Max-plus algebra and its application in spreading of information

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Abstract: In this paper circulant matrices in max-plus algebra are presented. Circulant matrices are special form of matrices which are entered by vector of inputs. For special types of matrices such as circulant matrices, the computation can often be performed in the simpler way than in the general case. The so-called max-plus algebra is useful for investigation of discrete events systems and the sequence of states in discrete time corresponds to powers of matrices in max-plus algebra. The eigenproblem for max-plus matrices describes the steady state of the system. Max-plus algebra has been intensively studied by many authors, see e.g. (Cunningham-Green, Gavalec, Plavka, Zimmerman). Max-plus algebra can solve the problem when looking for a steady system states shifted in time. Therefore, this article focuses on possible applications of max-plus artificial reverberation.

Key–Words: Max-plus algebra, eigenproblem, eigenvector, circulant matrices, steady states, artificial reverberation

1 Introduction

In many applications the models use operations of maximum and minimum together with further arithmetical operations. The max-plus algebra is useful for investigation of discrete events systems and the sequence of states in discrete time corresponds to powers of matrices in max-plus algebra. A typical application of discrete events systems are production lines, where every machine must wait with starting a new operation until the operations on other machines are completed. The eigenproblem for max-plus matrices describes the steady state of the system and, therefore, it has been intensively studied by many authors, see e.g. [1, 3, 4, 9, 12] or [7, 2, 5, 10, 11, 6]. For special types of matrices such as circulant, Toeplitz, Hankel or Monge matrix, the computation can often be performed in the simpler way than in the general case, hence the investigation of special cases is important from the computational point of view. In this paper the eigenspace structure for a special case of so-called circulant matrices is studied. Circulant matrices arise, for example, in applications involving the discrete Fourier transform and the study of cyclic codes for error correction.

2 Circulant matrices

In this section, we will consider special type of matrices in max-plus algebra, called circulant matrices. Matrix A of type $n \times n$ is circulant if

$$a_{ij} = a_{i'j'}$$

whenever

$$i - i' = j - j' \pmod{n}.$$  \hspace{1cm} (2)

Hence, matrix $A$ is fully determined by its inputs in the first row, denoted as $a_0, a_1, \ldots, a_{n-1}$. ($a_0$ is the common value of all diagonal inputs, and similarly, each $a_i$ is the common value of all inputs on a line parallel to the matrix diagonal). This notation means: $a_{ij} = a_0$ if $i = j$ otherwise $a_{ij} = a_k : k = j - i \pmod{n}$. For the notations above we should write the matrix as follows:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}$$ \hspace{1cm} (3)

In graph theory, a graph or digraph whose adjacency matrix is circulant is called a circulant graph (or digraph). Graph theoretic properties of circulant graphs have been analysed many times. A particular topic of investigation has been the question as to whether a known graph problem becomes easier when the general instance is forced to be a circulant graph. Circulant graphs have a vast number of applications in telecommunication net-working.
3 Max-plus algebra

By a max-plus algebra we understand a triple $(\mathcal{R}, \oplus, \otimes)$, where $\mathcal{R}$ is the set of all real numbers and $\oplus, \otimes$ are binary operations on $\mathcal{R}$ defined as

$$a \oplus b = \max(a, b), \quad a \otimes b = a + b \quad (4)$$

for all $a, b \in \mathcal{R}$. The operations $\oplus, \otimes$ are extended to matrices and vectors in the same way as in conventional linear algebra. Hence, in both types of extremal algebras, the matrix product $A \otimes B$ is defined for matrices $A \in \mathcal{R}^{m \times p}, B \in \mathcal{R}^{p \times n}$ as a matrix $C \in \mathcal{R}^{m \times n}$ by formula

$$c_{ij} = \bigoplus_{k=1}^{p} (a_{ik} \otimes b_{kj}) \quad (5)$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. The $k$th power of a square matrix $A \in \mathcal{R}^{n \times n}$ is denoted by $A^{(k)}$ and defined by recursion on $k = 2, 3, \ldots$

