Evaluation of the Lyapunov Exponent for Stochastic Dynamical Systems with Event Synchronization

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Abstract: We consider stochastic dynamical systems operating under synchronization constraints on system events. The system dynamics is represented by a linear vector equation in an idempotent semiring through second-order state transition matrices with both random and constant entries. As the performance measure of interest, the Lyapunov exponent defined as the asymptotic mean growth rate of the system state vector is examined. For a particular system, we derive a general expression for the exponent under the assumptions that the random matrices are independent and identically distributed, and their random entries have finite means. To illustrate, the exponent is calculated in the case when the random entries have exponential and continuous uniform distributions.

Key-Words: Stochastic dynamical system, Event synchronization, Idempotent semiring, Lyapunov exponent, Convergence in distribution

1 Introduction

Dynamical models describing time evolution of actual systems in engineering, manufacturing, economics, management, and other areas, frequently involve various synchronization mechanisms for events that may occur in the systems. Specifically, in a manufacturing system, production process of a new product cannot start before an appropriate raw material arrives. In a data communication system, a station initiates transmission of a new data packet to another station as soon as it receives an acknowledgment that the previous packet reached the destination.

The dynamics of many systems with event synchronization is described with a state vector representing the occurrence time of system events. The state evolution of the system can be then determined by a state transition matrix through an equation that is linear in the sense of an idempotent semiring [1–4]. Of particular interest and concern are stochastic models with random matrices, which can usually provide more adequate representation of actual systems.

In the analysis of system performance, one often needs to evaluate the mean growth rate of the state vector, which is normally referred to as the Lyapunov exponent of the system [1, 5]. The exponent presents the mean time of one system operation cycle (production cycle, communication session, etc), whereas its inverse can be regarded as the system throughput.

Exact evaluation of the Lyapunov exponent can be a rather cumbersome problem even for quite simple systems. Most of the known results are obtained for systems with second-order matrices that have exponentially distributed entries (see, e.g. [5–12]). There are also a few solutions obtained for discrete uniform, Bernoulli, and geometric distributions [5–7].

The aim of this paper is to overview and further extend the list of known results for systems with second-order matrices. We find the Lyapunov exponent when the matrix has its first row containing nonnegative random variable with finite mean and constant, while the other row consists entirely of zeros.

While being of independent interest, the system under study acquires evident practical and theoretical significance when considered in the context of all related models with known solution, which, taken together, can cover a range of actual systems and provide a basis for development of a general solution. Although it may appear that the system is quite simple, its related exact solution proves to be nontrivial.

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At the same time, a particular form of the matrix enables one to obtain the solution under general conditions and in a general form. Unlike known results, the solution does not rely on exponential distribution assumptions; instead, it is given by an expression that is independent of particular distribution and yet suitable for both numerical computations and further analysis.

The paper is organized as follows. We start with some motivating examples and an overview of known results. Furthermore, a new system with second-order matrix is described and its preliminary analysis is given. To compute the Lyapunov exponent, we first change variables from the state vector to a single state variable. A sequence of one-dimensional distribution functions associated with the variable is then introduced and convergence analysis for the sequence is performed. An expression for the exponent is derived as the mean of the limiting distribution. Finally, examples of calculation of the exponent in the case of exponential and continuous uniform distributions of the random entry in the state transition matrix are given.

2 Motivating Examples

We start with two example systems that are drawn from manufacturing and telecommunications. Other related examples can be found in [1, 5, 7].

2.1 Manufacturing

Consider a manufacturing system that consists of two production centers A and B. Each center produces its output based on use of an output from the other center. The operation of each center as well as of the entire system forms a sequence of production cycles.

Every cycle involves simultaneous production of a new output and transportation of the output from the previous cycle. A center completes its current cycle as soon as it finishes production of the current output and the output from the other center arrives. A current production cycle of the entire system comes to the end and the next cycle is initiated as soon as both centers A and B complete their related cycles.

Suppose that at the initial time, each center has its output available for delivery. Given production and transportation time at both centers, one is often interested in evaluating the mean time of production cycle as the number of cycles goes to infinity. The system throughput determined as the inverse of the mean cycle time is another performance measure of interest.

