On the Equivalence of Cost-Extended Control Systems on Lie Groups

RORY BIGGS
Rhodes University
Department of Mathematics
PO Box 94, 6140 Grahamstown
SOUTH AFRICA
rorybiggs@gmail.com

CLAUDIU C. REMSING
Rhodes University
Department of Mathematics
PO Box 94, 6140 Grahamstown
SOUTH AFRICA
c.c.remsing@ru.ac.za

Abstract: We consider the equivalence of cost-extended control systems corresponding to certain invariant optimal control problems. We prove that two equivalent cost-extended systems have the same optimal controlled trajectories and the same extremal curves (up to a diffeomorphism). We also prove that equivalent cost-extended systems are lifted (via the Pontryagin Maximum Principle) to linearly equivalent Hamilton-Poisson systems. A few illustrative examples are discussed.

Key–Words: Invariant control system, feedback equivalence, Hamilton-Poisson system.

1 Introduction

We consider an equivalence relation on a certain class of cost-extended control systems. Specifically, we consider equivalence on the class corresponding to left-invariant optimal control problems on Lie groups with fixed terminal time, affine dynamics, and affine quadratic cost. Formally, such problems are given by

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G} \qquad (1)$$

$$g(0) = g_0, \quad g(T) = g_1 \qquad (2)$$

$$\mathcal{J} = \int_0^T (u(t)-\mu)^\top Q\,(u(t)-\mu)\,dt \to \min\,. \qquad (3)$$

Here G is a (real, finite-dimensional) connected Lie group with Lie algebra $\mathfrak{g}$, A, $B_1, \ldots, B_\ell \in \mathfrak{g}$ where $B_1, \ldots, B_\ell$ are linearly independent, $u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell$, $\mu \in \mathbb{R}^\ell$, and Q is a positive-definite $\ell \times \ell$ matrix.

Such an equivalence (between two cost-extended systems) establishes a one-to-one correspondence between the associated optimal controlled trajectories. Likewise, it establishes a one-to-one correspondence between the associated (normal) extremal curves. Each cost-extended system can be lifted (via the Pontryagin Maximum Principle) to a quadratic Hamilton-Poisson system. If two cost-extended systems are equivalent, then the associated Hamilton-Poisson systems are linearly equivalent.

A number of illustrative examples, pertaining to the classification of cost-extended systems and Hamilton-Poisson systems, are discussed.

2 Preliminaries

2.1 The Lie-Poisson structure

Let $\mathfrak{g}$ be a (real) Lie algebra. The dual space $\mathfrak{g}^*$ has a natural Poisson structure, the *(minus) Lie-Poisson structure* (cf. [13]). This structure is given by $\{F, G\}(p) = -p\,([dF(p), dG(p)])$, $p \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$. (Here $dF(p), dG(p) \in \mathfrak{g}^{**} = \mathfrak{g}$.) The Poisson space $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir function* if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$.

To each function $H \in C^\infty(\mathfrak{g}^*)$, we associate a *Hamiltonian vector field* $\vec{H}$ on $\mathfrak{g}^*$ specified by $\vec{H}[F] = \{H, F\}$. Two vector fields $\vec{F}$ and $\vec{G}$ (on $\mathfrak{g}^*$ and $\mathfrak{h}^*$, respectively) are *compatible* with a smooth map $\phi : \mathfrak{g}^* \to \mathfrak{h}^*$ if $T_p\phi \cdot \vec{F}(p) = \vec{G}(\phi(p))$ for $p \in \mathfrak{g}^*$.

A linear map $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ is a *linear Poisson morphism* if $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson morphisms are exactly the dual maps of Lie algebra morphisms.

2.2 Invariant optimal control

Invariant control systems on Lie groups were first considered in 1972 by Brockett [9] and by Jurdjevic and Sussmann [12]. A *left-invariant control affine system* Σ is a control system of the form

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)$$

where $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$. Here the *parametrization map* $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine

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map. The "product" $g\,\Xi\,(\mathbf{1}, u)$ is to be understood as $T_{\mathbf{1}}L_g \cdot \Xi\,(\mathbf{1}, u)$, where $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ is the left translation by g. Note that the *dynamics* $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is invariant under left translations, i.e., $\Xi\,(g, u) = g\,\Xi\,(\mathbf{1}, u)$. We shall denote such a system by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [5]).

The image set $\Gamma = \operatorname{im} \Xi\,(\mathbf{1}, \cdot)$, called the *trace* of Σ, is an affine subspace of $\mathfrak{g}$. Accordingly, $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$. A system Σ is called *drift-free* if $A = 0$, *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Also, Σ is said to have *full rank* if its trace generates the whole Lie algebra (i.e., the smallest Lie algebra containing Γ is $\mathfrak{g}$).

The admissible controls are piecewise-continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A *controlled trajectory* is a pair $(g(\cdot), u(\cdot))$ where $g(\cdot) : [0, T] \to \mathsf{G}$ is an absolutely continuous curve such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. For more details about (invariant) control systems see, e.g., [2, 11, 12, 14].

Two systems are detached feedback equivalent if there exists a "detached" feedback transformation which transforms the first system to the second (see [6, 10]). Equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. More precisely, let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control affine systems. Σ and Σ' are called *detached feedback equivalent* (shortly DF-equivalent) if there exists a diffeomorphism $\Phi : \mathsf{G} \times \mathbb{R}^\ell \to \mathsf{G}' \times \mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \varphi(u))$ such that $T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$ for $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. Full-rank systems Σ and Σ' are DF-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$ [6].

Now, consider an *optimal control problem* given by the specification of (i) a left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$, (ii) an affine quadratic cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$, and (iii) boundary data $\mathcal{B}(g_0, g_1, T)$, consisting of an initial state $g_0 \in \mathsf{G}$, a target state $g_1 \in \mathsf{G}$ and a fixed terminal time $T > 0$. Explicitly, we want to *minimize* the functional $\mathcal{J} = \int_0^T \chi(u(t))\, dt$ over the trajectory-control pairs of Σ subject to the boundary conditions (2). To such a problem, we associate a *cost-extended system* (Σ, χ) via the boundary data $\mathcal{B}(g_0, g_1, T)$, and vice-versa.

The *Pontryagin Maximum Principle* is a necessary condition for optimality. To an optimal control problem (1)-(2)-(3), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$H_u^\lambda(\xi) = \lambda\, \chi(u) + p\,(\Xi(\mathbf{1}, u))\,. \tag{4}$$

Here $\xi = (g, p) \in T^*\mathsf{G}$. The Maximum Principle can then be stated as follows.

Maximum Principle. *Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ defined over the interval $[0, T]$ is a solution for the optimal control problem (1)-(2)-(3). Then, there exists a curve $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ with $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, and a real number $\lambda \leq 0$, such that the following conditions hold for almost every $t \in [0, T]$:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{5}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{6}$$

$$H^\lambda_{\bar{u}(t)}\,(\xi(t)) = \max_u H^\lambda_u\,(\xi(t)) = \textit{constant}. \tag{7}$$

3 Equivalence

Let (Σ, χ) and (Σ', χ') be two cost-extended systems. (Σ, χ) and (Σ', χ') are said to be *cost equivalent* (shortly C-equivalent) if there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$ and $\chi' \circ \varphi = r\chi$ for some $r > 0$.

Remark 1 The "dynamics-preserving" condition is just that of DF-equivalence (on full-rank systems). The "cost-preserving" condition is partially motivated by the following. Each cost χ on $\mathbb{R}^\ell$ induces a strict partial ordering $u < v \iff \chi(u) < \chi(v)$. It turns out that χ and χ' induce the same strict partial ordering on $\mathbb{R}^\ell$ if and only if $\chi = r\chi'$ for some $r > 0$.

The following result is easy to prove (cf. [5, 6]).

Proposition 2 *(Σ, χ) and (Σ', χ') are C-equivalent if and only if there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and $\chi' \circ \varphi = r\chi$ for some $r > 0$.*

Corollary 3 *If (Σ, χ) and (Σ', χ') are C-equivalent, then Σ and Σ' are DF-equivalent.*

Corollary 4 *Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be two full-rank systems. If Σ and Σ' are DF-equivalent with respect to a feedback transformation φ, then $(\Sigma, \chi \circ \varphi)$ and (Σ', χ) are C-equivalent for any cost χ.*

The cost χ of a system may always be transformed into $\chi(u) = u^\top u$ by "complicating" the parametrization map.

Proposition 5 *Any cost-extended system (Σ, χ) is C-equivalent to a system (Σ', χ'), where $\mathsf{G}' = \mathsf{G}$, $\ell' = \ell$, $\Gamma' = \Gamma$, and $\chi'(u) = u^\top u$.*

Proof: Let $\chi(u) = (u-\mu)^\top Q\,(u-\mu)$. As Q is symmetric and positive-definite, there exists (by Sylvester's law of inertia) a non-singular real matrix R such that $R^\top Q R = I$. Let $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$, $u \mapsto Ru + \mu$ and let $\Xi' : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$, $\Xi'(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$. Then $T_{\mathbf{1}} id_{\mathsf{G}} \cdot \Xi'(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$ and $\chi(\varphi(u)) = u^\top R^\top Q R u = u^\top u$. The result follows by proposition 2. □

Let $(g(\cdot), u(\cdot))$ be a controlled trajectory, defined over an interval $[0, T]$, of a cost-extended system (Σ, χ). $(g(\cdot), u(\cdot))$ is a *virtually optimal controlled trajectory* (shortly VOCT) if it is a solution for the associated optimal control problem with boundary data $\mathcal{B}(g(0), g(T), T)$. $(g(\cdot), u(\cdot))$ is a (normal) *extremal controlled trajectory* (shortly ECT) if it satisfies the necessary conditions of the Pontryagin Maximum Principle (with $\lambda < 0$). Clearly, any VOCT is an ECT. A map $\phi \times \varphi$ defining a C-equivalence between two cost-extended systems establishes a one-to-one correspondence between their respective VOCTs (and ECTs).

Theorem 6 *Suppose $\Phi = \phi \times \varphi$ defines a C-equivalence between (Σ, χ) and (Σ', χ'). Then $(g(\cdot), u(\cdot))$ is a VOCT if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT. Likewise, $(g(\cdot), u(\cdot))$ is an ECT if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT.*

Proof: Suppose $(g(\cdot), u(\cdot))$ is a controlled trajectory of (Σ, χ) and $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT of (Σ', χ'). Further suppose $(g(\cdot), u(\cdot))$ is not a VOCT of (Σ, χ). Then there exists another controlled trajectory $(h(\cdot), v(\cdot))$ such that $h(0) = g(0)$, $h(T) = g(T)$, and $\mathcal{J}(v(\cdot)) < \mathcal{J}(u(\cdot))$. Hence $\big(\phi \circ h(\cdot), \varphi \circ v(\cdot)\big)$ is a controlled trajectory of (Σ', χ'). It is then easy to show that $\int_0^T \chi'(\varphi(v(t)))\,dt < \int_0^T \chi'(\varphi(u(t)))\,dt$. However, this contradicts the fact that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT of (Σ', χ'). Thus if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT, then so is $(g(\cdot), u(\cdot))$. As $\Phi^{-1} = \phi^{-1} \times \varphi^{-1}$ defines a C-equivalence between (Σ', χ') and (Σ, χ), the same argument can be used to show the converse.

Now assume $(g(\cdot), u(\cdot))$ is a controlled trajectory of Σ such that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT of (Σ', χ'). The Hamiltonian functions (4), associated to (Σ, χ) and (Σ', χ'), are given by $H_u(g, p) = p(\Xi(\mathbf{1}, u)) - \chi(u)$ and $H'_{u'}(g', p') = p'(\Xi'(\mathbf{1}, u')) - \chi'(u)$, respectively (with $\lambda = -1$). Recall that $\xi(\cdot) : t \mapsto (g(t), p(t))$ is an integral curve of $\vec{H}^\lambda_{u(t)}$ if and only if $\dot g(t) = \Xi(g(t), u(t))$ and $\dot p(t) = \mathrm{ad}^* \,\Xi(\mathbf{1}, u(t)) \cdot p(t)$ (cf. [11]). Here $(\mathrm{ad}^* A \cdot p)(B) = p([A, B])$ for $A, B \in \mathfrak{g}$ and $p \in \mathfrak{g}^*$. Let $g'(\cdot) = \phi \circ g(\cdot)$ and $u'(\cdot) = \varphi \circ u(\cdot)$. As $(g'(\cdot), u'(\cdot))$ is an ECT, there exists a curve $p'(\cdot) : [0, T] \to \mathfrak{g}^*$ such that $\xi'(t) = (g'(t), p'(t))$ satisfies (6)-(7). Also, by (6), we have $\dot g'(t) = \Xi'(g'(t), u'(t))$ and $\dot p'(t) = \mathrm{ad}^* \,\Xi'(\mathbf{1}, u'(t)) \cdot p'(t)$.

Let $p(\cdot) = \frac{1}{r} (T_{\mathbf{1}}\phi)^* \cdot p'(\cdot)$ and $\xi(t) = (g(t), p(t))$. (Here $r > 0$ is the constant specified by $\chi' \circ \varphi = r\chi$.) We show that $\xi(\cdot)$ satisfies (6)-(7). By assumption we have that $\dot g(t) = \Xi(g(t), u(t))$. Thus, to satisfy (6), we are left to show that $\dot p(t) = \mathrm{ad}^*\Xi(\mathbf{1}, u(t)) \cdot p(t)$. We have (for $A \in \mathfrak{g}^*$)

$$\begin{aligned}
\big(\dot p(t)\big)(A) &= \tfrac{1}{r}\left((T_{\mathbf{1}}\phi)^* \cdot \dot p'(t)\right)(A) \\
&= \tfrac{1}{r}\, p'(t)\big(\big[\Xi'(\mathbf{1}, u'(t)), T_{\mathbf{1}}\phi \cdot A\big]\big) \\
&= \tfrac{1}{r}\, p'(t)(T_{\mathbf{1}}\phi \cdot [\Xi(\mathbf{1}, u(t)), A]) \\
&= (\mathrm{ad}^*\Xi(\mathbf{1}, u(t)) \cdot p(t))\,(A).
\end{aligned}$$

Next we show that $\xi(\cdot)$ satisfies (7). Suppose there exist $\tilde u \in \mathbb{R}^\ell$ and $\tilde t \in [0, T]$ such that $H_{u(\tilde t)}(g(\tilde t), p(\tilde t)) < H_{\tilde u}(g(\tilde t), p(\tilde t))$. Hence

$$\begin{aligned}
&\tfrac{1}{r}\,(T_{\mathbf{1}}\phi)^* \cdot p'(\tilde t) \cdot \Xi(\mathbf{1}, u(\tilde t)) - \chi(u(\tilde t)) \\
&\quad < \tfrac{1}{r}\,(T_{\mathbf{1}}\phi)^* \cdot p'(\tilde t) \cdot \Xi(\mathbf{1}, \tilde u) - \chi(\tilde u).
\end{aligned}$$

Using the available identities and simplifying we get $H'_{u'(\tilde t)}(p'(\tilde t)) < H'_{\varphi(\tilde u)}(p'(\tilde t))$. However, this contradicts the fact that $\xi'(\cdot)$ satisfies (7). Consequently, if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT, then so is $(g(\cdot), u(\cdot))$. Again, the converse may be found by a similar argument (utilizing Φ^{-1}). □

Let (Σ, χ) and (Σ, χ') be two cost-extended systems (with identical dynamics). Let $\mathcal{T}_\Sigma$ be the group of *feedback transformations leaving Σ invariant*. More precisely, $\mathcal{T}_\Sigma = \{\varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \,:\, \exists \psi \in d\mathsf{Aut}\,(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma,\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\}$. (Here $\mathsf{Aff}\,(\mathbb{R}^\ell)$ is the group of affine isomorphisms of $\mathbb{R}^\ell$, $\mathsf{Aut}\,(\mathsf{G})$ is the group of Lie group automorphisms of G, and $d\mathsf{Aut}\,(\mathsf{G}) = \{T_{\mathbf{1}}\phi \,:\, \phi \in \mathsf{Aut}\,(\mathsf{G})\}$.) The following result is easy to prove.

Proposition 7 *(Σ, χ) and (Σ, χ') are C-equivalent if and only if there exists an element $\varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.*

4 Pontryagin lift

To a cost-extended system (Σ, χ) we associate a (lifted) Hamilton-Poisson system on $\mathfrak{g}^*_-$, via the Pontryagin Maximum Principle (cf. [2, 11, 15]).

Let $\mathfrak{g}$ be an n-dimensional Lie algebra. Let $(E_i)_{1 \le i \le n}$ be an ordered basis for $\mathfrak{g}$ and let $(E_i^*)_{1 \le i \le n}$ be the corresponding dual basis for $\mathfrak{g}^*$. For an element $A \in \mathfrak{g}$, let $\widehat{A}$ denote the corresponding column vector in $\mathbb{R}^n$ with respect to $(E_i)_{1 \le i \le n}$. Similarly, for an element $p \in \mathfrak{g}^*$, let $\widehat{p}$ denote the

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corresponding row vector with respect to $(E_i^*)_{1\leq i\leq n}$. Thus $p(A) = \widehat{p}\,\widehat{A}$. Also, for a linear map $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ (assuming a basis for $\mathfrak{h}$ has been fixed) let $\widehat{\psi}$ denote the matrix associated with ψ (with respect to the corresponding bases), i.e., $\widehat{\psi \cdot A} = \widehat{\psi}\,\widehat{A}$.

A positive-semidefinite quadratic Hamilton-Poisson system is a pair $(\mathfrak{g}^*_-, H_{A,Q})$, where $H_{A,Q}(p) = \widehat{p}\,\widehat{A} + \widehat{p}\,Q\,\widehat{p}^\top$, $A \in \mathfrak{g}$, and Q is a positive-semidefinite $n \times n$ matrix. Two Hamilton-Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are said to be *linearly equivalent* (shortly L-equivalent) if there exists a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\vec{G}$ and $\vec{H}$ are compatible with ψ.

Proposition 8 *The following pairs of Hamilton-Poisson systems (on $\mathfrak{g}^*_-$, specified by their Hamiltonians) are L-equivalent:*

(i) $H_{A,Q} \circ \psi$ and $H_{A,Q}$, where $\psi : \mathfrak{g}^*_- \to \mathfrak{g}^*_-$ is a linear Lie-Poisson automorphism (the associated vector fields are compatible with ψ);

(ii) $H_{A,Q}$ and $H_{A,rQ}$, where $r > 0$ (the associated vector fields are compatible with the dilation $\delta_{1/r} : p \mapsto \frac{1}{r}p$);

(iii) $H_{A,Q}$ and $H_{A,Q} + f(C)$, where C is a Casimir function and $f \in C^\infty(\mathbb{R})$ (the associated vector fields are compatible with the identity map).

Let (Σ, χ) be a cost-extended system with parametrization map $\Xi(\mathbf{1}, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell$ and cost $\chi(u) = (u-\mu)^\top Q (u - \mu)$. An application of the Pontryagin Maximum Principle yields the following result.

Theorem 9 *Any ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) is given by $\dot{g}(t) = \Xi(g(t), u(t))$, $u(t) = Q^{-1}\,\mathbf{B}^\top\,\widehat{p(t)}^\top$. Here $\mathbf{B} = \begin{bmatrix}\widehat{B}_1 & \cdots & \widehat{B}_\ell\end{bmatrix}$ is a $n \times \ell$ matrix and $p(\cdot) : [0, T] \to \mathfrak{g}^*$ is an integral curve for the Hamilton-Poisson system on $\mathfrak{g}^*_-$ specified by*

$$H(p) = \widehat{p}\,(\widehat{A} + \mathbf{B}\,\mu) + \tfrac{1}{2}\,\widehat{p}\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\,\widehat{p}^\top. \quad (8)$$

Proof: The Hamiltonian (4), with $\lambda = -\frac{1}{2}$, is given by $H_u(p) = \widehat{p}\,\widehat{A} + \widehat{p}\,\mathbf{B}\,u - \frac{1}{2}\,(u-\mu)^\top\,Q\,(u-\mu)$. Now, $\frac{\partial H_u}{\partial u}(p) = \widehat{p}\,\mathbf{B} - (u-\mu)^\top\,Q$. By applying the maximum condition (7), we get $u^\top = \widehat{p}\,\mathbf{B}\,Q^{-1} + \mu^\top$. Hence the maximized Hamiltonian is given by

$$\begin{aligned} H(p) &= \widehat{p}\,A + \widehat{p}\,\mathbf{B}\,(Q^{-1}\,\mathbf{B}^\top\,\widehat{p}^\top + \mu) \\ &\quad - \tfrac{1}{2}\,\widehat{p}\,\mathbf{B}\,Q^{-1}\,Q\,Q^{-1}\,\mathbf{B}^\top\,\widehat{p}^\top \\ &= \widehat{p}\,(A + \mathbf{B}\,\mu) + \tfrac{1}{2}\,\widehat{p}\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\,\widehat{p}^\top. \end{aligned}$$

The conditions (6) and (7) then yield the result. □

Accordingly, the study of ECTs of a cost-extended system may effectively be reduced to the study of the associated Hamilton-Poisson system (8).

Theorem 10 *If two cost-extended systems are C-equivalent, then their associated Hamilton-Poisson systems, given by* (8)*, are L-equivalent.*

Proof: Let (Σ, χ) and (Σ', χ') be cost-extended systems with parametrization maps given by $\widehat{\Xi(\mathbf{1}, u)} = \widehat{A} + \mathbf{B}\,u$ and $\widehat{\Xi'(\mathbf{1}, u')} = \widehat{A'} + \mathbf{B}'\,u'$, respectively. The associated Hamilton-Poisson systems on $\mathfrak{g}^*_-$ and $(\mathfrak{g}')^*_-$, are given by $H_{(\Sigma,\chi)}(p) = \widehat{p}\ (\widehat{A} + \mathbf{B}\,\mu) + \frac{1}{2}\ \widehat{p}\ \mathbf{B}\ Q^{-1}\ \mathbf{B}^\top\ \widehat{p}^\top$ and $H_{(\Sigma',\chi')}(p) = \widehat{p}\ (\widehat{A'} + \mathbf{B}'\,\mu') + \frac{1}{2}\,\widehat{p}\,\mathbf{B}'\,(Q')^{-1}\,(\mathbf{B}')^\top\,\widehat{p}^\top$, respectively. Suppose $\phi \times \varphi$, $\varphi(u) = Ru + \varphi_0$, $R \in \mathbb{R}^{\ell\times\ell}$ defines a C-equivalence between (Σ, χ) and (Σ', χ'). Here $\chi' \circ \varphi = r\chi$ for some $r > 0$. A simple calculation yields

$$\begin{aligned} &\widehat{T_{\mathbf{1}}\phi} \cdot \widehat{A} = \widehat{A'} + \mathbf{B}'\,\varphi_0 \qquad && R\mu + \varphi_0 = \mu' \\ &\widehat{T_{\mathbf{1}}\phi} \cdot \mathbf{B} = \mathbf{B}'R && R\,Q^{-1}\,R^\top = \tfrac{1}{r}\,(Q')^{-1}. \end{aligned}$$

Hence $(H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*)(p) = \widehat{p}\ (\widehat{A'} + \mathbf{B}'\,\mu') + \frac{1}{2r}\,\widehat{p}\ \mathbf{B}'\,(Q')^{-1}\,\mathbf{B}'^\top\,\widehat{p}^\top$. Thus the vector fields associated with $H_{(\Sigma',\chi')}$ and $H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*$, respectively, are compatible with the dilation δ_r (by proposition 8). Moreover, the vector fields associated with $H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*$ and $H_{(\Sigma,\chi)}$, respectively, are compatible with the linear Poisson isomorphism $(T_{\mathbf{1}}\phi)^*$ (again by proposition 8). Consequently $r(T_{\mathbf{1}}\phi)^*$ defines a L-equivalence between $((\mathfrak{g}')^*, H_{(\Sigma',\chi')})$ and $(\mathfrak{g}^*, H_{(\Sigma,\chi)})$. □

5 Examples

The Euclidean group $\mathsf{SE}(2)$, the Heisenberg group H_3, and the rotation group $\mathsf{SO}(3)$ are (real) connected matrix Lie groups. The respective standard bases for their Lie algebras have commutator relations

	$\mathfrak{se}(2)$	$\mathfrak{h}_3$	$\mathfrak{so}(3)$
$[E_2, E_3]$	E_1	E_1	E_1
$[E_3, E_1]$	E_2	0	E_2
$[E_1, E_2]$	0	0	E_3

Let (E_1^*, E_2^*, E_3^*) denote the respective dual bases. An element $p = p_1 E_1^* + p_2 E_1^* + p_3 E_3^*$ will be written as $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. The equations of motion of a Hamiltonian H (on each of the respective associated Lie-Poisson spaces) takes the form $\dot{p}_i = -p([E_i, dH(p)])$, $i = 1, 2, 3$.

Example 11 *Any full-rank two-input drift-free cost-extended system* (Σ, χ) *with homogeneous cost on* $\mathsf{SE}(2)$ *(i.e.,* $\Xi(\mathbf{1}, u) = u_1 B_1 + u_2 B_2$ *and* $\chi = u^\top Q\, u$*) is* C*-equivalent to* (Σ_1, χ_1)*, where*

$$\Xi_1(\mathbf{1}, u) = u_1 E_2 + u_2 E_3, \quad \chi_1(u) = u_1^2 + u_2^2.$$

The associated Hamilton-Poisson system on $\mathfrak{se}(2)^*_-$ *is given by* $H_1(p) = \frac{1}{2}\left(p_2^2 + p_3^2\right)$.

The system Σ is DF-equivalent to the system $\Sigma_1 = (\mathsf{SE}(2), \Xi_1)$ (cf. [8]). Moreover, the feedback transformation φ for this equivalence is linear. Thus, by corollary 4, (Σ, χ) is C-equivalent to a cost-extended system (Σ_1, χ') for some $\chi' : u \mapsto u^\top Q' u$.

The group of Lie algebra automorphisms $\mathsf{Aut}(\mathfrak{se}(2))$ is given by

$$\left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \; : \; \begin{matrix} x, y, v, w \in \mathbb{R}, \varsigma = \pm 1 \\ x^2 + y^2 \neq 0 \end{matrix} \right\}.$$

It turns out that $\mathsf{Aut}(\mathfrak{se}(2)) = d\mathsf{Aut}(\mathsf{SE}(2))$. We now calculate the group $\mathcal{T}_{\Sigma_1}$ of feedback transformations leaving Σ_1 invariant. Let $\psi \in d\mathsf{Aut}(\mathsf{SE}(2))$ such that $\psi \cdot \Gamma_1 = \Gamma_1$. Then $\psi \cdot \langle E_2, E_3 \rangle = \langle E_2, E_3 \rangle$ and so $y = v = 0$. Now suppose $\varphi : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a feedback transformation such that $\psi \cdot \Xi_1(\mathbf{1}, u) = \Xi_1(\mathbf{1}, \varphi(u))$, i.e., $(\varsigma x u_1 + w u_2) E_2 + (\varsigma u_2) E_3 = (a_1 u_1 + a_2 u_2 + c_1) E_2 + (b_1 u_1 + b_2 u_2 + c_2) E_3$. By equating the coefficients (of E_2 and E_3, and then of u_1 and u_2) we get that $\mathcal{T}_{\Sigma_1}$ is (a group of linear isomorphisms) given by

$$\left\{ u \mapsto \begin{bmatrix} \varsigma x & w \\ 0 & \varsigma \end{bmatrix} u \; : \; x \neq 0, \, w \in \mathbb{R}, \, \varsigma = \pm 1 \right\}.$$

Let $Q' = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$. Now $\varphi_1 = \begin{bmatrix} 1 & -\frac{b}{a_1} \\ 0 & 1 \end{bmatrix} \in \mathcal{T}_{\Sigma_1}$ and $(\chi' \circ \varphi_1)(u) = u^\top \operatorname{diag}(a_1, \, a_2 - \frac{b^2}{a_1})\, u$. Let $a_2' = a_2 - \frac{b^2}{a_1}$ and let $\varphi_2 = \operatorname{diag}(\sqrt{\frac{a_2'}{a_1}}, 1) \in \mathcal{T}_{\Sigma_1}$. Then $(\chi' \circ (\varphi_1 \circ \varphi_2))(u) = a_2' u^\top u = a_2' \chi_1(u)$. Consequently, by theorem 7, (Σ, χ) is C-equivalent to (Σ_1, χ_1).

Similarly, we have the following classification of a class of *controllable* cost-extended systems on H_3. Any system on H_3 with trace $\Gamma = A + \langle B_1, \ldots, B_\ell \rangle$ is controllable if and only if $B_1, \ldots, B_\ell$ generates $\mathfrak{h}_3$ ([15]).

Example 12 (cf. [7]) *Any controllable two-input inhomogeneous cost-extended system on* H_3 *is* C*-equivalent to exactly one of the cost-extended systems* $(\Sigma_1, \chi_{1,\alpha})$*, where*

$$\begin{cases} \Xi_1(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3 \\ \chi_{1,\alpha}(u) = (u_1 - \alpha)^2 + u_2^2. \end{cases}$$

Here $\alpha \geq 0$ *parametrizes a family of (non-equivalent) class representatives. The associated Hamilton-Poisson systems on* $(\mathfrak{h}_3)^*_-$ *are given by* $H_{1,\alpha}(p) = p_1 + \alpha\, p_2 + \frac{1}{2}\left(p_2^2 + p_3^2\right)$.

Remark 13 The Hamilton-Poisson system specified by $H_{1,\alpha}(p) = p_1 + \alpha\, p_2 + \frac{1}{2}\left(p_2^2 + p_3^2\right)$ is L-equivalent to a system specified by $H_2(p) = p_2^2 + p_3^2$ or $H_3(p) = p_2 + p_2^2 + p_3^2$. In particular, this shows that the converse of theorem 10 does not hold.

Various classes of quadratic Hamilton-Poisson systems have been studied in the last few years (see, e.g., [3, 4, 16, 1]). We discuss L-equivalence on some classes of Hamilton-Poisson systems.

Example 14 *Any homogeneous Hamilton-Poisson system* $((\mathfrak{h}_3)^*_-, H_Q)$ *is* L*-equivalent to one of the Hamilton-Poisson systems on* $(\mathfrak{h}_3)^*_-$ *specified by* $H_0(p) = 0$*,* $H_1(p) = p_3^2$*, and* $H_2(p) = p_2^2 + p_3^2$.

