On the Equivalence of Cost-Extended
Control Systems on Lie Groups

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Abstract: We consider the equivalence of cost-extended control systems corresponding to certain invariant optimal control problems. We prove that two equivalent cost-extended systems have the same optimal controlled trajectories and the same extremal curves (up to a diffeomorphism). We also prove that equivalent cost-extended systems are lifted (via the Pontryagin Maximum Principle) to linearly equivalent Hamilton-Poisson systems. A few illustrative examples are discussed.

Key–Words: Invariant control system, feedback equivalence, Hamilton-Poisson system.

1 Introduction

We consider an equivalence relation on a certain class of cost-extended control systems. Specifically, we consider equivalence on the class corresponding to left-invariant optimal control problems on Lie groups with fixed terminal time, affine dynamics, and affine quadratic cost. Formally, such problems are given by

\[ \dot{g} = g(A + u_1B_1 + \cdots + u_\ell B_\ell), \quad g \in G \]  
(1)

\[ g(0) = g_0, \quad g(T) = g_1 \]  
(2)

\[ J = \int_0^T (u(t) - \mu)^T Q(u(t) - \mu) \, dt \to \min. \]  
(3)

Here \( G \) is a (real, finite-dimensional) connected Lie group with Lie algebra \( g \), \( A, B_1, \ldots, B_\ell \in g \) where \( B_1, \ldots, B_\ell \) are linearly independent, \( u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \), \( \mu \in \mathbb{R}^2 \), and \( Q \) is a positive-definite \( \ell \times \ell \) matrix.

Such an equivalence (between two cost-extended systems) establishes a one-to-one correspondence between the associated optimal controlled trajectories. Likewise, it establishes a one-to-one correspondence between the associated (normal) extremal curves. Each cost-extended system can be lifted (via the Pontryagin Maximum Principle) to a quadratic Hamilton-Poisson system. If two cost-extended systems are equivalent, then the associated Hamilton-Poisson systems are linearly equivalent.

A number of illustrative examples, pertaining to the classification of cost-extended systems and Hamilton-Poisson systems, are discussed.

2 Preliminaries

2.1 The Lie-Poisson structure

Let \( g \) be a (real) Lie algebra. The dual space \( g^* \) has a natural Poisson structure, the (minus) Lie-Poisson structure (cf. [13]). This structure is given by \( \{F, G\}(p) = -p(dF(p), dG(p)) \), \( p \in g^* \), \( F, G \in C^\infty(g^*) \). (Here \( dF(p), dG(p) \in g^{**} = g \).) The Poisson space \( (g^*, \{ \cdot, \cdot \}) \) is denoted by \( g^* \). A function \( C \in C^\infty(g^*) \) is a Casimir function if \( \{C, F\} = 0 \) for all \( F \in C^\infty(g^*) \).

To each function \( H \in C^\infty(g^*) \), we associate a Hamiltonian vector field \( \vec{H} \) on \( g^* \) specified by \( \vec{H}[F] = \{H, F\} \). Two vector fields \( \vec{F} \) and \( \vec{G} \) (on \( g^* \) and \( h^* \), respectively) are compatible with a smooth map \( \phi : g^* \to h^* \) if \( T_p\phi \cdot \vec{F}(p) = \vec{G}(\phi(p)) \) for \( p \in g^* \).

A linear map \( \psi : g^* \to h^* \) is a linear Poisson morphism if \( \{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\} \) for all \( F, G \in C^\infty(g^*) \). Linear Poisson morphisms are exactly the dual maps of Lie algebra morphisms.

2.2 Invariant optimal control

Invariant control systems on Lie groups were first considered in 1972 by Brockett [9] and by Jurdjevic and Sussmann [12]. A left-invariant control affine system \( \Sigma \) is a control system of the form

\[ \dot{g} = g \Xi(1, u) = g(A + u_1B_1 + \cdots + u_\ell B_\ell) \]

where \( g \in G, u \in \mathbb{R}^\ell \). Here the parametrization map \( \Xi(1, \cdot) : \mathbb{R}^\ell \to g \) is an injective affine
The “product” $g \Xi(1, u)$ is to be understood as $T_1 L_g \cdot \Xi(1, u)$, where $L_g : G \to G$, $h \mapsto gh$ is the left translation by $g$. Note that the dynamics $\Xi : G \times \mathbb{R}^l \to T^*G$ is invariant under left translations, i.e., $\Xi(g, u) = g \Xi(1, u)$. We shall denote such a system by $\Sigma = (G, \Xi)$ (cf. [5]).

The image set $\Gamma = \text{im} \Xi(1, \cdot)$, called the trace of $\Sigma$, is an affine subspace of $g$. Accordingly, $\Gamma = A + t^0 = A + \langle B_1, \ldots, B_r \rangle$. A system $\Sigma$ is called drift-free if $A = 0$, homogeneous if $A \in \mathbb{R}^l$, and inhomogeneous otherwise. Also, $\Sigma$ is said to have full rank if its trace generates the whole Lie algebra (i.e., the smallest Lie algebra containing $\Gamma$ is $g$).

The admissible controls are piecewise-continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^l$. A controlled trajectory is a pair $(g(\cdot), u(\cdot))$ where $g(\cdot) : [0, T] \to G$ is an absolutely continuous curve such that $g(t) = g(t) \Xi(1, u(t))$ for almost every $t \in [0, T]$. For more details about (invariant) control systems see, e.g., [2, 11, 12, 14].

Two systems are detached feedback equivalent if there exists a “detached” feedback transformation which transforms the first system to the second (see [6, 10]). Equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. More precisely, let $\Sigma = (G, \Xi)$ and $\Sigma' = (G', \Xi')$ be left-invariant control affine systems. $\Sigma$ and $\Sigma'$ are called detached feedback equivalent (shortly $DF$-equivalent) if there exists a diffeomorphism $\phi : G \times \mathbb{R}^l \to G' \times \mathbb{R}^l$, $(g, u) \mapsto (\phi(g), \varphi(u))$ such that $T_0 \phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in G$ and $u \in \mathbb{R}^l$. Full-rank systems $\Sigma$ and $\Sigma'$ are $DF$-equivalent if and only if there exists a Lie group isomorphism $\phi : G \to G'$ such that $T_0 \phi \cdot \Gamma = \Gamma'$ [6].

Now, consider an optimal control problem given by the specification of (i) a left-invariant control system $\Sigma = (G, \Xi)$, (ii) an affine quadratic cost function $\chi : \mathbb{R}^l \to \mathbb{R}$, and (iii) boundary data $B(g_0, g_1, T)$, consisting of an initial state $g_0 \in G$, a target state $g_1 \in G$ and a fixed terminal time $T > 0$. Explicitly, we want to minimize the functional $J = \int_0^T \chi(u(t)) \, dt$ over the trajectory-control pairs of $\Sigma$ subject to the boundary conditions (2). To such a problem, we associate a cost-extended system $(\Sigma, \chi)$ via the boundary data $B(g_0, g_1, T)$, and vice-versa.

The Pontryagin Maximum Principle is a necessary condition for optimality. To an optimal control problem (1)-(2)-(3), we associate, for each real number $\lambda$ and each control parameter $u \in \mathbb{R}^l$, a Hamiltonian function on $T^*G = G \times g^*$:

$$H_u^\lambda(\xi) = \lambda \chi(u) + p \cdot (\Xi(1, u)).$$

Here $\xi = (g, p) \in T^*G$. The Maximum Principle can then be stated as follows.

**Maximum Principle.** Suppose the controlled trajectory $(g(t), u(t))$ defined over the interval $[0, T]$ is a solution for the optimal control problem (1)-(2)-(3). Then, there exists a curve $(\xi(t) : [0, T] \to T^*G$ with $\xi(t) \in T_{\bar{g}(t)}G$, $t \in [0, T]$, and a real number $\lambda \leq 0$, such that the following conditions hold for almost every $t \in [0, T] :$

$$\lambda, \xi(t) \neq (0, 0) \quad (5)$$

$$\dot{\xi}(t) = H_u^\lambda(\xi(t)) \quad (6)$$

$$H_u^\lambda(\xi(t)) = \max_u H_u^\lambda(\xi(t)) = \text{constant}. \quad (7)$$

### 3 Equivalence

Let $(\Sigma, \chi)$ and $(\Sigma', \chi')$ be two cost-extended systems. $(\Sigma, \chi)$ and $(\Sigma', \chi')$ are said to be cost equivalent (shortly $C$-equivalent) if there exist a Lie group isomorphism $\phi : G \to G'$ and an affine isomorphism $\varphi : \mathbb{R}^l \to \mathbb{R}^l$ such that $T_0 \phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in G$, $u \in \mathbb{R}^l$ and $\chi' \circ \varphi = r \chi$ for some $r > 0$.

**Remark 1** The “dynamics-preserving” condition is just that of $DF$-equivalence (on full-rank systems). The “cost-preserving” condition is partially motivated by the following. Each cost $\chi$ on $\mathbb{R}^l$ induces a strict partial ordering $u < v \iff \chi(u) < \chi(v)$. It turns out that $\chi$ and $\chi'$ induce the same strict partial ordering on $\mathbb{R}^l$ if and only if $\chi = r \chi'$ for some $r > 0$.

The following result is easy to prove (cf. [5, 6]).

**Proposition 2** $(\Sigma, \chi)$ and $(\Sigma', \chi')$ are $C$-equivalent if and only if there exist a Lie group isomorphism $\phi : G \to G'$ and an affine isomorphism $\varphi : \mathbb{R}^l \to \mathbb{R}^l$ such that $T_0 \phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ and $\chi' \circ \varphi = r \chi$ for some $r > 0$.

**Corollary 3** If $(\Sigma, \chi)$ and $(\Sigma', \chi')$ are $C$-equivalent, then $\Sigma$ and $\Sigma'$ are $DF$-equivalent.

**Corollary 4** Let $\Sigma = (G, \Xi)$ and $\Sigma' = (G', \Xi')$ be two full-rank systems. If $\Sigma$ and $\Sigma'$ are $DF$-equivalent with respect to a feedback transformation $\varphi$, then $(\Sigma, \chi \circ \varphi)$ and $(\Sigma', \chi)$ are $C$-equivalent for any cost $\chi$.

The cost $\chi$ of a system may always be transformed into $\chi(u) = u^T \bar{u}$ by “complicating” the parametrization map.

**Proposition 5** Any cost-extended system $(\Sigma, \chi)$ is $C$-equivalent to a system $(\Sigma', \chi')$, where $G' = G$, $\ell' = \ell$, $\Gamma' = \Gamma$, and $\chi'(u) = u^T \bar{u}$. 

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Proof: Let $\chi(u) = (u - \mu)^T Q (u - \mu)$. As $Q$ is positive-definite, there exists a curve $\xi(t) = (g(t), p(t))$ such that $R \dot{\xi}(t) R^{-1} = I$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, $u \mapsto Ru + \mu$ and let $\Xi : \mathbb{R}^d \times \mathbb{R}^d \to TG$, $\Xi(1, u) = \Xi(1, \varphi(u))$ and $\chi(u) = u^T R^T Q R u = u^T u$. The result follows by proposition 2. □

Let $(g(\cdot), u(\cdot))$ be a controlled trajectory, defined over an interval $[0, T]$, of a cost-extended system $(\Sigma, \chi)$. Then $(g(\cdot), u(\cdot))$ is a virtually optimal controlled trajectory (shortly VOCT) if it is a solution for (6), we have $\dot{\xi}(t) = (g(\xi(t), u(t)))$ and $\dot{p}(t) = \text{ad}^* \Xi(1, u(t)) \cdot p(t)$. Let $p(\cdot) = \frac{1}{r} (T_1 \phi)^r \cdot p'(t)$ and $\chi(\cdot) = (g(\cdot), p(\cdot))$. Thus, to satisfy (6), we are left to show that $\dot{p}(t) = \text{ad}^* \Xi(1, u(t)) \cdot p(t)$. We have (for $A \in g^*$)

$$(p(t)) = \frac{1}{r} (T_1 \phi)^r \cdot p'(t)$$

Hence

Using the available identities and simplifying we get $H_u^*(p' \in) < H_u^*(\bar{p}')$. However, this contradicts the fact that $\chi^*(\cdot)$ satisfies (7). Consequently, if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT, then so is $(g(\cdot), u(\cdot))$.

As $\Phi^{-1} = \phi^{-1} \times \varphi^{-1}$ defines a $C$-equivalence between $(\Sigma', \chi')$ and $(\Sigma, \chi)$, the same argument can be used to show the converse.

Now assume $(g(\cdot), u(\cdot))$ is a controlled trajectory of $(\Sigma', \chi')$ such that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT of $(\Sigma', \chi')$. The Hamiltonian functions (4), associated to $(\Sigma, \chi)$ and $(\Sigma', \chi')$, are given by $H_u(g, p) = p(\Xi(1, u) - \chi(u))$ and $H_u'(g, p) = p(\Xi(1, u') - \chi(u'))$. The result follows by proposition 2.

□

4 Pontryagin lift

To a cost-extended system $(\Sigma, \chi)$ we associate a (lifted) Hamilton-Poisson system on $g^*$, via the Pontryagin Maximum Principle (cf. [2, 11, 15]). Let $g$ be an $n$-dimensional Lie algebra. Let $(E_1)_{1 \leq i \leq n}$ be an ordered basis for $g$ and let $(E_1^*)_{1 \leq i \leq n}$ be the corresponding dual basis for $g^*$. For an element $A \in g$, let $\tilde{A}$ denote the corresponding column vector in $\mathbb{R}^n$ with respect to $(E_1)_{1 \leq i \leq n}$. Similarly, for an element $p \in g^*$, let $\tilde{p}$ denote the
corresponding row vector with respect to \((E^*_i)_{1 \leq i \leq n}\). Thus \(p(A) = \hat{p} \hat{A}\). Also, for a linear map \(\psi: \mathfrak{g}^* \to \mathfrak{h}^*\) (assuming a basis for \(\mathfrak{h}\) has been fixed) let \(\hat{\psi}\) denote the matrix associated with \(\psi\) (with respect to the corresponding bases), i.e., \(\hat{\psi} \cdot A = \psi \hat{A}\).

A positive-semidefinite quadratic Hamilton-Poisson system is a pair \((\mathfrak{g}^*_s, H_{AQ})\), where \(H_{AQ}(p) = \hat{p} \hat{A} + \hat{p} Q \hat{p}^\top\), \(A \in \mathfrak{g}\), and \(Q\) is a positive-semidefinite \(n \times n\) matrix. Two Hamilton-Poisson systems \((\mathfrak{g}^*_s, Q)\) and \((\mathfrak{h}^*_s, H)\) are said to be linearly equivalent (shortly L-equivalent) if there exists a linear isomorphism \(\psi: \mathfrak{g}^* \to \mathfrak{h}^*\) such that \(\hat{G}\) and \(\hat{H}\) are compatible with \(\psi\).

**Proposition 8** The following pairs of Hamilton-Poisson systems (on \(\mathfrak{g}^*_s\), specified by their Hamiltonians) are L-equivalent:

(i) \(H_{AQ} \circ \psi\) and \(H_{AQ}\), where \(\psi: \mathfrak{g}^*_s \to \mathfrak{g}^*_s\) is a linear Lie-Poisson automorphism (the associated vector fields are compatible with \(\psi\));

(ii) \(H_{AQ}\) and \(H_{AQQ}\), where \(r > 0\) (the associated vector fields are compatible with the dilation \(\delta_{1/r}: p \mapsto \frac{1}{r^2} p\));

(iii) \(H_{AQ}\) and \(H_{AQ} + f(C)\), where \(C\) is a Casimir function and \(f \in C^\infty(\mathbb{R})\) (the associated vector fields are compatible with the identity map).

Let \((\Sigma, \chi)\) be a cost-extended system with parametrization map \(\Xi(1, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell\) and cost \(\chi(u) = (u - \mu)^\top Q (u - \mu)\). An application of the Pontryagin Maximum Principle yields the following result.

**Theorem 9** Any ECT \((g(\cdot), u(\cdot))\) of \((\Sigma, \chi)\) is given by \(\hat{g}(t) = \Xi(g(t), u(t)), u(t) = Q^{-1} B^\top p(t)\).

Here \(B = \bigg[\begin{array}{c} \hat{B}_1 \\ \vdots \\ \hat{B}_\ell \end{array}\bigg]\) is an \(n \times \ell\) matrix and \(p(\cdot): [0, T] \to \mathfrak{g}^*\) is an integral curve for the Hamilton-Poisson system on \(\mathfrak{g}^*_s\) specified by

\[
H(p) = \hat{p} (A + B \mu) + \frac{1}{2} \hat{p} B Q^{-1} B^\top \hat{p}^\top.
\]  

**Proof:** The Hamiltonian (4), with \(\lambda = -\frac{1}{2}\), is given by \(H_u(p) = \hat{p} A + \hat{p} B u - \frac{1}{2} (u - \mu)^\top Q (u - \mu)\). Now, \(\frac{\partial H_u}{\partial u}(p) = \hat{p} B - (u - \mu)^\top Q\). By applying the maximum condition (7), we get \(u^\top = \hat{p} B Q^{-1} + \mu^\top\). Hence the maximized Hamiltonian is given by

\[
H(p) = \hat{p} (A + \hat{p} B (Q^{-1} B^\top \hat{p}^\top + \mu)) - \frac{1}{2} \hat{p} B Q^{-1} Q^{-1} B^\top \hat{p}^\top = \hat{p} (A + B \mu) + \frac{1}{2} \hat{p} B Q^{-1} B^\top \hat{p}^\top.
\]

The conditions (6) and (7) then yield the result. \(\square\)

Accordingly, the study of ECTS of a cost-extended system may effectively be reduced to the study of the associated Hamilton-Poisson system (8).

**Theorem 10** If two cost-extended systems are C-equivalent, then their associated Hamilton-Poisson systems, given by (8), are L-equivalent.

**Proof:** Let \((\Sigma, \chi)\) and \((\Sigma', \chi')\) be cost-extended systems with parametrization maps given by \(\Xi(1, u) = \hat{A} + \hat{B} u\) and \(\Xi(1, u') = \hat{A}' + \hat{B}' u'\), respectively. The associated Hamilton-Poisson systems on \(\mathfrak{g}^*_s\) and \((\mathfrak{g}^*_s, Q')\) are given by \(H(\Sigma, \chi)(p) = \hat{p} (\hat{A} + B \mu) + \frac{1}{2} \hat{p} B Q^{-1} B^\top \hat{p}^\top\) and \(H(\Sigma', \chi')(p) = \hat{p} (\hat{A}' + B' \mu') + \frac{1}{2} \hat{p} B' Q'^{-1} B'^\top \hat{p}^\top\), respectively. Suppose \(\phi \times \varphi, \varphi(u) = R u + \varphi_0, R \in \mathbb{R}^{\ell \times \ell}\) defines a C-equivalence between \((\Sigma, \chi)\) and \((\Sigma', \chi')\). Here \(\chi' \times \varphi = r \chi\) for some \(r > 0\). A simple calculation yields

\[
\begin{align*}
\hat{T}_1 \phi \cdot \hat{A} &= \hat{A}' + \hat{B}' \varphi_0 \quad R \mu + \varphi_0 = \mu' \\
\hat{T}_1 \phi \cdot B &= B' R \\
R Q^{-1} R^\top &= \frac{1}{r} (Q')^{-1}.
\end{align*}
\]

Hence \((H(\Sigma, \chi) \circ (T_1 \phi)^*) (p) = \hat{p} (A' + B' \mu') + \frac{1}{2} \hat{p} B' (Q')^{-1} B'^\top \hat{p}^\top\). Thus the vector fields associated with \(H(\Sigma, \chi)\) and \(H(\Sigma, \chi) \circ (T_1 \phi)^*\), respectively, are compatible with the linear Poisson isomorphism \((T_1 \phi)^*\) (again by proposition 8). Moreover, the vector fields associated with \(H(\Sigma, \chi)\) and \(H(\Sigma, \chi)\), respectively, are compatible with the linear Poisson isomorphism \((T_1 \phi)^*\) (again by proposition 8). Consequently \(r (T_1 \phi)^*\) defines a L-equivalence between \(((\mathfrak{g}^*_s), H(\Sigma, \chi))\) and \((\mathfrak{g}^*_s, H(\Sigma, \chi))\). \(\square\)

**5 Examples**

The Euclidean group \(SE(2)\), the Heisenberg group \(H_3\), and the rotation group \(SO(3)\) are (real) connected matrix Lie groups. The respective standard bases for their Lie algebras have commutator relations

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<th>(E_2, E_3)</th>
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Let \((E^*_1, E^*_2, E^*_3)\) denote the respective dual bases. An element \(p = p_1 E^*_1 + p_2 E^*_2 + p_3 E^*_3\) will be written as \(p = (p_1 \ p_2 \ p_3)\). The equations of motion of a Hamiltonian \(H\) (on each of the respective associated Lie-Poisson spaces) takes the form \(\dot{p}_i = -p([E_i, dH(p)])\), \(i = 1, 2, 3\).
Example 11 Any full-rank two-input drift-free cost-extended system \((\Sigma_1, \chi)\) with homogeneous cost on SE(2) (i.e., \(\Xi(1, u) = u_1B_1 + u_2B_2\) and \(\chi = u^\top Q u\)) is \(C\)-equivalent to \((\Sigma_1, \chi_1)\), where
\[
\Xi(1, u) = u_1E_2 + u_2E_3, \quad \chi_1(u) = u_1^2 + u_2^2.
\]
The associated Hamilton-Poisson system on SE(2) is given by \(H_1(p) = \frac{1}{2} (p_2^2 + p_3^2)\).

The system \(\Sigma\) is \(DF\)-equivalent to the system \(\Sigma_1 = (SE(2), \Xi(1))\) (cf. [8]). Moreover, the feedback transformation \(\varphi\) for this equivalence is linear. Thus, by corollary 4, \((\Sigma_1, \chi)\) is \(C\)-equivalent to a cost-extended system \((\Sigma_1, \chi')\) for some \(\chi' : u \mapsto u^\top Q' u\).

The group of Lie automorphisms \(\text{Aut}(\psi(2))\) is given by
\[
\left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \right| x, y, v, w \in \mathbb{R}, \varsigma = \pm 1, \quad x^2 + y^2 \neq 0 \right\}.
\]
It turns out that \(\text{Aut}(\psi(2)) = d\text{Aut}(SE(2))\). We now calculate the group \(\mathcal{T}_{\Sigma_1}\) of feedback transformations leaving \(\Sigma_1\) invariant. Let \(\psi \in \text{dAut}(SE(2))\) such that \(\psi \cdot \Gamma_1 = \Gamma_1\). Then \(\psi \cdot (E_2, E_3) = (E_2, E_3)\) and so \(y = v = 0\). Now suppose \(\varphi : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\) is a feedback transformation such that \(\psi \cdot \Xi(1, u) = \Xi(1, \varphi(u))\), i.e., \((\chi x u_1 + u_2)E_2 + (\chi u_2)E_3 = (\varphi_1 u_1 + \varphi_2 u_2 + c_1)E_2 + (\varphi_1 u_1 + \varphi_2 u_2 + c_2)E_3\). By equating the coefficients of \((E_2\) and \(E_3\), and then of \(u_1\) and \(u_2\)) we get that \(\mathcal{T}_{\Sigma_1}\) is a group of linear isomorphisms given by
\[
\left\{ u \mapsto \begin{bmatrix} cz \\ 0 \\ \varsigma \end{bmatrix} u : x \neq 0, w \in \mathbb{R}, \varsigma = \pm 1 \right\}.
\]
Let \(Q' = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}\). Now \(\varphi_1 = \begin{bmatrix} 1 & -b \\ 0 & a_1 \end{bmatrix} \in \mathcal{T}_{\Sigma_1}\) and \((\chi' \circ \varphi_1)(u) = u^\top \text{diag}(a_1, a_2 - \frac{b^2}{a_1}) u\). Let \(a_2' = a_2 - \frac{b^2}{a_1}\) and let \(\Phi_2 = \text{diag}(\sqrt{\frac{a_1}{a_1}}, 1) \in \mathcal{T}_{\Sigma_1}\). Then \((\chi' \circ (\varphi_1 \circ \varphi_2))(u) = a_2'^\top u = a_2' \chi_1(u)\). Consequently, by theorem 7, \((\Sigma_1, \chi)\) is \(C\)-equivalent to \((\Sigma_1, \chi_1)\).

