Approximate polynomial solutions for the nonlinear temperature distribution equation of a thick rectangular fin

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Abstract: We present a method to obtain approximate polynomial solutions for a class of second order nonlinear differential equations arising in heat transfer problems. We apply this method to find an approximate polynomial solution for the nonlinear temperature distribution equation of a thick rectangular fin.

Key-Words: Nonlinear differential equation; boundary value problem; \( E \)-approximate analytical approximate polynomial solution; temperature distribution equation

1 Introduction

Most of the problems studied in the engineering fields, including heat transfer problems, are of a nonlinear nature, and thus are described by nonlinear equations, which are usually difficult to solve, especially using an analytical method.

Among the approximation methods previously used to find an approximate solution for equations of this type we mention the Adomian decomposition methods [1], collocation-type methods [2, 3] and methods based on Taylor polynomials [4], Chebyshev polynomials [5] and Bernstein polynomials [6]. Variational methods were also employed in the study of such problems, but usually in existence/unicity/multiplicity type-problems [7]. Another category of methods successfully employed to find approximate analytical solutions are perturbations methods, but the assumption of the small values of a parameter restricts their applications. New methods to approximately solve non-linear equations without small parameter were introduced in literature recently, such as the Homotopy Perturbation Method [9,10,11,12], the Homotopy Analysis Method [13,14], the Variational Iteration Method [15,16,17], the Parameter-Expanding Method [18,19,20], the Iteration Perturbation Method [21] the Optimal Iteration Parametrization Method [22].

The original method presented in this paper allows us to determine in an easy manner analytical polynomial approximate solutions with good precision for a class of nonlinear equations encountered in heat transfer problems.

2 Method of solution

In this paper we propose a method to find an approximate polynomial solution for the boundary value problem which models the temperature distribution in a thick rectangular fin radiating to free space with nonlinearity of higher order [23]:

\[
\begin{align*}
\theta'' &= -\alpha \theta^4, \quad 0 \leq t \leq 1 \\
\theta(1) &= 1 \\
\theta'(0) &= 0
\end{align*}
\]

The equation (1) is of the type:

\[
x'' = F(x(t))
\]

on the \([a, b]\) interval, where \(F\) is a continuously differentiable real function, and the boundary conditions can be considered of the type:

\[
x(a) = \alpha, \quad x(b) = \beta
\]

where \(\alpha, \beta\) are real constants.

We suppose that the problem (2, 3) admits a continuous solution \(f\) on the \([a,b]\) interval. Then there exists a sequence of polynomials \(P_n\) which converges to \(f\) for \(n \to \infty\) [5].

We consider the operator:

\[
D(x) = x^{(2)}(t) - F(x(t),t)
\]

In the process of finding approximate solutions for differential equations, we are interested in finding
approximate polynomial solutions \( x_{\text{app}} \) on the \([a, b]\) interval, solutions which satisfy the following condition:

\[
\left| R(t, x_{\text{app}}) \right| < \varepsilon
\]  

(4)

where

\[
R(t, x_{\text{app}}) = D(x_{\text{app}}(t)), \ t \in [a, b]
\]  

(5)

represents the error obtained by replacing the exact solution \( x \) with the polynomial approximation \( x_{\text{app}} \).

**Definitions:**

We call \( \varepsilon \)-approximate polynomial solution of the problem (2,3) an approximate polynomial solution \( x_{\text{app}} \) satisfying the relation (4).

We call weak \( \varepsilon \)-approximate polynomial solution of the problem (2,3) an approximate polynomial solution \( x_{\text{app}} \) satisfying the relation:

\[
\int_a^b \left| R(t, x_{\text{app}}) \right| dt < \varepsilon
\]

We will find a weak \( \varepsilon \)-polynomial solution of the type:

\[
\tilde{x}(t) = \sum_{k=0}^n c_k t^k
\]  

(6)

where the constants \( c_0, c_1, \ldots, c_n \) are calculated using the steps outlined in the following.

By substituting the approximate solution (6) in the equation (2) we obtain the following expression:

\[
\mathcal{R}(t, c_0, c_1, \ldots, c_n) = R(t, \tilde{x}) = \tilde{x}^{\prime\prime} - F(\tilde{x}, t)
\]

(7)

If we can find the constants \( c_0^0, c_1^0, \ldots, c_n^0 \) such that

\[
\mathcal{R}(t, c_0^0, c_1^0, \ldots, c_n^0) = 0
\]

for any \( t \geq 0 \), then by substituting \( c_0^0, c_1^0, \ldots, c_n^0 \) in (6) we obtain the exact solution of (2). In general this situation is rarely encountered in the polynomial approximation methods.

