Invariability of the Sections in a Fiber Bundles

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Abstract. Important fields in differential geometry, physics, etc., can be defined as sections in the fiber bundles. In this work, the conditions in which the sections in a G-fiber bundles \( \xi = (E, \pi, M, F, G) \) remain invariant to the different functions that generate morphisms or automorphisms of their G-fiber bundles, are studied. As to a G-fiber bundles \( \xi = (E, \pi, M, F, G) \) is associated in a natural way a principal fiber bundles \( \xi_a = (P, p, M, G) \), we used this opportunity to find an equivalent conditions for invariability of the sections.

Key-words: Manifolds, Lie Group, Fiber Bundles, Principal Fiber Bundle, Invariable Sections.

1. Introduction.
The sections of fiber bundles play an important role in differential geometry, in physic, etc. The fields of important geometric objects (connections, tensor fields, metrics, etc.) is the sections in fiber bundles associated to some principal fiber bundles.

By example, in the theory of forces interaction, the quarks field is the section of fiber bundle with standard fiber \( \mathbb{C}^3 \) associated to the principal fiber bundles over space-time with structure group \( SU(3) \). In the Hamiltonian mechanics, developed using symplectic geometry, the study of invariability for some objects (fields, forms, etc.), plays an important role.

Invariability study of sections in a fiber bundles may play an important role in the understanding some special properties of this geometric objects. In this paper, all geometric objects like manifolds, maps, sections, etc. is considered to be smooth.

2. Fiber bundles
Recall that a fiber bundle \( (E, \pi, M, F) \) consists of the smooth manifolds: \( E \) (total space), \( M \) (base space), \( F \) (standard fiber), and a smooth surjective submersion \( \pi : E \to M \) such that each \( x \in M \) has a fiber chart or a local trivialization of \( E \), i.e. an open neighborhood \( U \subset M \), \( x \in U \) and a diffeomorphism \( \psi : E_x = \pi^{-1}(U) \to U \times F \), such that the diagram

\[
\begin{array}{ccc}
E_x & \xrightarrow{\psi} & U \times F \\
\downarrow \pi & & \downarrow p \circ \pi \\
U & \xrightarrow{id_U} & U
\end{array}
\]

commute, i.e. \( p \circ \psi = id_U \circ \pi \).

A collection of fiber charts \( \{(U_i, \psi_i)\} \) such that \( \{U_i\} \) is an open cover of \( M \), is called a fiber bundle atlas [2].

Always, to a fiber bundles, we will assign an atlas \( \{(U_i, \psi_i)\} \) and then

\[(\psi_j \circ \psi_i^{-1})(x, s) = (x, \psi_j(x, s))\]

where \( \psi_j : (U_i \cap U_j) \times F \to F \) is smooth and the functions \( \psi_j(x, ) \), called the transition functions of the fiber bundle is a diffeomorphism of \( F \), for each \( x \in U_i \cap U_j \) if \( U_i \cap U_j \neq \emptyset \).

A G-fiber bundle \( \xi = (E, \pi, M, F, G) \), is a fiber bundle \( (E, \pi, M, F) \) such that there is a left action of the Lie group \( G \) on the standard fiber \( F \) and the fiber bundle atlas, whose transition functions act on \( F \) via the \( G \)-action [2].

In the following, we consider the sections in a G-fiber bundle \( \xi = (E, \pi, M, F, G) \) with standard fiber \( F \), and \( \xi_a = (P, p, M, G) \) its associated principal fiber bundle [2].

Recall that if \( \phi : E \to E \) is a homomorphism of fiber bundle \( \xi = (E, \pi, M, F, G) \), there is a uniquely determined smooth map \( \phi : M \to M \) such that the diagram

\[
\begin{array}{ccc}
E_x & \xrightarrow{\phi} & E_y \\
\downarrow \pi_x & & \downarrow \pi_y \\
x & \xrightarrow{\phi} & y
\end{array}
\]
\[
\begin{array}{c}
E \xrightarrow{\varphi} E \\
\pi \downarrow \downarrow \pi \\
M \xrightarrow{\varphi} M
\end{array}
\]

commute, i.e. \( \pi \circ \varphi = \varphi \circ \pi \)

Let \( \xi = (E, \pi M, F, G) \) be a \( G \)-fiber bundle with
standard fiber \( F \) and \( \xi_0 = (P, p, M, G) \) its associated
principal fiber bundle,

\[(g, v) \in G \times F \rightarrow gv \in F\]
is the left action of the structure group \( G \) on the
standard fiber \( F \) and

\[(g, r) \in G \times P \rightarrow rg \in P\]
is the right action of the structure group \( G \) on the
manifold \( P \).

We can define the right action:

\[(g, r, v) \in G \times (P \times F) \rightarrow (r, g^{-1} v) \in P \times F\]
is the left action of the structure group \( G \) on manifold \( P \times F \) and
identify the manifold \( E \) with \( P \times_G F \).

