^0},
\end{aligned} \quad (36)$$

$$\begin{aligned}
w_{20}(\theta) &= (w_{120}(\theta), w_{220}(\theta))^\top = \\
&= -\frac{g_{20}}{\lambda_1}ve^{\lambda_1\theta} - \frac{\bar{g}_{20}}{3\lambda_1}\bar{v}e^{\lambda_2\theta} + E_1e^{2\lambda_1\theta}\\
w_{11}(\theta) &= (w_{111}(\theta), w_{211}(\theta))^\top = \\
&= \frac{g_{11}}{\lambda_1}ve^{\lambda_1\theta} - \frac{\bar{g}_{11}}{\lambda_1}\bar{v}e^{\lambda_2\theta} + E_2.
\end{aligned}$$

$$E_1 = -\begin{pmatrix} \frac{b-2\lambda_1}{D_1} & \frac{2b}{D_1}e^{2\lambda_2\tau_2} \\ -\frac{a^2}{b^2D_1}e^{2\lambda_2\tau_1^0} & -\frac{(2\lambda_1 - \frac{a^2}{b^2})}{D_1} \end{pmatrix}\begin{pmatrix} F_{120} \\ F_{220} \end{pmatrix}$$

$$E_2 = -\begin{pmatrix} \frac{b^2}{a^2} & \frac{2b^2}{a^2} \\ -\frac{1}{b} & \frac{1}{b} \end{pmatrix}\begin{pmatrix} F_{111} \\ F_{211} \end{pmatrix},$$

$$D_1 = (2\lambda_1 + \tfrac{a^2}{b^2})(2\lambda_1 - b) + \tfrac{2a^2}{b}e^{2\lambda_2\tau_0}.$$

Therefore, we can compute the following parameters:

$$\begin{aligned}
&C(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}\\
&\mu_2 = -\frac{Re(C(0))}{M},\ T = -\frac{Im(C(0)) + \mu_2 N}{\omega_0},\\
&\beta_2 = 2Re(C(0))
\end{aligned} \quad (37)$$

where M and N are given by (16).

In the three formulas (37), μ_2 determines the directions of the Hopf bifurcation; β_2 determines the stability of the bifurcation periodic solutions; T_2 determines the period of the bifurcating periodic solution. Therefore:

1. If $\mu_2 > 0 (< 0)$ the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0 (< \tau_0)$

2. If $\beta_2 < 0 (> 0)$ the solutions are orbitally stable (unstable)

3. If $T_2 > 0 (< 0)$ the period increases (decreases).

According the above results, we conclude that there are three cases of financial crises contagions between two countries; if $b^3 < a^2$. (a) If $0 \le \tau_1 + \tau_2 < \tau_0$, where τ_0 is given by (11), the stationary solution of the system (1) is asymptotic stable. In this case, the financial crises contagions have evolved into a disaster and both the two countries have to endure a strong impact (b) If $\tau_1 + \tau_2 = \tau_0$, there is a limit cycle for the system (1). Its phase plane shows that it occurs an alternating oscillation of stock returns between two countries. However, this oscillation may not enlarge without any limitation due to there exists a limit cycle. The immunity ability and self-repair capacity of the economic system in both countries may limit the oscillation magnitude within some controllable size, which depends on the size of the limit cycle. So it is a contagion case with limit and controllable oscillation.

(c) If $\tau_1 + \tau_2 > \tau_0$, the stationary solution of the system (1) is unstable.

4 The stochastic model with delay

In this section we consider the stochastic perturbation of (1) given by the stochastic differential system with delay (3).

Linearizing (4) around the equilibrium $(x_{10}, x_{20})^\top$ yields the linear stochastic differential equation with delay

$$du(t) = (Au(t) + B_1u(t-\tau_1)+ \\ +B_2u(t-\tau_2)dt) + Cu(t)dw(t) \quad (38)$$

where $u(t) = (u_1(t), u_2(t))^\top$, and $A,\ B_1,\ B_2$ are given by (6) and $C = diag(\sigma, \sigma)$.

Using the method from [6], we analyze the first and second moments of the solutions for (38) with respect τ_1 and τ_2.

Proposition 3 *1. The first moment of the solution of (38) is given by*

$$\dot{E}(u(t)) = AE(u(t)) + B_1E(u(t-\tau_1))+ \\ +B_2E(u(t-\tau_2)) \quad (39)$$

2. The characteristic function for (39) is given by (7).

3. For $\tau \in [0, \tau_0)$ the equilibrium point $(0, 0)$ is locally asymptotically stable, where τ_0 is given by (11).