We will attach to the problem (2,3) the following real functional:

\[
J(c_0, c_1, \ldots, c_n) = \int_a^b \left| \mathcal{R}(t, c_0, c_1, \ldots, c_n) \right| dt, \ a \leq t \leq b
\]

(8)

The values \( c_0^0, c_1^0, \ldots, c_n^0 \) which give the minimum of the functional (8) will be computed from the conditions:

\[
\frac{\partial J}{\partial c_0} = 0, \ \frac{\partial J}{\partial c_1} = 0, \ldots, \ \frac{\partial J}{\partial c_n} = 0
\]

(9)

Using the constants \( c_0^0, c_1^0, \ldots, c_n^0 \) thus determined, we consider the polynomial:

\[
T_n(t) = \sum_{k=0}^n c_k^0 t^k
\]

(10)

The following convergence theorem holds:

**Convergence theorem:**

There exists \( n_0 \in N \) such that for any \( n \in N, \ n > n_0 \) it follows that \( T_n \) is a weak \( \varepsilon \)-approximate polynomial solution of the problem (2,3).

**Proof:** Based on the way the polynomial \( T_n \) is defined (10), the following inequality holds:

\[
\int_a^b \left| R(t, T_n) \right| dt < \int_a^b \left| R(t, P_n) \right| dt .
\]

Since \( P_n \) which converges to \( f \) for \( n \to \infty \), we have

\[
\lim_{n \to \infty} \int_a^b \left| R(t, T_n) \right| dt = 0 \text{ q.e.d.}
\]

**Remark:**

Any \( \varepsilon \)-approximate polynomial solution of the problem (2,3) is also a weak \( \varepsilon \cdot (b - a) \)-approximate polynomial solution, but the opposite is not always true.

Taking into account the above remark, for the numerical example considered in this paper we will determine first weak approximate polynomial solutions, \( \tilde{x}_{\text{app}} \).

Next we give an estimate of the error, computed as

\[
R_{\text{app}}(t, \tilde{x}_{\text{app}}) = \tilde{x}_{\text{app}}^{\prime\prime} - F(\tilde{x}_{\text{app}}^{\prime}, \tilde{x}_{\text{app}}^{\prime}, t), \ t \in [a, b].
\]

\( R_{\text{app}} \) represents the error obtained by replacing the exact solution \( x \) with the weak polynomial approximation \( \tilde{x}_{\text{app}} \).

If \( \left| R(t, x_{\text{app}}) \right| < \varepsilon \) then \( x_{\text{app}} \) is also a \( \varepsilon \)-approximate polynomial solution of the problem.

**Remarks:**

The constants \( c_0^0, c_1^0, \ldots, c_n^0 \) can also be determined using other method, such as, for example, collocation-type methods.

The method described above has the advantage, in comparison with other methods, that it can be
applied not only for weakly-nonlinear equations but also for strong nonlinear differential and integro-differential equations.

3 The temperature distribution in a thick rectangular fin radiating to free space

The boundary value problem (1) models the temperature distribution in a thick rectangular fin radiating to free space with nonlinearity of higher order [23].

For the case \( \alpha = 0.7 \), the 5th degree polynomial approximate analytical solution is:

\[
\hat{x}_{\text{app}} = 0.0177293 t^5 - 0.00350875 t^4 + 0.0118825 t^2 + 0.155253 t + 0.818644
\]

The following plot contains the graphical representation of this polynomial (green dashed line) together with the corresponding numerical solution of eq. (1) obtained by using a forth order Runge Kutta method (red line).

The graphs are practically overlapping.

Next we present the graphical representation of the error \( R \).

It follows that our 5th degree polynomial is a 0.002 approximate analytical solution of (1).

The following table presents a comparison between the numerical solution of eq. (1) obtained by using a forth order Runge Kutta method (Num_Sol, third column) and the 5th degree polynomial approximate solution computed here (\( \hat{x}_{\text{app}} \), second column). We compare the values of these approximations computed in \( t = 0.1, t = 0.2... t = 0.9 \). We also calculated the differences in absolute value between these values (\( \varepsilon_{\hat{x}_{\text{app}}} \), fourth column).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \hat{x}_{\text{app}} )</th>
<th>Num_Sol</th>
<th>( \varepsilon_{\hat{x}_{\text{app}}} )</th>
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</table>

4 Conclusion

The paper describes a method to determine an analytical polynomial approximate solution for a class of nonlinear problems of the type (2,3) arising in heat transfer. This method allows us to obtain approximate solutions with good precision, as confirmed by the chosen examples. A smaller value of the approximation error can be always obtained by using polynomials of higher degree.

This method can be easily extended for the case of nonlinear equations and systems, nonlinear differential systems, integral equations e.t.c., and as such it can be considered a powerful tool for the computation of approximate solutions for nonlinear problems.
References:


