On the Equivalence of Control Systems on the Orthogonal Group $\mathsf{SO}(4)$

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Abstract: We investigate a certain class of left-invariant control systems evolving on the Lie group $\mathsf{SO}(4)$. Two such systems are $\mathfrak{L}$-equivalent provided their traces are related by a Lie algebra automorphism. We produce structural results regarding $\mathfrak{L}$-equivalence of all homogeneous control affine systems on $\mathsf{SO}(4)$. An illustrative example is provided.

Key–Words: Invariant control affine system, $\mathfrak{L}$-equivalence, feedback equivalence, Lie algebra automorphism.

1 Introduction

Invariant control systems are (smooth, nonlinear) control systems evolving on (real, finite dimensional) Lie groups with dynamics invariant under translations. Such control systems have been studied by a number of authors over the last few decades (see, e.g., [2], [8], [9]). In order to understand the local geometry of (nonlinear) control systems it is useful to introduce natural equivalence relations. The most important (and useful) equivalence relations are state space equivalence and feedback equivalence (see, e.g., [7]). The general case of equivalences (of control systems) on Lie groups has been investigated by Biggs and Remsing [4]. (For some concrete cases, see [1], [5].)

In this paper we consider only left-invariant control affine systems, evolving on the (six-dimensional) orthogonal group $\mathsf{SO}(4)$. These systems have the form

$$\dot{g} = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad 1 \leq \ell \leq 6$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{so}(4)$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent.) Some interesting results concerning (invariant) optimal control problems on $\mathsf{SO}(4)$ have been obtained in recent years (see, e.g., [3], [6]). We study the structure of such *homogeneous* systems, with respect to detached feedback equivalence. Specifically, we prove that any homogeneous system is $\mathfrak{L}$-equivalent to one of a list of equivalence representatives.

A few remarks pertaining to the classification of systems, under $\mathfrak{L}$-equivalence, conclude the paper.

2 Invariant control systems

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g\,\Xi(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell)$$

where $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. Here G is a (real, finite dimensional) connected matrix Lie group with Lie algebra $\mathfrak{g}$. Also, the *parametrization map* $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \ldots, B_\ell$ are linearly independent). Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is invariant under left translations i.e., $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$. Such a system is denoted by $\Sigma = (\mathsf{G}, \Xi)$.

The *trace* of Σ, $\Gamma = \operatorname{im}\Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$, is an affine subspace of $\mathfrak{g}$. A system is called *homogeneous* if $A \in \langle B_1, \ldots, B_\ell \rangle$, and *inhomogeneous* otherwise. Σ has *full rank* if the Lie algebra generated by its trace coincides with $\mathfrak{g}$.

The admissible controls are piecewise-continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$.

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ be two systems on G. We say that Σ and Σ' are (locally) *detached feedback equivalent* if there exist open neighborhoods N and N' of (the unit element) $\mathbf{1}$ and a (local) diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$ such that $\phi(\mathbf{1}) = \mathbf{1}$ and $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. Two detached feedback equivalent systems have the same trajectories (up to a diffeomorphism in the state space), which are parametrized differently by admissible controls. We say that the systems Σ and Σ' (with traces $\Gamma \subseteq \mathfrak{g}$ and $\Gamma' \subseteq \mathfrak{g}$, respectively) are $\mathfrak{L}$-*equivalent* if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$. Suppose Σ and Σ' have full rank.

Proposition 1 ([4]) *Σ and Σ' are detached feedback equivalent if and only if they are $\mathfrak{L}$-equivalent.*

Remark 2 Suppose Σ and Σ' do not have full rank. If the systems are $\mathfrak{L}$-equivalent, then they are detached feedback equivalent, but not conversely.

3 The orthogonal group $\mathsf{SO}(4)$

The orthogonal group

$$\mathsf{SO}(4) = \left\{ g \in \mathsf{GL}(4,\mathbb{R}) \;:\; g^\top g = \mathbf{1},\ \det g = 1 \right\}$$

is a six-dimensional, non-commutative, semisimple, compact Lie group. Its Lie algebra

$$\mathfrak{so}(4) = \left\{ A \in \mathbb{R}^{4\times 4} \;:\; A^\top + A = \mathbf{0} \right\}$$

is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Let

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{E}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard (ordered) basis for $\mathfrak{so}(3)$. The map $\varsigma : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4)$, given by

$$\left(\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \right)$$

$$\mapsto \frac{1}{2} \begin{bmatrix} 0 & x_3 - y_3 & x_2 - y_2 & x_1 - y_1 \\ -x_3 + y_3 & 0 & x_1 + y_1 & -x_2 - y_2 \\ -x_2 + y_2 & -x_1 - y_1 & 0 & x_3 + y_3 \\ -x_1 + y_1 & x_2 + y_2 & -x_3 - y_3 & 0 \end{bmatrix}$$

is a Lie algebra isomorphism. The *natural basis* of $\mathfrak{so}(4)$ is given by

$$E_i = \varsigma \cdot (\mathbf{E}_i, \mathbf{0}) \qquad i = 1,2,3$$
$$E_j = \varsigma \cdot (\mathbf{0}, \mathbf{E}_{j-3}) \qquad j = 4,5,6.$$

The commutator table for $\mathfrak{so}(4)$ is given below.

Next we find the group of Lie algebra automorphisms $\mathsf{Aut}(\mathfrak{so}(4))$. With respect to the natural basis, each automorphism is identified with its matrix.

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3	$-E_2$	0	0	0
E_2	$-E_3$	0	E_1	0	0	0
E_3	E_2	$-E_1$	0	0	0	0
E_4	0	0	0	0	E_6	$-E_5$
E_5	0	0	0	$-E_6$	0	E_4
E_6	0	0	0	E_5	$-E_4$	0

Lemma 3 *The group of inner automorphisms of $\mathfrak{so}(4)$ is given by*

$$\mathsf{Int}(\mathfrak{so}(4)) = \left\{ \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \;:\; \psi_1, \psi_2 \in \mathsf{SO}(3) \right\}.$$

For convenience we will identify an inner automorphism $\psi = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}$ with the pair (ψ_1, ψ_2).

Proposition 4 *The group $\mathsf{Aut}(\mathfrak{so}(4))$ is generated by $\mathsf{Int}(\mathfrak{so}(4))$ and the automorphism $\zeta = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}$.*

Proof: Let $M \in \mathsf{Aut}(\mathfrak{so}(4))$. We show that there exist $N_1, \ldots, N_k \in \mathsf{Int}(\mathfrak{so}(4)) \cup \zeta$ such that $N_1 \cdots N_k \cdot M = \mathbf{1}$. Write

$$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{bmatrix}.$$

There exists a rotation $\psi_1 \in \mathsf{SO}(3)$ such that $\psi_1 \cdot \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^\top = \begin{bmatrix} a_1' & 0 & 0 \end{bmatrix}^\top$ with $a_1' \geq 0$. There exists another rotation ψ_1', preserving $\begin{bmatrix} a_1' & 0 & 0 \end{bmatrix}^\top$, such that $\psi_1' \cdot \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}^\top = \begin{bmatrix} a_2' & b_2' & 0 \end{bmatrix}^\top$ with $b_2' \geq 0$. Therefore the top-left block of $(\psi_1', \mathbf{1}) \cdot (\psi_1, \mathbf{1}) \cdot M$ is

$$\begin{bmatrix} a_1' & a_2' & a_3' \\ 0 & b_2' & b_3' \\ 0 & 0 & c_3' \end{bmatrix}.$$

Similarly, by the application of an automorphism (preserving the top-left block), the entries e_4, f_4, f_5 can be made zero and the entries d_4, e_5 can be made nonnegative. Thus there exists an automorphism N_1 such that $N_1 \cdot M = M'$, where

$$M' = \begin{bmatrix} a_1' & a_2' & a_3' & a_4' & a_5' & a_6' \\ 0 & b_2' & b_3' & b_4' & b_5' & b_6' \\ 0 & 0 & c_3' & c_4' & c_5' & c_6' \\ d_1' & d_2' & d_3' & d_4' & d_5' & d_6' \\ e_1' & e_2' & e_3' & 0 & e_5' & e_6' \\ f_1' & f_2' & f_3' & 0 & 0 & f_6' \end{bmatrix}$$

and $a_1', b_2', d_4', e_5' \geq 0$.

As M' is a Lie algebra automorphism, $M' \cdot [E_i, E_j] = [M' \cdot E_i, M' \cdot E_j]$, $i, j = 1, \ldots, 6$. Hence a_2', a_3', b_3', d_5', d_6', e_6' are all zero. Also $a_1' = b_2' c_3'$, $b_2' = a_1' c_3'$, $c_3' = a_1' b_2'$, $d_4' = e_5' f_6'$, $e_5' = d_4' f_6'$, and $f_6' = d_4' e_5'$. Consequently, the diagonal entries are either all zero or all one.

Suppose the diagonal entries are all one. Then

$$M' = \begin{bmatrix} 1 & 0 & 0 & a_4' & a_5' & a_6' \\ 0 & 1 & 0 & b_4' & b_5' & b_6' \\ 0 & 0 & 1 & c_4' & c_5' & c_6' \\ d_1' & d_2' & d_3' & 1 & 0 & 0 \\ e_1' & e_2' & e_3' & 0 & 1 & 0 \\ f_1' & f_2' & f_3' & 0 & 0 & 1. \end{bmatrix}.$$

Again, we impose the condition that M' preserves the Lie bracket. Simple calculations show that $M' = \mathbf{1}$.

Suppose the diagonal entries of M' are all zero. Then

$$\zeta \cdot M' = \begin{bmatrix} d_4' & d_5' & d_6' & 0 & 0 & 0 \\ e_4' & e_5' & e_6' & 0 & 0 & 0 \\ f_4' & f_5' & f_6' & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1' & a_2' & a_3' \\ 0 & 0 & 0 & b_1' & b_2' & b_3' \\ 0 & 0 & 0 & c_1' & c_2' & c_3'. \end{bmatrix}.$$

A similar argument shows that there exists an automorphism N_2 such that $N_2 \cdot \zeta \cdot M' = \mathbf{1}$. □

4 Equivalence

We shall consider $\mathfrak{L}$-equivalence of homogeneous systems on $\mathsf{SO}(4)$. Such a system $\Sigma = (\mathsf{SO}(4), \Xi)$ is uniquely determined by its parametrization map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{so}(4)$. Let $A = \sum_{i=1}^6 a_i E_i$ and $B = \sum_{i=1}^6 b_i E_i$. Any automorphism of $\mathfrak{so}(4)$ preserves the dot product $A \bullet B = \sum_{i=1}^6 a_i b_i$. Let $\Gamma^\perp$ denote the orthogonal complement of a subspace $\Gamma \subset \mathfrak{so}(4)$. The following result is easy to prove.

Proposition 5 *Let $\Gamma, \widetilde{\Gamma}$ be subspaces of $\mathfrak{so}(4)$ and let $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$. Then*

$$\psi \cdot \Gamma = \widetilde{\Gamma} \iff \psi \cdot \Gamma^\perp = \widetilde{\Gamma}^\perp.$$

For the sake of convenience we shall use $\rho_1(\theta)$, $\rho_2(\theta)$, and $\rho_3(\theta)$ to denote, respectively, the rotations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 6 Let $\Xi(\mathbf{1}, u) = u_1(E_1 + \sqrt{3}E_3 + E_4) + u_2(3E_2 + E_5 + E_6)$. The trace of this system is given by $\Gamma = \langle E_1 + \sqrt{3}E_3 + E_4, 3E_2 + E_5 + E_6 \rangle$. Now $(\rho_2(\frac{\pi}{3}), \mathbf{1})$ is an automorphism such that

$$\begin{aligned} (\rho_2(\tfrac{\pi}{3}), \mathbf{1}) \cdot \Gamma = \big\langle & \cos\tfrac{\pi}{3} E_1 - \sin\tfrac{\pi}{3} E_3 \\ & + \sqrt{3}\,(\sin\tfrac{\pi}{3} E_1 + \cos\tfrac{\pi}{3} E_3) + E_4, \\ & 3E_2 + E_5 + E_6 \big\rangle \\ = & \langle 2E_1 + E_4, 3E_2 + E_5 + E_6 \rangle. \end{aligned}$$

Also, $(\mathbf{1}, \rho_1(-\frac{\pi}{4}))$ is an automorphism such that

$$\begin{aligned} & (\mathbf{1}, \rho_1(-\tfrac{\pi}{4})) \cdot \langle 2E_1 + E_4, 3E_2 + E_5 + E_6 \rangle \\ & \qquad = \left\langle E_1 + \tfrac{1}{2}E_4, E_2 + \tfrac{\sqrt{2}}{3}E_5 \right\rangle = \widetilde{\Gamma}. \end{aligned}$$

Therefore, Σ is $\mathfrak{L}$-equivalent to $\widetilde{\Xi}(\mathbf{1}, u) = u_1(E_1 + \frac{1}{2}E_4) + u_2(E_2 + \frac{\sqrt{2}}{3}E_5)$.

By proposition 5, it follows that $\psi \cdot \Gamma^\perp = \widetilde{\Gamma}^\perp$, where $\psi = \big(\rho_2(\frac{\pi}{3}), \rho_1(-\frac{\pi}{4})\big)$. Hence any system with trace

$$\begin{aligned} \Gamma^\perp = \Big\langle & E_1 - E_4 + E_5 - E_6, \\ & 2E_2 + E_3 - \sqrt{3}E_4 - 6E_6, \\ & E_3 - \sqrt{3}E_4,\ E_1 + E_2 - E_4 - 3E_6 \Big\rangle \end{aligned}$$

is $\mathfrak{L}$-equivalent to one with trace

$$\widetilde{\Gamma}^\perp = \left\langle E_3, E_6, E_1 - 2E_4, E_2 - \tfrac{3}{\sqrt{2}}E_5 \right\rangle.$$

Theorem 7 *Any one-input system is $\mathfrak{L}$-equivalent to a system*

$$\Xi_1^1(\mathbf{1}, u) = u_1 E_1$$
$$\Xi_{2,\alpha}^1(\mathbf{1}, u) = u_1(E_1 + \alpha E_4)$$

for some $0 < \alpha \leq 1$.

Proof: Any single-input system has trace $\Gamma_1 = \langle A_1 \rangle$, $\Gamma_2 = \langle A_2 \rangle$, or $\Gamma_3 = \langle A_1 + A_2 \rangle$. Here $A_1 = \sum_{i=1}^3 a_i E_i$ and $A_2 = \sum_{i=4}^6 a_i E_i$ are nonzero. For

Γ_1, there exists an inner automorphism $\psi = (\psi_1, \mathbf{1})$ such that $\psi \cdot \Gamma_1 = \langle E_1 \rangle = \Gamma_1^1$. Similarly, there exists an inner automorphism $\psi = (\mathbf{1}, \psi_2)$ such that $\psi \cdot \Gamma_2 = \langle E_4 \rangle$. Hence $\zeta \cdot \psi \cdot \Gamma_2 = \langle E_1 \rangle = \Gamma_1^1$. For Γ_3, there exists a $\psi = (\psi_1, \psi_2)$ such that $\psi \cdot \Gamma_3 = \langle E_1 + \alpha E_4 \rangle$ for some $\alpha > 0$. If $\alpha \leq 1$, then $\psi \cdot \Gamma_3 = \Gamma^1_{2,\alpha}$. If $\alpha > 1$, then $\zeta \cdot \psi \cdot \Gamma_3 = \langle E_4 + \alpha E_1 \rangle = \left\langle E_1 + \frac{1}{\alpha} E_4 \right\rangle = \Gamma^1_{2,\frac{1}{\alpha}}$. □

By proposition 5, we get the following result.

Corollary 8 *Any five-input system is $\mathfrak{L}$-equivalent to a system*

$$\begin{aligned} \Xi_1^5\,(\mathbf{1}, u) &= u_1 E_2 + u_2 E_3 + u_3 E_4 \\ &\quad + u_4 E_5 + u_6 E_6 \\ \Xi_{2,\alpha}^5\,(\mathbf{1}, u) &= u_1 E_2 + u_2 E_3 + u_3 E_5 \\ &\quad + u_4 E_6 + u_5(\alpha E_1 - E_4) \end{aligned}$$

for some $0 < \alpha \leq 1$.

Remark 9 Any five-input system has full rank.

Lemma 10 *Any system* $\Xi(\mathbf{1}, u) = u_1(E_1 + \gamma_1 E_4 + \alpha E_5) + u_2(E_2 + \gamma_2 E_5)$, $\gamma_1, \gamma_2, \alpha > 0$ *is $\mathfrak{L}$-equivalent to a system* $\Xi'(\mathbf{1}, u) = u_1(E_1 + \gamma_1' E_4 + \alpha' E_5) + u_2(E_2 + \gamma_1' E_5)$ *for some* $\gamma_1', \alpha' > 0$.