$$A^{(k)} = A \otimes A^{(k-1)} \quad (6)$$

3.1 Eigenproblem in max-plus algebra

The eigenproblem in max-plus algebra is formulated as follows. For given $A \in \mathcal{R}^{n \times n}$, find $\lambda \in \mathcal{R}$ and $x \in \mathcal{R}^n$ satisfying

$$A \otimes x = \lambda \otimes x \quad (7)$$

The value $\lambda$ and the vector $x$ fulfilling the above equation are called the eigenvalue and the eigenvector of matrix $A$. The set of all eigenvectors is called the eigenspace of $A$. It has been shown in [3] that the eigenvalue of a given max-plus matrix can be efficiently described by considering cycles in specific digraphs.

The associated digraph $D_A$ of a matrix $A \in \mathcal{R}^{n \times n}$ is defined as a complete arc-weighted digraph with the node set $V = N = \{1, 2, \ldots, n\}$, and with the arc weights $w(i, j) = a_{ij}$ for every $(i, j) \in N \times N$. If $p$ is a path or a cycle in $D_A$, of length $r = |p|$, then the weight $w(p)$ is defined as the sum of all weights of the arcs in $p$. If $r > 0$, then the mean weight of $p$ is defined as $w(p)/r$. The maximal mean weight of a cycle in $D_A$ is denoted by $\lambda(A)$.

Lemma 1 [3] If $A \in \mathcal{R}^{n \times n}$, then $\lambda(A)$ is the unique eigenvalue of $A$.

Eigenvectors of a given max-plus matrix can be found using the following procedure: for any $B \in \mathcal{R}^{n \times n}$ we denote by $\Delta(B)$ the matrix $B \oplus B^{(2)} \oplus \ldots \oplus B^{(n)}$. Further, we denote $A_\lambda = -\lambda(A) \otimes A$ (in the formal product of a scalar value $-\lambda(A)$ and a matrix $A$ we have $[A_\lambda]_{ij} = -\lambda(A) + a_{ij}$ for any $(i, j) \in N \times N$). It is shown in [3] that matrix $\Delta(A_\lambda)$ contains at least one column, the diagonal element of which is 0. Every such a column is an eigenvector (so called: fundamental eigenvector) of the matrix $A$. Moreover, every eigenvector of $A$ can be expressed as a linear combination of fundamental eigenvectors.

Let us denote by $g_1, g_2, \ldots, g_n$ all columns of $\Delta(A_\lambda)$. We shall say that vectors $g_j, g_k$ are equivalent, if there is $\alpha \in G$ such that $g_j = \alpha \otimes g_k$. It has been shown in [3] that vectors $g_j, g_k$ are equivalent if and only if the vertices $j, k$ are contained in a common cycle $c$ with $w(c) = \lambda(A)$ in $D_A$. The eigenspace dimension of matrix $A$ is defined as the maximal number of non-equivalent fundamental eigenvectors. In general case, the eigenspace dimension can be found by computing matrix $\Delta(A_\lambda)$ in $O(n^3)$ time. In this section we show that for circulant matrices the eigenspace dimension can be computed in $O(n)$ time.

The eigenproblem for circulant max-plus matrices was studied in [8] and following lemmas were presented.

Lemma 2 [8] If $A$ and $B$ are circulant max-plus matrices of the same type $n \times n$, then $A \otimes B$ is also a circulant matrix of type $n \times n$.

Lemma 3 [8] For a circulant matrix $A$ the eigenvalue $\lambda(A)$ is equal to the maximal value of the elements in the first row of $A$.

Theorem 4 The eigenspace dimension of a given circulant matrix $A$ is equal to the greatest common divisor of all positions of the maximal value $\lambda(A)$ in the first row of matrix $A$ and the size $n$ of $A$.

3.2 Proof of Theorem 4

Let $A \in \mathcal{R}^{n \times n}$ be a fixed circulant matrix. Without any loss of generality we can assume that the maximal value in the first row of $A$ is equal to zero. Then $\lambda(A) = 0$, $A_\lambda = A$, and any two fundamental eigenvectors $g_j, g_k$ are equivalent if and only if vertices $j, k$ are contained in a common cycle $c$ with $w(c) = 0$ in $D_A$. Such a cycle will be called zero-cycle. For reader’s convenience, the proof is divided into three cases.

3.2.1 Case 1.

In the first case we assume that the maximal value 0 in the first row of matrix $A$ is contained on the position 0 and nowhere else. In other words, $a_0 = 0$ and $a_i < 0$ for $i = 1, 2, \ldots, n - 1$. Then matrix $A$ contains zeros on the main diagonal and negative values on other
positions. Hence, the only zero-arcs in the associated digraph \( D_A \) are the loops on all vertices, which are also the only zero-cycles. Hence, and all fundamental eigenvectors are pairwise non-equivalent. As a consequence, the eigenspace dimension is \( n = \gcd(0, n) \), and the assertion of Theorem 4 in Case 1 is true.

### 3.2.2 Case 2.

In this case we assume that the maximal value 0 in the first row is situated on a single position \( p > 0 \). Then the zero-arcs in the associated digraph \( D_A \) are exactly the arcs of span \( p \), i.e., the arcs connecting every vertex \( i \) with vertex \( i + p \). Hence, every zero-cycle is composed of arcs of span \( p \) and, therefore, the length of every zero-cycle is a multiple of \( p \). At the same time, the length of any cycle is a multiple of \( n \), and a multiple of the greatest common divisor \( d = \gcd(p, n) \), as well. By a well-known theorem from number theory, every sufficiently large multiple of \( d \) can be expressed as a linear combination of numbers \( p, n \) with positive coefficients. As a consequence, any two vertices \( i, j \in D_A \) are contained in a common zero-cycle if and only if \( i - j \equiv d \pmod{n} \). By this, there are exactly \( d \) classes of eigenvector equivalence. In other words, the eigenspace dimension is equal to \( d = \gcd(p, n) \).

### 3.2.3 Case 3.

In fact, the last case is the most general and covers also the cases 1 and 2. We assume that the maximal value 0 in the first row of \( A \) is situated on \( k \) positions \( p_1, p_2, \ldots, p_k \). Let us denote \( d = \gcd(p_1, p_2, \ldots, p_k, n) \). By similar arguments as in the previous two cases, we get that any two vertices \( i, j \in D_A \) are contained in a common zero-cycle if and only if \( i - j \equiv d \pmod{n} \), i.e., the number of equivalence classes in the eigenspace of \( A \) is exactly \( d \). By this, the proof of Theorem 4 is complete.

### 3.3 Example of circulant matrices in max-plus algebra

We have circulant matrix \( A \) and initial state \( x_0 \)

\[
A = \begin{pmatrix}
8 & 7 & 4 & 2 & 3 \\
3 & 8 & 7 & 4 & 2 \\
2 & 3 & 8 & 7 & 4 \\
4 & 2 & 3 & 8 & 7 \\
7 & 4 & 2 & 3 & 8
\end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 5 \\ 4 \end{pmatrix}
\]

From definition 3 we know, that \( \lambda = 8 \). For steady states (eigenvectors) the equation \( A \otimes x = \lambda \otimes x \) holds true. \( A \otimes x^{(k)} = x^{(k+1)} \) if for any \( r \) the equation \( x^{(r+1)} = x_r \otimes \lambda \) hold true, then \( x^{(r+1)} \) is steady state (eigenvector) of given circulant matrix.

In the table below is the system states progression shown.

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( \Delta_1 )</th>
<th>( x_1 )</th>
<th>( \Delta_2 )</th>
<th>( x_2 )</th>
<th>( \Delta_3 )</th>
<th>( x_3 )</th>
<th>( \Delta_4 )</th>
<th>( x_4 )</th>
<th>( \Delta_5 )</th>
<th>( x_5 )</th>
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<td>8</td>
<td>36</td>
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<td>44</td>
</tr>
</tbody>
</table>

The state \( x_4 \) and the following states are steady states.

For the eigenspace we need a reduced form of given circulant matrix.

\[
B = \begin{pmatrix}
0 & -1 & -4 & -6 & -5 \\
-5 & 0 & -1 & -4 & -6 \\
-4 & -6 & -5 & 0 & -1 \\
-1 & -4 & -6 & -5 & 0
\end{pmatrix}
\]

by a multiplication of matrix powers we can find a steady state base - state of fundamental vectors (eigenspace) \( \Delta(B) \).

\[
\Delta(B) = \begin{pmatrix}
f_1 & f_2 & f_3 & f_4 & f_5 \\
0 & -1 & -2 & -3 & -4 \\
-4 & 0 & -1 & -2 & -3 \\
-3 & -4 & 0 & -1 & -2 \\
-2 & -3 & -4 & 0 & -1 \\
-1 & -2 & -3 & -4 & 0
\end{pmatrix}
\]

 Eigenvalue is on position 0 then by 4 we know, that dimension of given circulant matrices is 5.

\[
\Delta(B) \otimes x_0 = y
\]

\[
y \otimes 4\lambda = x_4
\]

\[
\begin{pmatrix}
0 & -1 & -2 & -3 & -4 \\
-4 & 0 & -1 & -2 & -3 \\
-3 & -4 & 0 & -1 & -2 \\
-2 & -3 & -4 & 0 & -1 \\
-1 & -2 & -3 & -4 & 0
\end{pmatrix} \otimes \begin{pmatrix} 0 \\ 2 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 5 \end{pmatrix}
\]

\[
\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 5 \end{pmatrix} \otimes 4\lambda = \begin{pmatrix} 34 \\ 35 \\ 36 \\ 37 \\ 36 \end{pmatrix}
\]
4 Application

Max-plus algebra is used in some special problems of the Operational Research. For example, we give a dynamic programming, finding ways traffic problems, special problems of planning, etc. Among the known applications Max-plus algebra can be classified as “JobShop Scheduling”, determination of steady state behavior of a set of machines. This problem is often reduced to only finding eigenvectors. With suitability of max-plus algebra to solve various problems that occur or are expressed in discrete time such as synchronization, scheduling, or scheduling, it is sometimes referred to as ”schedule algebra.” Other use of max-plus algebra is offered to describe the communication network synchronization problems of production and transport, the shortest travel, etc.

4.1 Spreading of information

Sound propagation and artificial reverberation systems in linear algebra are very complex systems. In this application we use extremal algebra, namely max-plus for performance modeling of sound propagation in an artificial environment. The system is solved for a special model presented circulant matrix whose inputs for determining sound propagation between sources can be enter, thanks to its special properties, only the vector of inputs.

4.2 Artificial reverberation

The general concept is considered as reverberation deliver audio signals to listeners ear to 0.1 seconds after turning off the sound source. We will use max-plus algebra and special matrix. In max-plus algebra is equation of eigenproblem:

\[ A \otimes x = \lambda \otimes x \]

It follows that the states are always stable for \( \lambda \) differ. \( \lambda \) is therefore very important in the system of artificial reverberation, as determined by indentation period (sending signal) reverberation. The value of variable \( \lambda \) is the maximum mean weight cycle. In our model, we need to create a reverb to sound waves reflected back to the original point, that is the source for taking the time to 0.1 seconds.

There must be input \( a_2 = \lambda \) such that \( \lambda = 0.1 + 8 \). In our model, for 8 different sound sources in the room comply with this requirement will be the fourth starting point will be a reflection despite all the reflection points is worth \( a_3 = \lambda \) will be equal 0.0125 and the other input values will be smaller. In addressing the reverberation we used the human ear properties to compose sounds that get in the ear 0.1 second together and therefore there is no echo. Therefore we are looking for the maximum average weight of the largest cycle in the matrix, because it ensures us that other sounds are reflected back to the source location for less than 0.1 seconds, thus will consist of reverberation.

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