The quotient map:

\[c : (r, v) \in P \times F \rightarrow [r, v] \in P \times_G F = E\]
is a submersion and the diagram

\[
\begin{array}{c}
P \times F \xrightarrow{c} E = P \times_G F \\
pr_1 \downarrow \downarrow \pi \\
P \xrightarrow{pr} M
\end{array}
\]

commute, \( \pi \circ c = p \circ pr_1 \)

If \( \Phi \) is a \( G \)-homomorphism of principal fiber bundle \( \xi_0 \) i.e. a smooth map \( \Phi : P \rightarrow P \), \( G \)-equivariant \((\Phi(rg) = \Phi(r)g \quad \forall \ r \in P, \ g \in G) \) then,
there is a uniquely determined smooth map\( \Phi : M \rightarrow M \), so that the diagram

\[
\begin{array}{c}
P \xrightarrow{\Phi} P \\
p \downarrow \downarrow p \\
M \xrightarrow{\Phi} M
\end{array}
\]

commute, \( p \circ \Phi = \Phi \circ p \)

If \( \Phi = id_M \), the \( \Phi \) is the \( \Phi \) automorphism of \( \Phi \) fiber bundle \( \xi_0 \). For each \( x \in M \) the map:

\[\Phi : P \rightarrow P \]
is \( G \)-equivariant and therefore a diffeomorphism.

Let be \( \Phi \) a \( G \)-homomorphism of principal fiber bundle \( \xi_0 \). If the map:

\[h : F \rightarrow F\]
is smooth \( G \)-equivariant i.e. \( h(gv) = gh(v) \ \forall \ g \in G, \ v \in F \), then

\[
\Phi \times h : P \times F \rightarrow P \times F \text{ is } G \text{-equivariant and induced map }
\]

\[\varphi = \Phi \times h : E = P \times_G F \rightarrow E = P \times_G F\]
is a homomorphism of fiber bundle \( \xi_0 \). The diagram

\[
\begin{array}{c}
E = P \times_G F \xrightarrow{\varphi} E = P \times_G F \\
\pi \downarrow \downarrow \pi \\
P \xrightarrow{\varphi} M
\end{array}
\]

commute for a uniquely determined smooth map

\[\varphi : M \rightarrow M \text{ and } \Phi \times h \text{. [2].}\]

3. Invariable sections

Let \( \xi = (E, \pi M, F, G) \) be a \( G \)-fiber bundle with standard
fiber \( F \), \( \varphi : E \rightarrow E \) a homomorphism of fiber bundle \( \xi \)
and \( \sigma : M \rightarrow E \) a section of \( \xi \).

Definition. The section \( \sigma : M \rightarrow E \) is called \( \varphi \)-
invariable if the diagram

\[
\begin{array}{c}
E \xrightarrow{\varphi} E \\
\sigma \downarrow \downarrow \sigma \\
M \xrightarrow{\varphi} M
\end{array}
\]

commute i.e.

\[\sigma \circ \varphi = \varphi \circ \sigma .\]

If \( f : M \rightarrow M \) is a smooth map, we call the section \( \sigma \), \( f \)-invariable if there is a homomorphism \( \varphi \) of the fiber
bundle \( \xi \) which cover \( f \) , i.e. \( f = \varphi \), and the section \( \sigma \)
is \( \varphi \)-invariable.

Remark.

If the smooth section \( \sigma : M \rightarrow E \) of a fiber bundle \( \xi \) is
\( \varphi \) and \( \psi \)-invariable then \( \sigma \) is \( \varphi \circ \psi \)-invariable.

If the smooth section \( \sigma : M \rightarrow E \) of a fiber bundle \( \xi \) is
\( \varphi \)-invariable and \( \varphi \) is an automorphism of \( \xi \), then \( \sigma \)
is also \( \varphi^{-1} \)-invariable.

So, we have two invariance group of the section \( \sigma \):

1) The set of automorphisms \( \varphi \) of a fiber bundle \( \xi \) with properties that \( \sigma \) is \( \varphi \)-invariable.

2) The set of diffeomorphisms \( f \) of base manifold \( M \) of a fiber bundle \( \xi \) with properties that \( \sigma \) is \( f \)-invariable is a group.
Exemples.
1. Let $G$ be a Lie group and $(TG,p,G)$ the tangent bundle of $G$.

A vector field $X$ on $G$ is a section of the tangent bundle $(TG,p,G)$.

The vector field $X$ on $G$ is left invariant if the diagram

$$\begin{array}{ccc}
TG & \xrightarrow{TL_a} & TG \\
X & \Uparrow & X \\
G & \xrightarrow{L_a} & G
\end{array}$$

commute for any $a \in G$.

$L_a$ is the left translation of $G$ determined by $a \in G$ and $TL_a$ is tangent map of $L_a$.

Thus a vector field is left invariant if is $TL_a$ -invariant and $L_a^{-1}$ -invariant for any $a \in G$.