Proof: It suffices to show that there exists an automorphism $\psi = (\rho_3(\theta_1), \rho_3(\theta_2))$ such that

$$\begin{aligned} \psi \cdot (E_1 + \gamma_1 E_4 + \alpha E_5) &= r_1(E_1 + \gamma_1' E_4 + \alpha' E_5) \\ &\quad + r_2(E_2 + \gamma_1' E_5) \\ \psi \cdot (E_2 + \gamma_2 E_5) &= r_3(E_1 + \gamma_1' E_4 + \alpha' E_5) \\ &\quad + r_4(E_2 + \gamma_1' E_5) \end{aligned}$$

for some $\alpha', \gamma_1' > 0$, $r_1, r_2, r_3, r_4 \in \mathbb{R}$. (Here (r_1, r_2) and (r_3, r_4) are required to be linearly independent.) This reduces to showing that (the system of equations)

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$$

and

$$\begin{aligned} &\begin{bmatrix} \alpha\sin\theta_2 + \gamma_1\cos\theta_2 & \alpha\cos\theta_2 - \gamma_1\sin\theta_2 \\ \gamma_2\sin\theta_2 & \gamma_2\cos\theta_2 \end{bmatrix} \\ &\qquad = \begin{bmatrix} r_1\gamma_1' & r_1\alpha' + r_2\gamma_1' \\ r_3\gamma_1' & r_3\alpha' + r_4\gamma_1' \end{bmatrix} \end{aligned}$$

has a solution.

The values r_1, r_2, r_3, r_4 are fixed by the first equation (in terms of θ_1). From the second equation we then get (by equating the first columns)

$$\begin{bmatrix} \alpha\sin\theta_2 + \gamma_1\cos\theta_2 \\ \gamma_2\sin\theta_2 \end{bmatrix} = \begin{bmatrix} \gamma_1'\cos\theta_1 \\ \gamma_1'\sin\theta_1 \end{bmatrix}.$$

For any θ_2 there always exist θ_1 and $\gamma_1' > 0$ satisfying this equation. Hence $\cos\theta_1 = \frac{1}{\gamma_1'}(\alpha\sin\theta_2 + \gamma_1\cos\theta_2)$ and $\sin\theta_1 = \frac{1}{\gamma_1'}(\gamma_2\sin\theta_2)$.

It remains to be shown that

$$\begin{aligned} &\begin{bmatrix} \alpha\cos\theta_2 - \gamma_1\sin\theta_2 \\ \gamma_2\cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma_1'}(\alpha\sin\theta_2 + \gamma_1\cos\theta_2)\alpha' - \frac{1}{\gamma_1'}(\gamma_2\sin\theta_2)\gamma_1' \\ \frac{1}{\gamma_1'}(\gamma_2\sin\theta_2)\alpha' + \frac{1}{\gamma_1'}(\alpha\sin\theta_2 + \gamma_1\cos\theta_2)\gamma_1' \end{bmatrix} \end{aligned}$$

has a solution (for some α' and θ_2). This reduces to

$$\begin{bmatrix} \gamma_1\alpha' - \alpha\gamma_1' & \alpha\alpha' + \gamma_1'(\gamma_1 - \gamma_2) \\ \gamma_1'(\gamma_1 - \gamma_2) & \gamma_2\alpha' + \alpha\gamma_1' \end{bmatrix} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} = \mathbf{0}.$$

For $\alpha' = \frac{\sqrt{\alpha^2 + (\gamma_1 - \gamma_2)^2}(\gamma_1)'}{\sqrt{\gamma_1}\sqrt{\gamma_2}}$ the determinant of the matrix is zero. Thus there exists a solution (for some θ_2). □

A similar argument yields the following result.

Lemma 11 *Any system* $\Xi(\mathbf{1}, u) = u_1(E_1 + \gamma_1 E_4) + u_2(E_2 + \gamma_2 E_5)$, $\gamma_1, \gamma_2 > 0$ *is $\mathfrak{L}$-equivalent to a system* $\Xi'(\mathbf{1}, u) = u_1(E_1 + \gamma_1' E_4 + \alpha' E_5) + u_2(E_2 + \gamma_1' E_5)$ *for some* $\gamma_1' > 0$, $\alpha' \geq 0$.

Theorem 12 *Any two-input system is $\mathfrak{L}$-equivalent to a system*

$$\begin{aligned} \Xi_1^2\,(\mathbf{1}, u) &= u_1 E_1 + u_2 E_2 \\ \Xi_2^2\,(\mathbf{1}, u) &= u_1 E_1 + u_2 E_4 \\ \Xi_{3,\alpha}^2\,(\mathbf{1}, u) &= u_1 E_1 + u_2(E_2 + \alpha E_5) \\ \Xi_{4,\alpha\beta}^2\,(\mathbf{1}, u) &= u_1(E_1 + \alpha E_4 + \beta E_5) \\ &\quad + u_2(E_2 + \alpha E_5) \end{aligned}$$

for some $\alpha > 0$, $\beta \geq 0$.

Proof: Let Ξ be a two-input system with trace $\Gamma = \langle A_1 + A_2, B_1 + B_2 \rangle$. Here $A_1 = \sum_{i=1}^3 a_i E_i$, $A_2 = \sum_{i=4}^6 a_i E_i$, $B_1 = \sum_{i=1}^3 b_i E_i$, and $B_2 = \sum_{i=4}^6 b_i E_i$. Suppose $A_1, A_2, B_1, B_2 \neq 0$, $\langle A_1 \rangle \neq \langle B_1 \rangle$ and $\langle A_2 \rangle \neq \langle B_2 \rangle$. There exists $(\psi_1, \mathbf{1}) \in \mathsf{Int}(\mathfrak{so}\,(4))$ such that $(\psi_1, \mathbf{1}) \cdot A_1 = rE_1$, $r = \sqrt{A_1 \bullet A_1}$. Thus $(\psi_1, \mathbf{1}) \cdot \Gamma = \Gamma'$, where

$$\Gamma' = \left\langle rE_1 + A_2, b_1' E_1 + b_2' E_2 + b_3' E_3 + B_2 \right\rangle$$

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for some $b'_1,\, b'_2,\, b'_3 \in \mathbb{R}$. Hence

$$\Gamma' = \left\langle E_1 + \tfrac{1}{r}A_2, b'_2E_2 + b'_3E_3 + B'_2 \right\rangle$$

where $B'_2 = B_2 - \frac{b'_1}{r}A_2$. Similarly, there exists $(\mathbf{1}, \psi_2) \in \mathsf{Int}(\mathfrak{so}\,(4))$ such that

$$\Gamma'' = (\mathbf{1}, \psi_2)\cdot \Gamma' = \left\langle E_1 + A'_2, b''_1E_2 + b''_3E_3 + E_5 \right\rangle$$

for some $a'_4,\, a'_5,\, a'_6,\, b''_1,\, b''_3 \in \mathbb{R}$ with $A'_2 = \sum_{i=4}^{6} a'_i E_i$.

Now $b''_1 = \gamma\cos\theta$ and $b''_3 = \gamma\sin\theta$ for some $\gamma > 0$ and $\theta \in \mathbb{R}$. Applying the automorphism $\psi = (\rho_1(-\theta), \mathbf{1})$ we obtain

$$\begin{aligned}\psi \cdot \Gamma'' &= \big\langle E_1 + A'_2, \gamma\cos\theta(\cos\theta E_2 - \sin\theta E_3) \\ &\qquad + \gamma\sin\theta(\sin\theta E_2 + \cos\theta E_3) + E_5 \big\rangle \\ &= \left\langle E_1 + A'_2, \gamma E_2 + E_5 \right\rangle.\end{aligned}$$

Similarly, there exists an automorphism $\psi' = (\mathbf{1}, \rho_2(\theta))$ such that

$$\psi' \cdot \psi \cdot \Gamma'' = \langle E_1 + \gamma_1E_4 + \alpha E_5, E_2 + \gamma_2E_5 \rangle$$

for some $\gamma_1,\, \gamma_2 > 0$ and $\alpha \in \mathbb{R}$. If $\alpha < 0$, then $\psi'' = (\rho_2(-\pi), \rho_2(-\pi))) \cdot \psi' \cdot \psi$ is an automorphism such that

$$\psi'' \cdot \Gamma'' = \langle E_1 + \gamma_1E_4 - \alpha E_5, E_2 + \gamma_2E_5 \rangle .$$

By lemma 10 and 11, it then follows that Ξ is $\mathfrak{L}$-equivalent to the system

$$\begin{aligned}\Xi^2_{\alpha'\beta'}\,(\mathbf{1}, u) &= u_1(E_1 + \alpha'E_4 + \beta'E_5) \\ &\quad + u_2(E_2 + \alpha'E_5)\end{aligned}$$

for some $\alpha' > 0,\ \beta' \geq 0$.

Suppose one or more of $A_1,\, A_2,\, B_1,\, B_2$ are zero, $\langle A_1\rangle = \langle B_1\rangle$ or $\langle A_2\rangle = \langle B_2\rangle$. Similarly, there exists an automorphism ψ such that $\psi \cdot \Gamma$ equals $\Gamma^2_1 = \langle E_1, E_2\rangle$, $\Gamma^2_2 = \langle E_1, E_4\rangle$, or $\Gamma^2_{3,\alpha} = \langle E_1, E_2 + \alpha E_5\rangle$ for some $\alpha > 0$. □

Corollary 13 *Any four-input system is $\mathfrak{L}$-equivalent to a system*

$$\begin{aligned}\Xi^4_1\,(\mathbf{1}, u) &= u_1E_3 + u_2E_4 + u_3E_5 + u_4E_6 \\ \Xi^4_2\,(\mathbf{1}, u) &= u_1E_2 + u_2E_3 + u_5E_5 + u_6E_6 \\ \Xi^4_{3,\alpha}\,(\mathbf{1}, u) &= u_1E_3 + u_2E_4 + u_3E_6 \\ &\quad + u_4(\alpha E_2 - E_5) \\ \Xi^4_{4,\alpha\beta}\,(\mathbf{1}, u) &= u_1E_3 + u_2E_6 + u_3\big(\beta E_1 + \alpha E_2 \\ &\quad - E_5\big) + u_4(\alpha E_1 - E_4)\end{aligned}$$

for some $\alpha > 0$ and $\beta \geq 0$.

Likewise, a somewhat more involved argument yields the following result. The proof is omitted.

Theorem 14 *Any three-input homogeneous system is $\mathfrak{L}$-equivalent to a system*

$$\begin{aligned}\Xi^3_{1,\boldsymbol{\alpha}\beta}\,(\mathbf{1}, u) &= u_1(E_1 + \alpha_1E_4) + u_2(E_2 + \alpha_2E_5) \\ &\quad + u_3(E_3 + \beta E_6) \\ \Xi^3_{2,\gamma}\,(\mathbf{1}, u) &= u_1(E_1 + \gamma E_4) + u_2(E_2 + \gamma E_5) \\ &\quad + u_3(E_3 \pm \gamma E_6) \\ \Xi^3_{3,\gamma}\,(\mathbf{1}, u) &= u_1(E_1 + \gamma E_4) + u_2E_2 + u_3E_6\end{aligned}$$

for some $\alpha_1 \geq \alpha_2 \geq |\beta| \geq 0$ and $0 \leq \gamma \leq 1$. Here $\alpha_1 \neq \alpha_2$ or $\alpha_2 \neq |\beta|$.

Remark 15 There is only one 6-dimensional affine subspace of $\mathfrak{so}\,(4)$, namely $\mathfrak{so}\,(4)$. Therefore any six-input system is $\mathfrak{L}$-equivalent to the system

$$\begin{aligned}\Xi^6(\mathbf{1}, u) &= u_1E_1 + u_2E_2 + u_3E_3 \\ &\quad + u_4E_4 + u_5E_5 + u_6E_6.\end{aligned}$$

5 An illustrative example

Any system

$$\begin{aligned}\Xi_{\boldsymbol{\gamma}}\,(\mathbf{1}, u) &= \gamma_1E_4 + u_1(\gamma_2E_1 + \gamma_3E_2 + \gamma_4E_3) \\ &\quad + u_2(\gamma_5E_3 + \gamma_6E_4) \\ &\quad + u_3(\gamma_7E_4) + u_4(\gamma_8E_5)\end{aligned}$$

is $\mathfrak{L}$-equivalent to the system

$$\widetilde{\Xi}(\mathbf{1}, u) = u_1E_2 + u_2E_3 + u_3E_4 + u_4E_5.$$

Here $\gamma_i > 0,\ i = 1, \ldots, 9$. As these systems have full rank, they are in fact detached feedback equivalent.

