Signal Processing Techniques

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Abstract: Due to this Fourier series exclusive interpretation applied to periodical signals, which shows that a stationary harmonic signal with a single pulsation $\omega_0$ cannot contain components of other pulsations but its own pulsation $\omega_0$, there is the opinion that stationary harmonic signals cannot stimulate physical systems which have their own pulsations different from $\omega_0$. The authors’ research have demonstrated that this theory is not true. More than this, signals considered identical, that is with identical amplitudes, phases and pulsations, are not identical from the spectral point of view if the signals have different durations and this can be noticed in many practical situations.

Key-Words: periodical signal, Fourier series, harmonics, amplitude, errors, spectral component, spectrum

1 Introduction

A signal is considered and interpreted as a variable that defines a time dependent physical phenomenon. As a generalized form, the signal is analytically defined, as a real function $f(t)$, with a single real variable that is $t$ (=time). While with respect to commonly accepted criteria, the signals were classified into several categories [1], for this paper there were selected only the following categories:

<table>
<thead>
<tr>
<th>Signal</th>
<th>Deterministic</th>
<th>Random</th>
</tr>
</thead>
<tbody>
<tr>
<td>Periodical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Harmonic</td>
<td>Stationary</td>
<td>Non-Stationary</td>
</tr>
<tr>
<td>Non-harmonic</td>
<td>Stationary</td>
<td>Non-Stationary</td>
</tr>
</tbody>
</table>

The best known and also the most commonly used periodical signals will be shortly presented below.

The signal $f(t) = A(t) \sin(\omega_0 t + \phi)$ is a harmonic signal, where $A(t)$ is the amplitude, $\omega_0$ is its own pulsation, $\phi$ is initial phase. Time $t$ is only considered for positive values, $t \geq 0$.

Mathematics creates the possibility that every stationary periodical non-harmonic signal with $\omega_0$ pulsation be interpolated with an infinite series of stationary harmonic functions whose pulsation is a multiple of the signal pulsation $\omega = n\omega_0$, $n = 1,2,3,\ldots\infty$. Based on this mathematical artifice, we are talking about Fourier series, one can explain the fact that any periodical non-harmonic signal has within sources able to stimulate a wide range of physical systems with own pulsations different from $\omega_0$, or $n\omega_0$. The authors’ research demonstrated that this theory is not true [1], [3]. More than this, signals considered identical, that is with identical amplitudes, phases and pulsations, are not identical from the spectral point of view if the signals have different durations and this can be noticed in many practical situations.
2 Fourier Series and its Properties

A stationary periodical signal \( g(t) \) with the period \( T_0 = \frac{2 \cdot \pi}{\omega_0} \), where \( \omega_0 \) is the signal pulsation, which fulfills the Dirichlet conditions, can be represented by a mathematical series whose terms are harmonic functions with pulsations multiple of the \( \omega_0 \) pulsation. The \( \omega_0 \) pulsation is called fundamental pulsation and the harmonic function with the pulsation equal to \( \omega_0 \) is called fundamental harmonic. Harmonic functions with pulsations \( n \cdot \omega_0 \), \( n = 2, 3, \ldots \) are called \( n \) order harmonics. The general form for the series of harmonic functions is:

\[
f(t) = \sum_{n=0}^{\infty} \left[ a_n \cdot \cos(n \cdot \omega_0 \cdot t) + b_n \cdot \sin(n \cdot \omega_0 \cdot t) \right]
\]

where the series has an infinite number of members and their values are:

\[
a_0 = \frac{1}{T_0} \int_0^{T_0} g(t) \cdot dt; \quad b_0 = 0;
\]

\[
a_n = \frac{2}{T_0} \int_0^{T_0} g(t) \cdot \cos(n \cdot \omega_0 \cdot t) \cdot dt;
\]

\[
b_n = \frac{2}{T_0} \int_0^{T_0} g(t) \cdot \sin(n \cdot \omega_0 \cdot t) \cdot dt; \quad n = 1, 2, \ldots
\]

The constant \( a_0 \), which is a measure of signal \( g(t) \) asymmetry with respect to the abscissa. Each harmonic component contains two terms from the same pulsation creating a trigonometric equivalent function:

\[
f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cdot \sin(n \cdot \omega_0 \cdot t + \phi_n)
\]

where the amplitude \( A_n \) and phase \( \phi_n \) result from the coefficients identification:

\[
\begin{align*}
A_n &= \sqrt{a_n^2 + b_n^2} \\
\phi_n &= \arctg \frac{a_n}{b_n}
\end{align*}
\]

By using the (3) formula, with coefficients calculated according to the relation (4), one finds that a specific order \( n \) harmonic is a stationary sinusoidal signal with its own amplitude and phase. In expressions (3) and (4) one notices that a certain harmonic of order \( n \) is actually a signal produced by a rotating vector \( \vec{v}_n \) (complex number), having modulus \( A_n \) and phase \( \phi_n \):

\[
\vec{v}_n = A_n \cdot e^{j \left( n \cdot \omega_0 \cdot t + \phi_n \right)}
\]

where components \( b_n \) are on the real axis and \( a_n \) are on the imaginary axis.

To determine the harmonics of a periodic signal is not only a theoretical problem which allows decomposing a periodic function into other periodic functions. Harmonics existence are strongly felt in practice because a non harmonic periodic signal generates an infinite number of excitation sources having frequencies equal to multiples of the basic signal frequency and these sources produce obvious effects by stimulating the physical systems with pulsations (frequencies) equal to any multiple of the signal’s own pulsation (frequency) [1] (pp.127).

Frequency multipliers used in radiotechnics are based solely on this obviously very real phenomenon. For multiplication, a stationary harmonic signal is distorted to generate harmonic components with pulsations \( n \omega_0 \) whence, by filtration, the component with the value \( n \) that is desired is extracted, \( n \) usually being equal to 2,..5.

This phenomenon is also found when referring to the mechanical systems, respectively a non-harmonic signal generating sources of excitation which get in resonance with components of the system. A good example is given by the non-harmonic signals such as earthquakes which produce damages to constructions or parts of constructions that have their own pulsation equal to harmonics of the earthquake.

A non-harmonic periodic signal is the better defined in Fourier series, the more components of the series are identified, respectively the higher \( n \) gets. In reality, one also gets a limitation here: in calculating the coefficients of the Fourier series, and even in the series itself, one uses the harmonic functions \( \sin(n \omega_0 t) \), \( \cos(n \omega_0 t) \) for which, the higher the value of \( n \) gets, the higher is the value of the parameter one needs to evaluate, while in
mathematics, it is well known that expressions
\[ \sin(\infty), \cos(\infty) \] are indeterminate.
By using a computer to calculate the series elements the non-determination situation
described above is rapidly reached due to the
way in which numbers are represented by the
operating system or programming language,
since numbers are only represented with a finite
number of figures. The below examples clearly
show this fact.

Be a saw tooth signal which is analytically
defined in the above, having the period \( T_0 = 1 \)
second and its amplitude \( =1 \), figure 1. In figure
2, in the upper side we show the shape of the
first eight harmonics and in the lower side the
initial signal, reconstructed from ten harmonics.
The value of the constant component \( a_0 \) is
0,99401 and the amplitude of the tenth order
harmonic \( A_{10} \) is 0,01593.

By increasing the number of harmonics one
would expect the signal reconstruction to be
more accurate, maybe with the exception of
interval limits, where the mathematical
discontinuity is also more pronounced. By
increasing the number of harmonics, like in
figure 3 and 4, one notices a contradictory
situation: reconstruction out of 300 harmonics is
more precise than of 500, where large errors
appear.

\[ a_0 = 0.99401 \; ; \; A_{300} = 0.00105 \]

Fig. 3 - Reconstruction out of 300 harmonics

To find the source of this deviation one has to
research the evolution of the harmonics
amplitude in the studied cases, figure 5.

If for a number up to 300-400 harmonics their
amplitude continually decreases at the same
time with the increase of the given harmonics
number of order, by continuing to increase at
number of harmonics, their amplitude begins to increase
again up to the value \( A_{500} = 0.497 \). Continuing
to increase the number of harmonics, their
amplitude decreases again followed once more
by an increase for order high value. This
variation of the amplitude by the increase of the
number of order, is not due to the structure of
known relations of calculation but actually to the
computer having to operate with hard
conditioned relations. The cause of the
deviations from figure 4, where the signal is
reconstructed out of 300 harmonics, can be thus
found in the particular way in which the digital
computers operate with numbers which have a
finite number of figures and, due to the
truncation errors that appear when a large
number of computations is being done, one may
end up experiencing big errors. Consider now
the signal from figure 6 which is a harmonic
pulse signal \( \omega = 10 \; \text{s}^{-1} \) described by the
equation \( g(t) = \sin(10t) \)
Subsequently, one can say of the Fourier series that:
- It is a series made of harmonic functions with pulsations equal to multiples of the pulsation of the non-harmonic periodic signal whence it originates;
- The harmonic functions which compose the Fourier series are real sources of excitation for the physical systems which have their own pulsations equal to one of the harmonic pulsation;
- Each harmonics amplitude has a finite value, usually considered as continuously decreasing by the order of the harmonic, although there may occur situations where the decrease is not continued;
- The harmonic functions come together in a discrete spectrum of pulsations contained in the base signal;
- The characteristics of each spectrum’s harmonic, meaning amplitude and phase, are independent of the signal duration;
- The Fourier series doesn’t show if, among the discrete harmonic components, it has others able to excite various other oscillating systems.

### 3 Comparison with the Fourier Transform

Let’s now compare the discrete spectral components given by the harmonics from the Fourier series with the continuous spectral components provided by the Fourier transform. For a signal described by the real function \( g(t) \), the Fourier transform leads to the function \( G(j\omega) \) [4] (pp. 14):

\[
G(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} g(t) dt \tag{6}
\]

with \( j = \sqrt{-1} \) which, after being submitted to the below transformation:

\[
G(j\omega) = \int_{-\infty}^{+\infty} \cos(\omega t) g(t) dt - j \int_{-\infty}^{+\infty} \sin(\omega t) g(t) dt = \Re(\omega) + j \Im(\omega) \tag{7}
\]

becomes a complex function with a real part \( \Re(\omega) \) and an imaginary one \( \Im(\omega) \). The modulus of this function \( S(\omega) = |G(j\omega)| \) is calculated by:

\[
S(\omega) = \sqrt{\Re^2(\omega) + \Im^2(\omega)} \tag{8}
\]

and one obtains a function dependent on the pulsation (or frequency), which is either named frequency characteristic should it refer to the behavior of one element of the signal’s propagation chain or spectral function should it refer to the spectral components of a signal \( g(t) \). For any time function \( g(t) \), the values from (6)---(8) are to be found easily on the numerical way [2](pp. 123-124). Let’s consider the pure harmonic signal:

\[
g(t) = \sin(\omega_0 t) \tag{9}
\]
which is fully known for \( t \in \left[ 0 \cdots 2 \pi / \omega_0 \right] \). For the spectrum determined via the Fourier transform using (8), the integration will be done between variable limits, those being 0 and \( 2 \pi / \omega_0 \) \((n \text{ is a multiple of the period } 2 \pi / \omega_0)\), in order to detect a possible influence of the signal’s duration upon its spectrum, duration which doesn’t appear in the Fourier series. The form below of the spectral function is obtained on the analytical calculation:

\[
S(\omega) = 2 \omega_0 \frac{\sin\left(\frac{n \pi \omega}{\omega_0} \right)}{\omega^2 - \omega_0^2} \tag{10}
\]

Figure 7 shows the (10) function for \( n=4 \). Analyzing expression (10) one notices some interesting aspects:

- In the point determined by \( \omega = \omega_0 \), on the abscissa, the function has the value:

\[
S(\omega_0) = 2 \omega_0 \lim_{\omega \to \omega_0} \frac{\sin\left(\frac{n \pi \omega}{\omega_0} \right)}{\omega - \omega_0} = n \frac{\pi}{\omega_0} \tag{11}
\]

- The peak of the spectral function, meaning the maximum amplitude, is not obtained for the pulsation \( \omega = \omega_0 \).

- Amplitude \( S \) for \( \omega = \omega_0 \) is increasing by the signal duration, so that for \( n \to \infty \), \( S \to \infty \).

Table 3 – Values for \( A=1, \ \omega_0 = 100 \pi \), \( \phi = 0 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta )</td>
<td>48.7</td>
<td>1.8</td>
<td>0.48</td>
<td>0.02</td>
<td>0.006</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>( B% )</td>
<td>124</td>
<td>24.3</td>
<td>12.1</td>
<td>2.53</td>
<td>1.3</td>
<td>0.65</td>
<td>0.16</td>
</tr>
</tbody>
</table>

The above shows that even a pure harmonic signal, which ought to have a frequency spectrum with a single pulsation, in reality has a very large spectrum, with many pulsations, the spectrum becoming narrow only for a single value \( \omega = \omega_0 \) and only if the signal is infinitely lasting, \( n \to \infty \). Since the signals have, in practice, a finite duration and are thus lasting for only several periods such as in the case of earthquakes or mechanical shocks, it means that their spectrum is actually very rich and they can excite an infinite number of devices which could have their own pulsation equal to some pulsation generated by the signal in its spectrum. Having a wide spectrum for short durations for a signal has one more inconvenient.

Let’s consider a signal composed of two stationary harmonic components:

\[
g(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) \tag{12}
\]

where we can use for example: \( A_1 = 1, \ A_2 = 1, \ \omega_{01} = 100 \pi, \ \omega_{02} = 110 \pi, \ \phi_1 = \phi_2 = 0 \) and which is lasting initially for only one period, and then for 50 periods with the value of \( 2 \pi / \omega_{01} \).

The spectrum from figure 8 shows that for the short length signal the spectrum’s width is so big that it doesn’t allow anymore the identification of its components’ pulsations, the bandwidth exceeds the difference between the two pulsations \( \omega_{01}, \ \omega_{02} \).
If the signal is lasting more, for example 50 periods, the spectral bandwidths get narrow and they show very clearly the existence of the two spectral components.

To see if this event is a characteristic of spectral analysis based on Fourier transform or it is a property of the signal let consider the reverse reconstruction

\[ f(t) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ F(j\omega) e^{j\omega t} \right] d\omega \]  

(13)

that is Fourier transform based on short and long signals to obtain the original signal.

For a range between \( \omega = 0 \) and \( \omega = \omega_0 \) reconstructions are shown in figures 9 and 10.

It turns out that spectrum rich in harmonics of short signals (polychromatic spectrum, by analogy with the spectra of light) is not a feature of Fourier transform in the form of generating false information, but a property depending on the duration of signals: signals short generates a rich spectrum of harmonics even if they are purely harmonic.

4 Conclusions

The elements of the Fourier series represent the spectral components of a given signal, only if the signal is lasting for very long time. For signals with a short duration of time the larger the spectrum the shorter the duration is. This explains the multitude of systems with various own

pulsations excited for very short periods by short duration signals such as earthquakes (mechanical systems) or radio electrical interferences produced by the engines ignition. That is why earthquakes destroy walls, pillars, consoles or chimneys with various dimensions and shapes and radio electrical interferences are detected up to frequencies of hundreds MHz.

References