Then, $\{TL_a / a \in G\}$ and $\{L_a / a \in G\}$ are invariance group of $X$.

2. Symplectic manifolds arise naturally in abstract formulations of classical mechanics as the cotangent bundles of configuration manifolds.

Let $(M, \omega)$ be a symplectic manifold. The simplectic 2-form $\omega$ is a section of fiber bundle $\mathbb{A}\mathbb{M}$.

A symplectomorphism of symplectic manifold $(M, \omega)$ is a diffeomorphism $f : M \to M$ which preserves the symplectic form $f^* \omega = \omega$ so $\omega$ is $f$-invariable.

Then $\text{Sp}(M, \omega)$, the group of symplectomorphisms of a symplectic manifold $(M, \omega)$ is an invariance group of $\omega$.

If $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is standard symplectic form on $\mathbb{R}^{2n}$ with linear coordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, the invariance group of $\omega_0$, $\text{Sp}(\mathbb{R}^{2n}, \omega_0)$ is the symmetry group of “classical mechanics”.

From the Darboux theorem follows that all symplectic manifolds of dimension $2n$ looks locally like $(\mathbb{R}^{2n}, \omega_0)$.

3. Let $\xi = (E, \pi, M, O, L_m)$ a fiber bundle of geometric objects of type $\Phi$ with $N$ dimensions and order $r$ on the manifold $M$ ($\Phi$ is action of Lie group $L_m$ on the fiber type $O$).

The base space $M$ is a smooth manifold, the fiber $O$ is an open set of $\mathbb{R}^N$, $L_m = \text{Inv} J_0^1(R^n, R^n)$ is the Lie Group of invertible $r$-jets from $R^n$ to $R^n$ with source and target $0 \in R^n$.

If $\varphi$ is a diffeomorphism of a manifold $M$, there is a canonical automorphism $\varphi$ of $\xi$ [5].

The section $\sigma$ is $\varphi$-invariable if and only if $\sigma$ is $\varphi^{-1}$-invariable.

The set of this diffeomorphisms $\varphi$ so that $\sigma$ is $\varphi^{-1}$-invariable is a group.
4. Let $\xi = (T^r_k M, \pi^r_k, M, L^r_{\text{rm}}, L^r_m)$ be the $(k, r)$ velocities fiber bundle on a manifold $M$ [8]. $T^r_k M = J^r_0 (R^k, M)$ is the set of $(k, r)$ velocities fiber bundle on a manifold $M$, i.e. the set of $r$-jets from $R^k$ to $M$ with source $0 \in R^n$.

$I^r_{m,n} = J^r_0 (R^n, R^n)$ is the set of $r$-jets from $R^n$ to $R^n$ with source $0 \in R^n$ and target $0 \in R^n$.

The diffeomorphism $\varphi : M \to M$ determine the canonical automorphism $\varphi : J^r_0 f \in T^r_k M \to J^r_0 (\varphi \circ f) \in T^r_k M$.

For $r=k=1$ the fiber bundle $\xi$ is tangent bundle $(TM, \pi, M, R^n, L^r_m)$ and $\varphi = \varphi : TM \to TM$ is classical differential map of $\varphi$.

5. Let $\xi = (T^r_k M, \pi^r_k, M, L^r_{\text{rm}}, L^r_m)$ be the $(k, r)$ covelocities fiber bundle on a manifold $M$.

The diffeomorphism $\varphi : M \to M$ determine the canonical automorphism $\varphi : J^r_0 f \in T^r_k \ast M \to J^r_0 (f \circ \varphi) \in T^r_k \ast M$.

For $r=k=1$ the fiber bundle $\xi$ is cotangent bundle $(T^k \ast M, \pi^k, M, R^n, L^r_m)$ and $\varphi = \varphi : T^k \ast M \to T^k \ast M$.

A vector field $X$ (a section of tangent bundle) is $\varphi$-invariable if and only if $\varphi \circ X = X \circ \varphi$,

$$\varphi \circ X(x) = X \circ \varphi(x) \quad \forall x \in M$$

A covector field $\omega$ (a section of cotangent bundle) is $\varphi$-invariable if and only if

$$\varphi^* \omega(x) = \omega(\varphi(x)) \quad \forall x \in M$$

According with precedent proposition a vector (covector) field is $\varphi$-invariable if and only its coordinates in a frame $u$ coincide with the coordinate in the frame $\Phi(u)$.

The local study of invariance of sections may use the similar method with method of mobile frame, taken a local section of principal associated fiber bundles.

5. Conclusion.

Studying invariance of sections in $G$-fiber bundles, we highlighted the invariance group of this sections and also have shown that we can replace a section under invariance study with a function defined on the total space of associated principal fiber bundle, with values in the fiber type. To illustrate the theory, some examples, describing invariance of classical sections (vector fields, forms, etc.), are presented.

References:


