Stationary Heat Transfer in System with Double Wall and Double Fins

TABITA BOBINSKA
[1]Faculty of Physics and Mathematics
University of Latvia
Zellu Street 8, LV1002 Riga
LATVIA
tabita.bobinska@gmail.com

MARGARITA BUIKE[1,2], ANDRIS BUIKIS[1,2]
[2]Institute of Mathematics and Computer Science
University of Latvia
Raina Blvd 29, LV1459 Riga
LATVIA
mbuike@lanet.lv, buikis@latnet.lv
http://ww3.lza.lv/scientists/buikis.htm

HYUNG HEE CHO
Department of Mechanical Engineering,
Yonsei University,
262 Seong Sanno, Seodaemun-Gu,
Seoul 120-749
KOREA
hhcho@yonsei.ac.kr

Abstract: - In this paper, we consider steady state heat conduction for double wall with double fins in 2D geometry. The stationary heat conduction problem is examined, when assigning third type linear boundary conditions. An approximate analytical solution is constructed by using conservative averaging method for L-type domains. Finally, the numerical results are presented.

Key-Words: - double wall with double fin, L-type domain, temperature fields, conservative averaging method, stationary heat conduction problem, boundary value problem

1 Introduction

There have been numerous studies on systems with extended surfaces where the entire structure is made of the same material. In many research areas, e.g., on modern computers, more complex elements have to be employed. Here, the element (see Fig.1), we'll call it *a double wall with double fins*, consists of a plain surface, that is roughened by adding densely distributed vertical nanowires, and then covered with some kind of coating, e.g., fluorine carbon.

Fig.1: 2D system with fins

Such micro/nano structures are often developed to enhance the performance of boiling heat transfer, (see, e.g., [6], [7]).

With this article we are beginning a new series of publications on systems with double wall and double fins. These types of mathematical models are new and have not been considered in the literature, e.g., [1] - [5], [8] - [10].

In this paper we focus on the simplest case when the process is stationary. Here the assembly is 2D and it has constant properties. And the process is linear and there are no heat sources or sinks present. As part of future work, we intend to consider transient problems where both linear and non-linear (when boiling process is present) conditions are examined, and to propose a different kind of 3D geometry (wall with pin fins of uniform cross-section).

2 Problem Formulation in 2D

Since the given system (see Fig.1) can be divided into several symmetrical parts, we will describe the problem for only one of those L-shaped parts (see Fig.2).

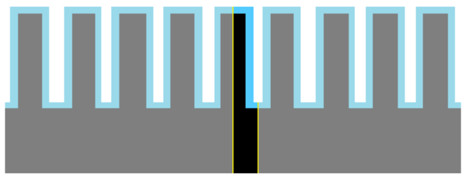

Fig.2: L-type domain

We are going to represent the original L-type domain as a finite union of canonical subdomains with appropriate conjugation conditions along the lines connecting two neighbour domains. We may therefore suppose that this L-shaped sample is made up from five rectangles (see Fig.3).

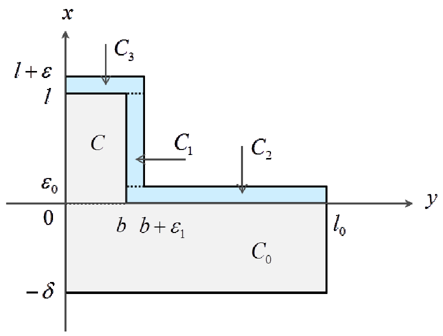

Fig.3: Definition of geometrical parameters for the sample

Let's assume that $V_i(x, y)$ denotes the temperature in the domain C_i with thermal conductivity k_i, and h_i is heat transfer coefficient. Here $k = k_0$ and $k_2 = k_3 = k_1$.

The temperature fields are described by

$$\frac{\partial^2 V_i}{\partial x^2} + \frac{\partial^2 V_i}{\partial y^2} = 0, \ x, y \in C_i.$$

Besides the equations, the following boundary conditions are imposed. We have a heat flux at $x = -\delta$:

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0(y).$$

Along the lines of symmetry, $y = 0$ and $y = l_0$ symmetry boundary conditions must be applied:

$$\frac{\partial V_i}{\partial n} = 0,$$

where n denotes the exterior normal to the boundary of the domains C_i. But at the other sides of the sample there is a heat exchange with the surrounding medium:

$$\frac{\partial V_i}{\partial n} + \beta_1^1 V_i = 0,$$

where $\beta_1^1 = \dfrac{h_1}{k_1}$.

Assuming that there is no contact resistance between the connected parts, we also add conjugation conditions:

$$V_i\big|_{x=\ldots} = V_j\big|_{x=\ldots},$$

$$\left.\frac{\partial V_i}{\partial x}\right|_{x=\ldots} = \frac{k_j}{k_i}\left.\frac{\partial V_j}{\partial x}\right|_{x=\ldots},$$

$$V_i\big|_{y=\ldots} = V_j\big|_{y=\ldots},$$

$$\left.\frac{\partial V_i}{\partial y}\right|_{y=\ldots} = \frac{k_j}{k_i}\left.\frac{\partial V_j}{\partial y}\right|_{y=\ldots}.$$

3 Approximate Solution of Problem

As the upper layer is quite thin, from now on we are going to assume that the temperature is uniform across the layer thickness. Hence from appropriate conjugation conditions we get these expressions:

$$V_2(x, y) = v_2(y) = V_0(0, y), \quad (1)$$

$$V_1(x, y) = v_1(x) = V(x, b), \quad (2)$$

$$V_3(x, y) = v_3(y) = V(l, y). \quad (3)$$

Because of that, we only need to solve the problem defined for the basic layer. So, we have the Laplace equations

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (4)$$

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} = 0 \quad (5)$$

with boundary conditions at the six sides. As we know from Section 2,

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta} = -Q_0(y) \quad (6)$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = 0, \ \left.\frac{\partial V_0}{\partial y}\right|_{y=l_0} = 0, \quad (7)$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = 0, \quad (8)$$

But to get boundary conditions at $x = 0$, $x = l$, and $y = b$, we use appropriate conjugation conditions

ISBN: 978-1-61804-118-0

and expressions (1) – (3). For example, at $x = 0$ we have

$$\left.\left(\frac{\partial V_2}{\partial x} + \beta_1^1 V_2\right)\right|_{x=\varepsilon_0} = \frac{1}{k_1}\left.\left(k_0\frac{\partial V_0}{\partial x} + h_1 V_0\right)\right|_{x=0} = 0$$

or

$$\left.\left(\frac{\partial V_0}{\partial x} + \beta_0^1 V_0\right)\right|_{x=0} = 0,\ y \in (b, l_0). \tag{9}$$

And

$$\left.\left(\frac{\partial V}{\partial x} + \beta_0^1 V\right)\right|_{x=l} = 0,\ y \in (0, b) \tag{10}$$

$$\left.\left(\frac{\partial V}{\partial y} + \beta_0^1 V\right)\right|_{y=b} = 0,\ x \in (0, l) \tag{11}$$

at $x = l$, $y = b$.

We also add conjugation conditions that state continuity of temperature and heat flux at the interface $x = 0$:

$$V_0\big|_{x=-0} = V\big|_{x=+0}, \tag{12}$$

$$\left.\frac{\partial V_0}{\partial x}\right|_{x=-0} = \left.\frac{\partial V}{\partial x}\right|_{x=+0}. \tag{13}$$

Using conservative averaging method (see [3], etc.), we are going to construct an approximate solution for the given problem.

3.1 Solution for the Fin

Let's use an exponential approximation in the y-direction for the 2D temperature field $V(x, y)$ in the fin. The general form of the function is given by

$$V(x,y) = \\ = f_0(x) + \left(e^{\rho y} - 1\right) f_1(x) + \left(1 - e^{-\rho y}\right) f_2(x), \tag{14}$$

where $\rho = b^{-1}$.

From symmetry condition (8) we find that $f_2(x) = -f_1(x)$. So, (14) assumes the following form:

$$V(x,y) = f_0(x) + 2\left(\cosh(\rho y) - 1\right) f_1(x). \tag{15}$$

Defining the function $v(x)$ as integral average value

$$v(x) = \rho\int_0^b V(x,y)\,dy, \tag{16}$$

and integrating the expression (15) with respect to y, we can find the function $f_1(x)$:

$$f_1(x) = \frac{v(x) - f_0(x)}{2\left(\sinh(1) - 1\right)}.$$

Let's substitute this in (14):

$$V(x,y) = \frac{\cosh(\rho y) - 1}{\sinh(1) - 1} v(x) + \\ + \frac{\sinh(1) - \cosh(\rho y)}{\sinh(1) - 1} f_0(x). \tag{17}$$

Applying the boundary condition (11), we have

$$\left(\rho\sinh(1) + \beta_0^1\left(\cosh(1) - 1\right)\right) v(x) + \\ + \left(-\rho\sinh(1) + \beta_0^1\left(\sinh(1) - \cosh(1)\right)\right) f_0(x) = 0.$$

And hence

$$f_0(x) = \psi v(x) \tag{18}$$

with

$$\psi = \frac{\sinh(1) + \beta_0^1 b\left(\cosh(1) - 1\right)}{\sinh(1) + \beta_0^1 b\left(\cosh(1) - \sinh(1)\right)}.$$

It follows immediately from (18) that

$$V(x,y) = v(x)\Phi(y), \tag{19}$$

where

$$\Phi(y) = \frac{\sinh(1) + \beta_0^1 b\left(\cosh(1) - \cosh(\rho y)\right)}{\sinh(1) + \beta_0^1 b\left(\cosh(1) - \sinh(1)\right)}. \tag{20}$$

Now the expression (14) (or (19)) contains only one unknown function – $v(x)$. In order to determine it, we use the definition (16) and from the partial differential equation (4) obtain an ordinary one for the unknown function:

$$\frac{d^2 v}{dx^2} + \rho\left.\frac{\partial V}{\partial y}\right|_{y=0}^{y=b} = 0,\ x \in (0, l).$$

The difference of the derivatives may be found via the boundary conditions (8) and (11), and (19):

$$\frac{d^2 v}{dx^2} - \lambda^2 v(x) = 0, \tag{21}$$

where

$$\lambda^2 = \rho\beta_0^1\Phi(b).$$

Applying the same operator (16) to (10) we get a boundary condition

$$v'(l) + \beta_0^1 v(l) = 0. \tag{22}$$

A solution to the problem (21), (22) is hence found to be

$$v(x) = c_1\left(e^{\lambda x} + \mu e^{-\lambda x}\right), \tag{23}$$

where

$$\mu = \frac{\lambda + \beta_0^1}{\lambda - \beta_0^1} e^{2\lambda l}$$

and c_1 is an unknown constant.

Therefore

$$V(x,y) = c_1\left(e^{\lambda x} + \mu e^{-\lambda x}\right)\Phi(y). \tag{24}$$

ISBN: 978-1-61804-118-0

3.2 Solution for the Wall

We act almost equally for the wall, and approximate the 2D temperature field $V_0(x,y)$ using exponential approximation in the x – direction:

$$V_0(x,y)= \\ = g_0(y)+\left(e^{-dx}-1\right)g_1(y)+\left(1-e^{dx}\right)g_2(y), \quad (25)$$

with $d=\delta^{-1}$.

Once again we use the properties of the function to solve for the unknown functions $g_i(y)$, $i=0,1,2$.

We obtain average value function by the integral

$$v_0(y)=d\int_{-\delta}^{0}V_0(x,y)dx. \quad (26)$$

Integrating (25) over the segment $(-\delta,0)$, gives

$$v_0(y)=g_0(y)+(e-2)g_1(y)+e^{-1}g_2(y). \quad (27)$$

The function $V_0(x,y)$ also satisfies the boundary condition (6):

$$eg_1(y)+e^{-1}g_2(y)=\delta Q_0(y). \quad (28)$$

So, combining (27) and (28), we have

$$g_1(y)=\frac{1}{2}\left(g_0(y)-v_0(y)+\delta Q_0(y)\right). \quad (29)$$

Finally, we conclude from (28) and (29) that formula (25) becomes

$$V_0(x,y)= \\ = g_0(y)\left(1+\frac{1}{2}\left(e^{-dx}-1\right)-e^2\frac{1}{2}\left(1-e^{dx}\right)\right) \\ + v_0(y)\left(-\frac{1}{2}\left(e^{-dx}-1\right)+e^2\frac{1}{2}\left(1-e^{dx}\right)\right) \\ + Q_0(y)\begin{pmatrix}\frac{1}{2}\delta\left(e^{-dx}-1\right)+ \\ +\left(\delta e-e^2\frac{1}{2}\delta\right)\left(1-e^{dx}\right)\end{pmatrix}. \quad (30)$$

3.2.1 Solution for the Right Part of the Wall

Here we examine the part of the base that occupies the domain $x\in(-\delta,0)$, $y\in(b,l_0)$.

Applying the boundary condition (9) to (30),

$$g_0(y)\left(-\frac{1}{2}d+\beta_0^1+\frac{1}{2}de^2\right) \\ + v_0(y)\left(\frac{1}{2}d-\frac{1}{2}de^2\right) \\ + Q_0(y)\left(-\frac{1}{2}-e+\frac{1}{2}e^2\right)=0.$$

We can rewrite this identity as

$$g_0(y)=v_0(y)a_0+Q_0(y)b_0, \quad (31)$$

where

$$a_0=\frac{d-de^2}{d-2\beta_0^1-de^2},\ b_0=-\frac{1+2e-e^2}{d-2\beta_0^1-de^2}.$$

Substituting (31) into the representation (30), yields

$$V_0(x,y)= \\ = v_0(y)\begin{pmatrix}a_0+\frac{1}{2}\left(a_0-1\right)\left(e^{-dx}-1\right)+ \\ +e^2\frac{1}{2}\left(1-a_0\right)\left(1-e^{dx}\right)\end{pmatrix} \\ + Q_0(y)\begin{pmatrix}b_0+\frac{1}{2}\left(b_0+\delta\right)\left(e^{-dx}-1\right)+ \\ +\left(\delta e-e^2\frac{1}{2}\left(\delta+b_0\right)\right)\left(1-e^{dx}\right)\end{pmatrix}. \quad (32)$$

This shows that the function now depends only on one unknown - function $v_0(y)$.

Let's integrate the main equation (5) in the x-direction

$$d\left.\frac{\partial V_0}{\partial x}\right|_{x=-\delta}^{x=0}+\frac{d^2v_0}{dy^2}=0. \quad (33)$$

The first addend can be found directly from the boundary condition (6), and (32). So (33) results in an ordinary differential equation

$$\frac{d^2v_0}{dy^2}-\kappa^2v_0(y)=\gamma Q_0(y),\quad y\in(b,l_0) \quad (34)$$

where

$$\kappa^2=-\frac{1}{2}d^2\left(1-e^2\right)\left(1-a_0\right)=d\beta_0^1a_0,$$

$$\gamma=-\frac{1}{2}d\left(b_0d\left(e^2-1\right)+\left(1-e\right)^2\right).$$

For simplicity reasons we henceforth assume the function $Q_0(y)$ to be constant, that is, $Q_0(y)=Q_0$.

Integrating (7_2), we obtain a boundary condition:

$$v_0'(l_0)=0. \quad (35)$$

The solution of (34), (35) is

$$v_0(y)=c_2\left(e^{\kappa y}+\mu_0e^{-\kappa y}\right)-\frac{\gamma Q_0}{\kappa^2} \quad (36)$$

with

$$\mu_0=e^{2\kappa l_0}$$

and c_2 as an unknown constant.

3.2.2 Solution for the Left Part of the Wall

To find the equation for the left part of the base, where $y\in(0,b)$, let us remind you that for $x=0$ the functions $V_0(x,y)$, $V(x,y)$ satisfy the conjugation conditions (12), (13). So, from (13) and (24) it follows that

$$d\frac{\partial V_0}{\partial x}\bigg|_{x=-0} = = d\frac{\partial V}{\partial x}\bigg|_{x=+0} = dc_1\lambda(1-\mu)\Phi(y). \quad (37)$$

If we now use (37) and the boundary condition (6), then equation (33) becomes

$$\frac{d^2 v_0}{dy^2} = -dc_1\lambda(1-\mu)\Phi(y) - dQ_0, \quad y \in (0,b).$$

Let us rewrite it as

$$\frac{d^2 v_0}{dy^2} = -c_1 B_1 - dQ_0 + c_1 B_2 \cosh(\rho y), \quad (38)$$

where

$$B_1 = d\lambda(1-\mu)\Phi_0, \quad B_2 = \beta_0^1 b d\lambda(1-\mu)\Phi_1,$$
$$\Phi_1 = \left(\sinh(1) + \beta_0^1 b(\cosh(1) - \sinh(1))\right)^{-1},$$
$$\Phi_0 = \left(\sinh(1) + \beta_0^1 b\cosh(1)\right)\Phi_1.$$

From (7_1) we get a boundary condition:

$$v_0'(0) = 0. \quad (39)$$

So, the solution of the problem (38), (39) is

$$v_0(y) = = c_1 B_2 \cosh(\rho y) b^2 + + \frac{1}{2}\left(-c_1 B_1 - dQ_0\right)y^2 + c_3. \quad (40)$$

When finding temperature for the left part of the base, one should take into account, that the function $g_0(y)$ is still unknown. We find that from the conjugation condition (12). Indeed, putting $x = 0$ in (30) and (24), we get that

$$g_0(y) = c_1(1+\mu)\Phi(y). \quad (41)$$

3.2.3 Conjugation of Solutions

We have just found solution to the given problem. But we are still left with finding the unknown constants in the formulas (23), (36) and (40). To determine those, we will need several requirements to be fulfilled. First, the temperatures $V_0(x,y)$, $V(x,y)$ must coincide at the contact point $x = 0, y = b$ between the fin and the right part of the wall, so

$$c_1(1+\mu)\Phi(b) = = c_2 a_0\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - a_0\frac{\gamma Q_0}{\kappa^2} + Q_0 b_0 \quad (42)$$

Second, the mean temperature values in the wall have continuity at $y = b$:

$$c_2\left(e^{\kappa b} + \mu_0 e^{-\kappa b}\right) - \frac{\gamma Q_0}{\kappa^2} = = c_1\left(B_2\cosh(1)b^2 - \frac{1}{2}B_1 b^2\right) - \frac{1}{2}dQ_0 b^2 + c_3. \quad (43)$$

Third, we claim that the mean fluxes also coincide at $y = b$:

$$c_2\kappa\left(e^{\kappa b} - \mu_0 e^{-\kappa b}\right) = = c_1\left(B_2\sinh(1)b - B_1 b\right) - dQ_0 b. \quad (44)$$

All the constants can be found from the system (42), (43), and (44). So, the approximate analytical solution to the problem is uniquely determined.

5 Numerical Results

To get numerical results for temperature distribution in the sample (see Fig.4 – 6), we used the following geometrical parameters:

$\delta = 500\mu m$

$l = 1\mu m$

$b = 5\cdot 10^{-2}\ \mu m$

$l_0 = 1\cdot 10^{-1}\ \mu m$

But for the termophysical properties we chose:

$h_1 = 11.085\cdot 10^{-4}\,W\mu m^{-2}K^{-1}$

$k_0 = 9\cdot 10^{-6}\,W\mu m^{-1}K^{-1}$

$Q_0 = 50K\mu m^{-1}$

although these are not quite precise.

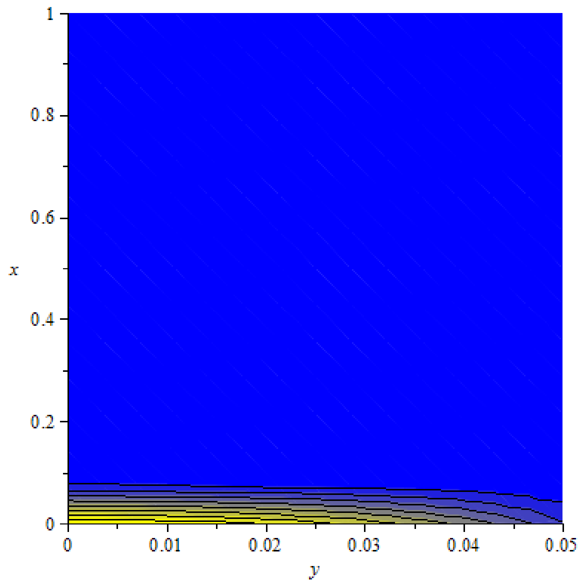

Fig.4: Temperature distribution in the fin

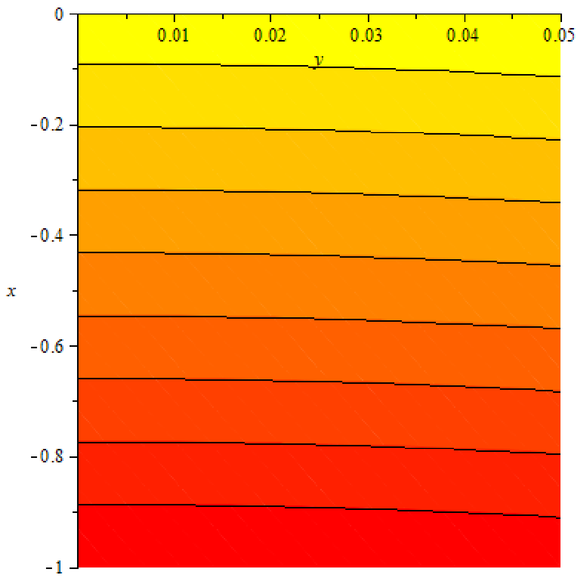

Fig.5: Temperature distribution in the left side of the base

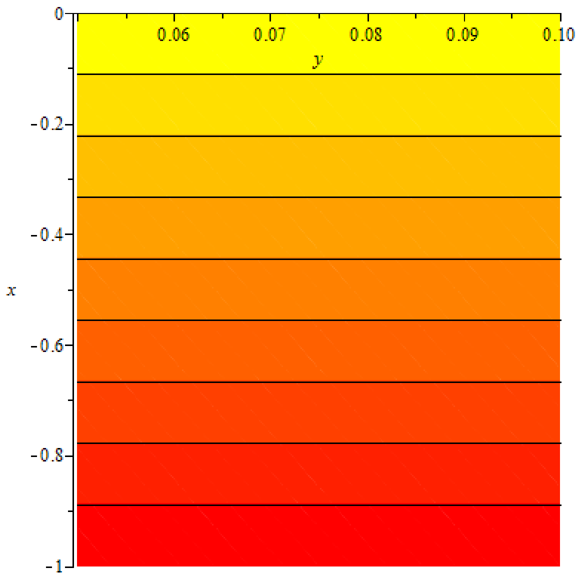

Fig.6: Temperature distribution in the right side of the base

6 Conclusion

We have given a formulation of a problem for stationary heat conduction in 2D double wall with double fins. And have constructed approximate analytical solution by using conservative averaging method for L-type domains.

Acknowledgments:

IEGULDĪJUMS TAVĀ NĀKOTNĒ

This work has been supported by the European Social Fund within the project No. 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008 and by the grant No. 09.1572 of the Latvian Council of Science.

References:

[1] A.H.M. Al-Essa, One Dimensional Finite Element Solution of the Rectangular Fin with Rectangular Perforations, *WSEAS Transactions on Heat and Mass Transfer*, Vol.1, No.10, 2006, pp. 762-768.

[2] M. Buike, Simulation of steady-state heat process for the rectangular fin-containing system, *Mathematical Modelling and Analysis*, Vol.4, 1999, pp. 33-43.

[3] M. Buike, A. Buikis, Approximate Analytical Two-Dimensional Solution for a Longitudinal Fin of Rectangular Profile, *Acta Universitatis Latviensis*, Vol.612, 1998, pp. 33-44.

[4] A. Buikis, M. Buike, N. Ulanova, Analytically-Numerical Solution for Transient Process in the System with Rectangular Fins, *Proc. of the 5th WSEAS Int. Conf. on HEAT and MASS TRANSFER, Acapulco, Mexico*, WSEAS Press, 2008, pp. 31-36.

[5] D.Q. Kern, A.D. Kraus, *Extended Surface Heat Transfer*, McGraw Hill Book Company, 1972.

[6] B.S. Kim, S. Shin, S.J. Shin, K.M. Kim, H.H. Cho, Control of Superhydrophilicity/ Superhydrophobicity using Silicon Nanowires via Electroless Etching Method and Fluorine Carbon Coatings, *Langmuir*, Vol.27, No.16, 2011, pp. 10148-10156.

[7] B.S. Kim, S. Shin, S.J. Shin, K.M. Kim, H.H. Cho, Micro-nano hybrid structures with manipulated wettability using a two-step silicon etching on a large area, *Nanoscale Research Letters, a SpringerOpen Journal*, Vol.6 (ID:333), 2011.

[8] A.D. Kraus, *Analysis and evaluation of extended surface thermal systems*, Hemisphere Publishing Corporation, 1982.

[9] M. Manzoor. *Lecture Notes in Engineering. Heat Flow Through Extended Surfaces Heat Exchangers*. Edited by C.A. Brebbia and S.A. Orszag. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[10] A.S. Wood, G.E. Tupholme., M.I.H. Bhatti, P.J. Heggs, Performance indicators for steady-state heat transfer through fin assemblies, *Trans. ASME Journal of Heat Transfer*, Vol.118, 1996, pp. 310-316.