Similarly, we have the following classification of a class of controllable cost-extended systems on \(H_3\). Any system on \(H_3\) with trace \(\Gamma = A + (B_1, \ldots, B_\ell)\) is controllable if and only if \(B_1, \ldots, B_\ell\) generates \(h_3\) ([15]).

Example 12 (cf. [7]) Any controllable two-input inhomogeneous cost-extended system on \(H_3\) is \(C\)-equivalent to exactly one of the cost-extended systems \((\Sigma_1, \chi_{1, \alpha})\), where
\[
\begin{cases}
\Xi(1, u) = E_1 + u_1 E_2 + u_2 E_3 \\
\chi_{1, \alpha}(u) = (u_1 - \alpha)^2 + u_2^2.
\end{cases}
\]
Here \(\alpha \geq 0\) parametrizes a family of (non-equivalent) class representatives. The associated Hamilton-Poisson systems on \((h_3)^\ast\) are given by \(H_1(\alpha)(p) = p_1 + \alpha p_2 + \frac{1}{2} (p_2^2 + p_3^2)\).

Remark 13 The Hamilton-Poisson system specified by \(H_1(\alpha)(p) = p_1 + \alpha p_2 + \frac{1}{2} (p_2^2 + p_3^2)\) is \(L\)-equivalent to a system specified by \(H_2(p) = p_2^2 + p_3^2\) or \(H_3(p) = p_2^2 + p_3^2\). In particular, this shows that the converse of theorem 10 does not hold.

Various classes of quadratic Hamilton-Poisson systems have been studied in the last few years (see, e.g., [3, 4, 16, 1]). We discuss \(L\)-equivalence on some classes of Hamilton-Poisson systems.

Example 14 Any homogeneous Hamilton-Poisson system \((h_3)^\ast, H_Q)\) is \(L\)-equivalent to one of the Hamilton-Poisson systems on \((h_3)^\ast\) specified by \(H_0(p) = 0\), \(H_1(p) = p_3^2\), and \(H_2(p) = p_2^2 + p_3^2\).

The group of linear Poisson automorphisms of \((h_3)^\ast\) is given by
\[
\left\{ p \mapsto p \begin{bmatrix} y_1 z_2 - y_2 z_1 & x_1 & x_2 \\ 0 & y_1 & y_2 \\ 0 & z_1 & z_2 \end{bmatrix} : x, y, z \in \mathbb{R}^2, y_1 z_2 \neq y_2 z_1 \right\}.
\]
Note that \(C(p) = p_1\) is a Casimir function.

Let \(H_Q(p) = pq^\top\), where
\[
Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.
\]
Suppose \(a_3 = 0\). The \(2 \times 2\) principle minors of \(Q\) are then \(a_1 a_2 - b_2^2, -b_3^2, \) and \(-b_3^2\). As \(Q\) is positive-semidefinite, the principle minors are non-negative. Thus \(b_2 = b_3 = 0\). Assume \(a_2 = 0\). Then \(b_1 = 0\) and so \(H_Q(p) = a_1 p_1^2 = a_1 C(p)^2\). Therefore, the system specified by \(H_Q\) is \(L\)-equivalent to the one specified by \(H_0\). Now suppose \(a_2 \neq 0\). Then \(\psi_1 : p \mapsto p \psi_1\),
\[
\psi_1 = \begin{bmatrix} 1 & -b_1 \\ 0 & a_2 \\ \frac{1}{\sqrt{a_2}} & 0 \end{bmatrix},
\]
is a linear Poisson automorphism such that \((H_Q \circ \psi_1)(p) = (a_1 a_2 - b_2^2)\). Thus the system specified by \(H_Q\) is \(L\)-equivalent to the one specified by \(H_1\) (by proposition 8).
Suppose \( a_3 \neq 0 \). Then
\[
\psi_2 : p \mapsto p\Psi_2, \quad \Psi_2 = \begin{bmatrix} 1 & 0 & -b_2 \\ 0 & 1 & -a_2 \\ 0 & 0 & 1 \end{bmatrix}
\]
is a linear Poisson automorphism such that
\[
(H_Q \circ \psi_2)(p) = p \begin{bmatrix} a_1 - \frac{b_2}{a_3} & b_1 - \frac{b_2a_3}{a_4} & 0 \\ b_1 - \frac{b_2a_3}{a_4} & a_2 - \frac{b_2^2}{a_4} & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top.
\]

Similar computations show that the system specified by \( H_Q \) is \( L \)-equivalent to one specified by \( H_1 \) or \( H_2 \).

Likewise, we have the following results for quadratic Hamilton-Poisson systems on \( \mathfrak{se}(2)_{-}\) and \( \mathfrak{so}(3)_{-}\).

**Example 15 (cf. [1])** Any homogeneous Hamilton-Poisson system \((\mathfrak{se}(2)_{-}, H_Q)\) is \( L \)-equivalent to one of the Hamilton-Poisson systems on \( \mathfrak{se}(2)_{-}\) specified by \( H_0(p) = 0 \), \( H_1(p) = p_2^2 \), \( H_2(p) = p_3^2 \), and \( H_3(p) = p_2^2 + p_3^2 \).

**Example 16** Any homogeneous Hamilton-Poisson system \((\mathfrak{so}(3)_{-}, H_Q)\) is \( L \)-equivalent to one of the Hamilton-Poisson systems on \( \mathfrak{so}(3)_{-}\) specified by \( H_0(p) = 0 \) and \( H_{1,\alpha}(p) = p_1^2 + \alpha p_2^2 \). Here \( 0 \leq \alpha \leq 1 \) parametrizes a family of distinct equivalence representatives.

**Remark 17** In the three foregoing examples one would need to verify that no two of the equivalence representatives are \( L \)-equivalent in order to obtain a classification.

A number of quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces have been shown to be \( L \)-equivalent to the relaxed free rigid body dynamics (see [16])
\[
\begin{align*}
\dot{p}_1 &= (\nu_3 - \nu_2)p_2p_3 \\
\dot{p}_2 &= (\nu_1 - \nu_3)p_1p_3 \\
\dot{p}_3 &= (\nu_2 - \nu_1)p_1p_2
\end{align*}
\]
These dynamics correspond to a Hamilton-Poisson system \((\mathfrak{so}(3)_{-}, H_\nu)\), where \( H_\nu(p) = \nu_1p_1^2 + \nu_2p_2^2 + \nu_3p_3^2 \). We may assume that \( \nu_1, \nu_2, \nu_3 > 0 \) (by adding a constant multiple of the Casimir function \( C(p) = p_1^2 + p_2^2 + p_3^2 \)). Then \((\mathfrak{so}(3)_{-}, H_\nu)\) is \( L \)-equivalent to one of the above specified Hamilton-Poisson systems on \( \mathfrak{so}(3)_{-}\).

**References:**