The group of linear Poisson automorphisms of $(\mathfrak{h}_3)^*_-$ is given by

$$\left\{ p \mapsto p \begin{bmatrix} y_1 z_2 - y_2 z_1 & x_1 & x_2 \\ 0 & y_1 & y_2 \\ 0 & z_1 & z_2 \end{bmatrix} \; : \; \begin{matrix} x, y, z \in \mathbb{R}^2 \\ y_1 z_2 \neq y_2 z_1 \end{matrix} \right\}.$$

Note that $C(p) = p_1$ is a Casimir function.

Let $H_Q(p) = p\, Q\, p^\top$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 = 0$. The 2×2 principle minors of Q are then $a_1 a_2 - b_1^2$, $-b_2^2$, and $-b_3^2$. As Q is positive-semidefinite, the principle minors are non-negative. Thus $b_2 = b_3 = 0$. Assume $a_2 = 0$. Then $b_1 = 0$ and so $H_Q(p) = a_1 p_1^2 = a_1 C(p)^2$. Therefore the system specified by H_Q is L-equivalent to the one specified by H_0. Now suppose $a_2 \neq 0$. Then $\psi_1 : p \mapsto p \Psi_1$,

$$\Psi_1 = \begin{bmatrix} -\frac{1}{\sqrt{a_2}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{a_2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{b_1}{a_2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that $(H_Q \circ \psi_1)(p) = (\frac{a_1 a_2 - b_1^2}{a_2^2}) C(p)^2 + p_3^2$. Thus the system specified by H_Q is L-equivalent to the one specified by H_1 (by proposition 8).

Suppose $a_3 \neq 0$. Then

$$\psi_2 : p \mapsto p\Psi_2, \quad \Psi_2 = \begin{bmatrix} 1 & 0 & -\frac{b_2}{a_3} \\ 0 & 1 & -\frac{b_3}{a_3} \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$(H_Q \circ \psi_2)(p) = p \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} p^\top.$$

Similar computations show that the system specified by H_Q is L-equivalent to one specified by H_1 or H_2.

Likewise, we have the following results for quadratic Hamilton-Poisson systems on $\mathfrak{se}\,(2)^*_-$ and $\mathfrak{so}\,(3)^*_-$.

Example 15 (cf. [1]) *Any homogeneous Hamilton-Poisson system* $(\mathfrak{se}\,(2)^*_-, H_Q)$ *is* L*-equivalent to one of the Hamilton-Poisson systems on* $\mathfrak{se}\,(2)^*_-$ *specified by* $H_0(p) = 0$*,* $H_1(p) = p_2^2$*,* $H_2(p) = p_3^2$*, and* $H_3(p) = p_2^2 + p_3^2$*.*

Example 16 *Any homogeneous Hamilton-Poisson system* $(\mathfrak{so}\,(3)^*_-, H_Q)$ *is* L*-equivalent to one of the Hamilton-Poisson systems on* $\mathfrak{so}\,(3)^*_-$ *specified by* $H_0(p) = 0$ *and* $H_{1,\alpha}(p) = p_1^2 + \alpha\, p_2^2$*. Here* $0 \leq \alpha \leq 1$ *parametrizes a family of distinct equivalence representatives.*

Remark 17 In the three foregoing examples one would need to verify that no two of the equivalence representatives are L-equivalent in order to obtain a classification.

A number of quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces have been shown to be L-equivalent to the relaxed free rigid body dynamics (see [16])

$$\begin{cases} \dot p_1 = (\nu_3 - \nu_2) p_2 p_3 \\ \dot p_2 = (\nu_1 - \nu_3) p_1 p_3 \qquad p \in \mathbb{R}^3,\ \nu_1, \nu_2, \nu_3 \in \mathbb{R}. \\ \dot p_3 = (\nu_2 - \nu_1) p_1 p_2 \end{cases}$$

These dynamics correspond to a Hamilton-Poisson system $(\mathfrak{so}\,(3)^*_-, H_\nu)$, where $H_\nu(p) = \nu_1 p_1^2 + \nu_2 p_2^2 + \nu_3 p_3^2$. We may assume that $\nu_1, \nu_2, \nu_3 > 0$ (by adding a constant multiple of the Casimir function $C(p) = p_1^2 + p_2^2 + p_3^2$). Then $(\mathfrak{so}\,(3)^*_-, H_\nu)$ is L-equivalent to one of the above specified Hamilton-Poisson systems on $\mathfrak{so}\,(3)^*_-$.

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