For every cycle $k = 1, 2, \ldots$ we introduce the following notation. By $x(k)$ and $y(k)$ we denote the respective cycle completion epochs at centers A and B. Let α_k and δ_k be the production time at centers A and B, and let β_k and γ_k be the transportation time from center B to A and from A to B.

With the condition that $x(0) = y(0) = 0$, the dynamics of the system is represented by two equations

$$x(k) = \max(x(k-1) + \alpha_k, y(k-1) + \beta_k), \quad (1)$$

$$y(k) = \max(x(k-1) + \gamma_k, y(k-1) + \delta_k). \quad (2)$$

The mean cycle time for the system is given by the limit

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \max(x(k), y(k)), \quad (3)$$

where λ can be considered as the mean asymptotic growth rate of the state vector, and it is normally referred to as the Lyapunov exponent for the system.

2.2 Telecommunications

Consider a system of two work stations A and B, which exchange messages over a communication network. Each station generates and sends messages in response to messages received from the other station. If a message is sent from a station, it immediately enters the network and then goes from one intermediate node to another until it arrives at the other station.

The operation of each station and of the whole system consists in a sequence of communication sessions. A new session at a station begins with generation of a new message and completes by sending the message to the other station. Once the message is generated, it is sent from the station as soon as a message from the other station arrives. A current communication session of the system starts when both nodes A and B complete their previous sessions, and lasts until they complete their sessions.

For every session $k = 1, 2, \ldots$ we denote by $x(k)$ and $y(k)$ the completion epochs for stations A and B. Let α_k and δ_k be the message generation time at station A and B, and β_k and γ_k be the message transmission time from B to A and from A to B.

Suppose that α_k, β_k, γ_k, and δ_k are given for all $k = 1, 2, \ldots$ Consider the problem of evaluating the mean communication session time which one can take as a natural performance measure for the system. As it is easy to see, the dynamic equations for the system have the same form as (1)-(2). Moreover, the mean communication session time can be represented as (3).

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3 System Representation

First note that equations (1)-(2) can be rewritten in terms of an idempotent semiring with the operation of maximum as semiring addition $\oplus$, and arithmetic addition as semiring multiplication $\otimes$, in the form

$$\begin{aligned} x(k) &= \alpha_k \otimes x(k-1) \oplus \beta_k \otimes y(k-1), \\ y(k) &= \gamma_k \otimes x(k-1) \oplus \delta_k \otimes y(k-1). \end{aligned}$$

Let us introduce a system state vector and a state transition matrix,

$$\boldsymbol{z}(k) = \begin{pmatrix} x(k) \\ y(k) \end{pmatrix}, \qquad A(k) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}.$$

We assume that for all $k = 1, 2, \dots$ the matrices $A(k)$ are independent and identically distributed.

Now we can replace the above scalar equations with one vector equation

$$\boldsymbol{z}(k) = A(k) \otimes \boldsymbol{z}(k-1),$$

where the matrix-vector multiplication $\otimes$ is thought of in the sense of scalar operations $\oplus$ and $\otimes$.

Consider the problem of evaluating the Lyapunov exponent defined by (3). Provided that all entries in $A(k)$ have finite means, it is not difficult to verify the existence of the limit at (3) by using the ergodic theorem in [14]. Application of the theorems to the system under consideration leads to the conclusions that the limit exists with probability one, and that the limiting value can be evaluated as

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \mathsf{E} \max(x(k), y(k)). \qquad (4)$$

Note that the last result allows one to reduce the problem of evaluating limits for sequences of random variables to that for sequences of their mean values.

3.1 Preliminary Results

Assume the entries α_k, β_k, γ_k, and δ_k in each matrix $A(k)$ to be independent random variables that have exponential distributions with respective parameters μ, ν, σ, and τ. A general solution to the problem is given in [10], where evaluation of the exponent reduces to solution of a system of linear equations and calculation of a linear functional on the solution. Another approach based on the joint Laplace transform of the state vector components is proposed in [5].

There are some cases when the Lyapunov exponent are derived in closed form as rational functions of the distribution parameters, including those with $\mu = \nu = \sigma = \tau$ [6, 7] and with $\mu = \tau$, $\nu = \sigma$ [5, 8].

Assume that the time to perform certain operations in a system (e.g., production of output at a center or transmission of messages between stations) is sufficiently short as compared to that of the other operations. Under this assumption, one can usually set the duration of the operations equal to zero with negligible loss of accuracy, and then consider the matrix $A(k)$ with zero in place of one or more its entries.

As a more general case, one can consider a situation when there is an operation time that remains constant for all cycles in a system, which leads to matrices with arbitrary nonnegative constant elements.

Suppose that some of the random entries in each matrix $A(k)$ are replaced with nonnegative constants. Systems with matrices having zero constants are analyzed in [5, 9, 10]. Examples of the considered matrices together with related results are given in Table 1.

$A(k)$	λ
$\begin{pmatrix} \alpha_k & 0 \\ 0 & \delta_k \end{pmatrix}$	$\frac{\mu^4+\mu^3\tau+\mu^2\tau^2+\mu\tau^3+\tau^4}{\mu\tau(\mu+\tau)(\mu^2+\tau^2)}$
$\begin{pmatrix} 0 & \beta_k \\ \gamma_k & 0 \end{pmatrix}$	$\frac{4\nu^2+7\nu\sigma+4\sigma^2}{6\nu\sigma(\nu+\sigma)}$
$\begin{pmatrix} \alpha_k & \beta_k \\ 0 & 0 \end{pmatrix}$	$\frac{2\mu^4+7\mu^3\nu+10\mu^2\nu^2+11\mu\nu^3+4\nu^4}{\mu\nu(\mu+\nu)^2(3\mu+4\nu)}$

Table 1: Results for matrices with zero entries.

Let c be an arbitrary nonnegative constant. Related solutions (see, e.g. [12, 13]) are listed in Table 2.

4 A Stochastic Dynamical System

Now we extend the above results to systems that have matrices of the form

$$A(k) = \begin{pmatrix} \alpha_k & c \\ 0 & 0 \end{pmatrix}.$$

that gives a more general solution where the assumption of exponential distribution is no longer needed.

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$A(k)$	λ
$\begin{pmatrix} \alpha_k & 0 \\ 0 & c \end{pmatrix}$	$c + \frac{2e^{-3\mu c}}{\mu(2-4\mu c e^{-\mu c}+\mu^2 c^2 e^{-2\mu c})}$
$\begin{pmatrix} c & \beta_k \\ 0 & 0 \end{pmatrix}$	$c + \frac{2e^{-2\nu c}}{\nu(2+e^{-2\nu c})}$
$\begin{pmatrix} \alpha_k & c \\ c & c \end{pmatrix}$	$c + \frac{2e^{-\mu c}}{\mu(2+e^{-\mu c}-2e^{-2\mu c}+e^{-3\mu c})}$

Table 2: Results for matrices with a constant $c \geq 0$.

We assume α_k to be a nonnegative random variable with the distribution function $F(t)$ and finite mean a, and c to be a nonnegative constant.

To solve the problem of evaluating the Lyapunov exponent, we first refine equations (1)-(2) to get

$$x(k) = \max(x(k-1) + \alpha_k, y(k-1) + c), \quad (5)$$
$$y(k) = \max(x(k-1), y(k-1)). \quad (6)$$

Since the conditions $\alpha_k \geq 0$ and $c \geq 0$ are fulfilled, for all $k = 0, 1, \ldots$ we have

$$\begin{aligned} x(k) &= \max(x(k-1) + \alpha_k, y(k-1) + c) \\ &\geq \max(x(k-1), y(k-1)) = y(k). \end{aligned}$$

With this inequality, dynamic equations (5)-(6) take the form

$$x(k) = \max(x(k-1) + \alpha_k, y(k-1) + c), \quad (7)$$
$$y(k) = x(k-1), \quad (8)$$

whereas (4) reduces to the limit

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \mathsf{E} x(k). \quad (9)$$

Consider two particular cases when the solution can be immediately obtained based on simple algebra. Let us first assume α_k to be bounded below such that $\alpha_k \geq c$. Through iteration of (7), we get

$$x(k) = x(k-1) + \alpha_k = \cdots = \alpha_1 + \cdots + \alpha_k,$$

and then immediately arrive at $\lambda = \mathsf{E}\alpha_1 = a$.

Furthermore, we examine a system where the random variables α_k have a finite support such that the condition $\alpha_k \leq c/2$ holds for all k. It is not difficult to verify based on (5)-(6) that

$$x(k) = \begin{cases} \frac{c}{2}k + \max(\alpha_2, \alpha_4, \ldots, \alpha_k), & \text{if } k \text{ is even;} \\ \frac{c}{2}(k+1), & \text{if } k \text{ is odd.} \end{cases}$$

Application of (9) gives the result $\lambda = c/2$.

5 Lyapunov Exponent Evaluation

We now turn to a general case of a nonnegative random variable α_k with either infinite support or a finite support that does not satisfy the above condition.

5.1 New State Variable and Its Distribution

First we introduce a new state variable

$$X(k) = x(k) - x(k-1)$$

for all $k = 1, 2, \ldots$, and note that

$$x(k) = X(1) + \cdots + X(k).$$

Furthermore, we further reduce equations (7)-(8) to one equation in the new variable

$$X(k) = \max(\alpha_k, c - X(k-1)),$$

and rewrite (9) in the form

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathsf{E} X(k). \quad (10)$$

Let us define the probability distribution function

$$\Phi_k(t) = \mathsf{P}\{X(k) < t\}.$$

It is easy to verify that

$$\Phi_0(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

For each $k \geq 1$ we have

$$\Phi_k(t) = \mathsf{P}\{\max(\alpha_k, c - X(k-1)) < t\}.$$

After evaluation of the probability, we arrive at the recursive equation

$$\Phi_k(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ F(t)(1 - \Phi_{k-1}(c-t)), & \text{if } 0 < t \leq c; \\ F(t), & \text{if } t > c. \end{cases}$$

5.2 Convergence of Distributions

Let us consider the above recursive equation when $0 < t \leq c$ under the condition that the distribution defined by $F(t)$ has either infinite support or a finite support that is not bounded from above by $c/2$.

We introduce $G(t) = F(t)F(c-t)$ and verify that $G(t) < 1$. It is clear that the inequality is valid in the case of the distribution with infinite support.

Suppose the distribution has a finite support with an upper bound $U > c/2$. To show that $G(t) < 1$, assume to the contrary that $G(t) = F(t)F(c-t) = 1$. From this equality we have $U \leq t \leq c-U$ and then $U \leq c/2$, which is a contradiction.

For all even $k = 2m$, by iterating the equation, we arrive at the function

$$\Phi_{2m}(t) = 1-(1-F(t))(1+G(t)+\cdots+G^{m-1}(t)).$$

The sum on the right-hand side presents an initial part of the geometric series with the common ratio $G(t) < 1$. In the limit as m tends to infinity, for all t under consideration, the subsequence $\Phi_{2m}(t)$ goes to

$$\Phi(t) = 1 - \frac{1-F(t)}{1-G(t)} = \frac{F(t)-G(t)}{1-G(t)}.$$

In the same way, for all $k = 2m+1$, we verify that $\Phi_{2m+1}(t) \to \Phi(t)$ as $m \to \infty$.

From the convergence of the subsequences to a common limit, it follows that when $0 < t \leq c$, the entire sequence $\Phi_k(t)$ also converges to

$$\Phi(t) = \frac{F(t)-G(t)}{1-G(t)} = \frac{F(t)(1-F(c-t))}{1-F(t)F(c-t)}.$$

Taking into account that $\Phi_k(t) = F(t)$ if $t > c$, we finally conclude that for all t, as $k \to \infty$, the sequence of functions $\Phi_k(t)$ converges to the function

$$\Phi(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \frac{F(t)(1-F(c-t))}{1-F(t)F(c-t)}, & \text{if } 0 < t \leq c; \\ F(t), & \text{if } t > c. \end{cases}$$

It is clear that $\Phi(t)$ is a distribution function for a random variable, which will be referred to as X.

5.3 Evaluation of the Lyapunov Exponent

Since $X(k)$ converges to X in distribution, we see that $\mathsf{E}X(k) \to \mathsf{E}X$ as $k \to \infty$. Therefore, the limit at (10) becomes

$$\lambda = \mathsf{E}X = \int_0^\infty t d\Phi(t).$$

Substitution of $\Phi(t)$ and rearrangement of terms gives the representation

$$\lambda = \int_0^\infty t dF(t) + \int_0^c t d\left(\frac{F(t)(1-F(c-t))}{1-F(t)F(c-t)} - F(t)\right),$$

where the first integral is equal to a.

Denote the second integral by I. By applying integration by parts, we get

$$I = \int_0^c \frac{F(t)F(c-t)(1-F(t))}{1-F(t)F(c-t)} dt.$$

Finally, we arrive at the solution

$$\lambda = a + \int_0^c \frac{F(t)F(c-t)(1-F(t))}{1-F(t)F(c-t)} dt. \qquad (11)$$

Note that the last result is derived when the probability distribution has either infinite support or a finite support with its upper boundary $U > c/2$. In the case of finite support with $U \leq c/2$, there is the solution $\lambda = c/2$ obtained in a previous section.

6 Examples

In this section we give examples of calculating λ with (11) for some particular probability distributions.

6.1 Exponential Distribution

First consider the case of the exponential distribution with distribution function

$$F(t) = \max(0, 1-e^{-\mu t}).$$

In (11), we have $a = 1/\mu$, while the integral takes the form

$$I = \int_0^c \frac{dt}{1-e^{-\mu(c-t)}(1-e^{\mu t})} - \frac{1}{\mu}(1-e^{-\mu c}).$$

Calculation of the integral gives

$$\lambda = \frac{c}{2} + \frac{e^{-\mu c}}{\mu} + \frac{3\arctan\sqrt{4e^{\mu c}-1}-\pi}{\mu\sqrt{4e^{\mu c}-1}}.$$

6.2 Continuous Uniform Distribution

Consider the uniform distribution on the segment $[0, 2a]$ with the distribution function

$$F(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \frac{t}{2a}, & \text{if } 0 < t \leq 2a; \\ 1, & \text{if } t > 2a. \end{cases}$$

To evaluate λ, we examine three cases. First we assume that $c \le 2a$ and apply (11). The integral in (11) reduces to that of the form

$$I = \frac{1}{2a}\int_0^c \frac{t(c-t)(2a-t)}{4a^2 - t(c-t)}dt.$$

After calculating the integral, we get

$$\lambda = \frac{(2a-c)^2}{4a} + \frac{4a(4a-c)}{\sqrt{16a^2-c^2}} \arctan\frac{c}{\sqrt{16a^2-c^2}}.$$

Suppose that $2a < c \le 4a$. Now we have

$$I = \frac{1}{2a}\int_0^{c-2a} t dt + \frac{1}{2a}\int_{c-2a}^{2a} \frac{t(c-t)(2a-t)}{4a^2 - t(c-t)}dt.$$

Evaluation of the integrals gives

$$\lambda = c - 2a + \frac{4a(4a-c)}{\sqrt{16a^2-c^2}} \arctan\frac{4a-c}{\sqrt{16a^2-c^2}}.$$

When $c > 4a$, we arrive at the case of finite support with the upper boundary $U = 2a \le c/2$. The solution now takes the form $\lambda = c/2$.

By combining the results for all cases, we finally get the solution given by

$$\lambda = \begin{cases} \frac{(2a-c)^2}{4a} + 4a\sqrt{\frac{4a-c}{4a+c}} \arctan\frac{c}{\sqrt{(4a-c)(4a+c)}}, & \text{if } c \le 2a; \\ c - 2a + 4a\sqrt{\frac{4a-c}{4a+c}} \arctan\sqrt{\frac{4a-c}{4a+c}}, & \text{if } 2a < c \le 4a; \\ \frac{c}{2}, & \text{if } c > 4a. \end{cases}$$

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