4. The value τ_0 given by (11) is a Hopf bifurcation

5. The solution of (40) with $\tau_1 + \tau_2 = \tau_0$ on the center manifold are given by

$$E(t) = z(t)v + \bar{z}(t)\bar{v} \quad (40)$$

where $v = (v_1, v_2)^\top$ is given by (25), $E(t) = E(u(t)) = (E(u_1(t)), E(u_2(t)))^\top$ and

$$z(t) = \lambda_1 z(t) \quad \lambda_1 = i\omega_0 \quad (41)$$

and ω_0 is a positive root of (10).

To examine the stability of the second moment of $u(t)$ for the linear stochastic differential delay equation (38) we use Ito's rule to give the stochastic differential of $u(t)\ u(s)^\top$.

Let $R(t, s) = E(u(t)u(s)^\top)$ be the covariance matrix of the process $u(t)$ so that $R(t, t)$ satisfies:

$$\dot{R}(t,t) = E(du(t)u^\top(t) + u(t)du^\top(t)+ \\ +Cu(t)u^\top(t)C) = AR(t,t) + R(t,t)A^\top+ \\ +B_1R(t,t-\tau_1) + R(t,t-\tau_1)B_1^\top+ \\ +B_2R(t,t-\tau_2) + R(t,t-\tau_2)B_2^\top + CR(t,t)C. \quad (42)$$

From (42) and $R\dot{y}(t, s) = E(u_j(t), u_j(s)),\quad i, j = 1, 2$, we get

Proposition 4 *1. The differential system (43) can be written as:*

$$\dot{R}_{11}(t,t) = (-\frac{2a^2}{b^2} + \sigma^2)R_{11}(t,t) - 4bR_{12}(t,t-\tau_2)$$

$$\dot{R}_{22}(t,t) = (2b + \sigma^2)R_{22}(t,t) + \tfrac{2a^2}{b^2}R_{12}(t,t-\tau_1) \quad (43)$$

$$\dot{R}_{12}(t,t) = (-\frac{a^2}{b^2} + \sigma^2)R_{12}(t,t) + \frac{a^2}{b^2}R_{11}(t,t-\tau_1)- \\ -2bR_{22}(t,t-\tau_2).$$

2. The characteristic function of (43) is

$$h(\tau,\tau) = 4\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + 4q_1(4\lambda + q_0)e^{-\lambda\tau}. \quad (44)$$

where

$$\tau = \tau_1 + \tau_2, p_2 = \frac{4a^2}{b^2} - 4b - 6\sigma^2,$$

$$p_1 = (4 - \frac{6a^2}{b^2})\sigma^4 + (\frac{4a^4}{b^3} - \frac{6a^2}{b^2} + 4b)\sigma^2 - \frac{4a^2}{b^2}$$

$$p_0 = -\tfrac{a^2}{b^2}(\tfrac{2a^2}{b^2} - 2\sigma^2)(2b + \sigma^2)\sigma^2 \quad (45)$$

$$q_1 = \frac{a^2}{b},\ \ q_0 = \frac{2a^2}{b^2} - 2b - 2\sigma^2.$$

Proposition 5 *1. If $\tau = 0$, and σ satisfy*

$$p_2 > 0,\ \ p_1 + 4q_1 > 0,\ \ p_0 + q_1q_0 > 0$$

$$p_2(p_1 + 4q_1) - 4(p_0 + q_1q_0) > 0 \quad (46)$$

then the equation $h(\lambda, 0) = 0$ has the roots with a negative real part.

2. If $\tau \neq 0$, and σ satisfy (46) then for $\tau \in [0, \tau_{00})$ the equation $l(\lambda, \tau) = 0$ has the roots with a negative real part, where

$$\tau_{00} = \tfrac{1}{\omega_{00}} arctg \tfrac{4\omega_{00}(p_2\omega_{00}^2 - p_0) - q_0(4\omega_{00}^3 - p_1\omega_{00})}{q_0(p_2\omega_{00} - p_0) + 4\omega_{00}(4\omega_{00}^3 - p_1\omega_{00})} \quad (47)$$

and ω_{00} is a positive root of equation

$$16\omega^6 + (p_2^2 - 8p_1)\omega^4 + (p_1^2 - 2p_2p_0+ \\ +256q_1^2)\omega^2 + p_0^2 - 16q_1^2q_0^2 = 0 \quad (48)$$

3. τ_{00} is a Hopf bifurcation.

From Proposition 5, we obtain that for $\tau_1 + \tau_2 < \tau_{00}$ the square mean values of the variables of (38) are asymptotically stable, and τ_{00} is a Hopf bifurcation. The solution of (43) on the center manifold are given by

$$R(t) = z(t)v + \bar{z}(t)\bar{v} \quad (49)$$

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where $R(t) = (R_{11}(t,t), R_{22}(t,t), R_{12}(t,t))^\top$ and v is the eigenvector corresponding the eigenvalue, $\lambda_3 = i\omega_{00}$, and

$$\dot{z}(t) = \lambda_3 z(t), \quad z(t) = x(t) + iy(t). \tag{50}$$

The vector's components $v = (v_{11}, v_{22}, v_{12})^\top$ are given by $v_{11} = -4be^{-\lambda_3\tau_2^{00}}(2\lambda_3 - 2b - \sigma^2)$, $v_{22} = \frac{2a^2}{b^2}(2\lambda_3 + \frac{2a^2}{b^2} - \sigma^2)e^{-\lambda_3\tau_1^{00}}$, $v_{12} = (2\lambda_3 + \frac{2\lambda^2}{b^2} - \sigma^2)(2\lambda_3 - 2b - \sigma^2)$ where $\tau_1^{00} = \tau_{00} - \tau_2^{00}$ and $\tau_2^{00} < \tau_{00}$ is given.

5 Numerical Simulation

For the parameters $a = 2,\ b = 1.5$, we obtain the equilibrium point $x_0 = 1.125, \quad y_0 = 1.125$. If $\tau_1 = \tau_2 = 0$ the equilibrium point is asymptotically stable. The bifurcation Hopf is given by $\tau_0 = 0.028$. Let $\tau_2 = 0.013$. Result $\tau_1 = 0.015$. The parameters $\mu_1,\ T_2,\ \beta_2$ are given by: $\mu_2 = -3588,\ \beta_2 = 1777,\ T_2 = 1173$. Because $\mu_2 < 0$ the Hopf bifurcation is subcritical and the bifurcating periodic solution exists for $\tau < \tau_0$. Because $\beta_2 > 0$ the solutions are orbitally unstable. Because $T_2 > 0$ the period decreases.

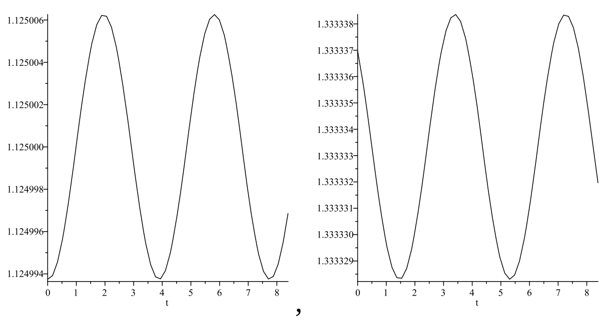

Figure 1: $(t, x_1(t)), \qquad (t, y_2(t))$

In what follows we have the numerical simulation of the differential stochastic system (3). For $a = 2,\ b = 1.5,\ \tau_1 = \tau_2 = 0,\ \sigma_1 = \sigma_2 = 0.3$ in We obtain the orbits $(n, x_1(n,\omega)), (n, x_2(n,\omega)), (x_1(n,\omega), x_2(n,\omega))$.

The value of τ_{00} (47) is given by $\tau_{00} = 1.01112$. For $\tau_{002} = 0.5$, result $\tau_{001} = 0.511$.

The numerical simulations verify the theoretical results.

6 Conclusion

In the present paper, a deterministic and stochastic dynamic model with delays of the financial crises contagions was considered. The deterministic system was

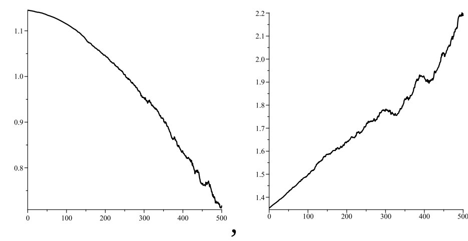

Figure 2: $(n, x_1(n,\omega)), \quad (n, x_2(n,\omega))$.

analyzed both analytically and numerically. It has been established that the system of the model is stable without delays, but by introducing the delay parameter, a Hopf bifurcation takes place.

Using Wiener process, the stochastic system was built in the neighborhood of the equilibrium point. We have studied the dynamics of the square mean value for the variables of the linearized system obtained by the stochastic system. We have determined the value of the delays for which the equilibrium point is asymptotically stable in square mean. Also, the value of the delay for which there is a Hopf bifurcation was determined.

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