The automorphism relating the traces of these systems is given by $\psi = (\psi_1, \mathbf{1})$, where

$$\psi_1 = \begin{bmatrix} \frac{\gamma_3}{\sqrt{\gamma_2^2+\gamma_3^2}} & -\frac{\gamma_2}{\sqrt{\gamma_2^2+\gamma_3^2}} & 0 \\ \frac{\gamma_2}{\sqrt{\gamma_2^2+\gamma_3^2}} & \frac{\gamma_3}{\sqrt{\gamma_2^2+\gamma_3^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The corresponding feedback transformation φ, defined by

$$\psi \cdot \Xi_{\boldsymbol{\gamma}}\,(\mathbf{1}, u) = \widetilde{\Xi}\,(\mathbf{1}, \varphi(u))$$

is given by

$$\varphi : u \mapsto \begin{bmatrix} \sqrt{\gamma_2^2+\gamma_3^2} & 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & 0 & 0 \\ 0 & \gamma_6 & \gamma_7 & 0 \\ 0 & 0 & 0 & \gamma_8 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \gamma_1 \\ 0 \end{bmatrix}.$$

There exists a group automorphism ϕ such that $T_{\mathbf{1}}\phi = \psi$. The map ϕ establishes a one-to-one correspondence between trajectories of Ξ_γ and $\widetilde{\Xi}$. Specifically, $(g(\cdot), u(\cdot))$ is a trajectory-control pair of Ξ_γ if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a trajectory-control pair of $\widetilde{\Xi}$. This reduces the study of trajectories of Ξ_γ to the study of trajectories of $\widetilde{\Xi}$. In particular, any ϕ-invariant property (e.g., periodicity) can be investigated on $\widetilde{\Xi}$ rather than on Ξ_γ.

6 Concluding remarks

In this paper we obtained an exhaustive list of equivalence representatives for homogeneous systems on $\mathsf{SO}(4)$. In order to get a classification of systems, it is required to make sure that no two equivalence representatives are equivalent.

It is easy to verify that the systems Ξ^1_1 and $\Xi^1_{2,\alpha}$ are not equivalent (for any $0 < \alpha \leq 1$). It is also easy to show that the systems $\Xi^1_{2,\alpha}$ and $\Xi^1_{2,\alpha'}$ $(0 < \alpha, \alpha' \leq 1)$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. Hence theorem 7 (and so also corollary 8) provides a classification of systems.

For the two-input systems such a verification becomes quite involved. Clearly,

$$\Xi^2_1(\mathbf{1}, u) = u_1 E_1 + u_2 E_2$$

is not $\mathfrak{L}$-equivalent to

$$\Xi^2_2(\mathbf{1}, u) = u_1 E_1 + u_2 E_4$$

as $\psi \cdot \langle E_1, E_2 \rangle \subset \langle E_1, E_2, E_3 \rangle$ or $\psi \cdot \langle E_1, E_2 \rangle \subset \langle E_4, E_5, E_6 \rangle$ for any automorphism ψ. Likewise, it is fairly easy to show that none of Ξ^2_1, Ξ^2_2, $\Xi^2_{3,\alpha}$, or $\Xi^4_{3,\alpha\beta}$ are $\mathfrak{L}$-equivalent to one another. Now suppose $\Xi^2_{3,\alpha}$ and $\Xi^2_{3,\alpha'}$ are $\mathfrak{L}$-equivalent. Any element of $\mathsf{SO}(3)$ can be written (in terms of Euler angles) as

$$R_{\theta_3\theta_2\theta_1} = \rho_3(\theta_3)\rho_2(\theta_2)\rho_1(\theta_1).$$

Hence, any automorphism ψ can be written as $\psi = (R_{\theta_3\theta_2\theta_1}, R_{\theta_6\theta_5\theta_4})$ or $\psi = \zeta \circ (R_{\theta_3\theta_2\theta_1}, R_{\theta_6\theta_5\theta_4})$. The system $\Xi^2_{3,\alpha}$ is $\mathfrak{L}$-equivalent to $\Xi^2_{3,\alpha'}$ if and only if there exist linearly independent vectors (r_1, r_2), $(r_3, r_4) \in \mathbb{R}^2$ such that $\psi \cdot E_1 = r_1 E_1 + r_2(E_2 + \alpha' E_5)$ and $\psi \cdot (E_2 + \alpha E_5) = r_3 E_1 + r_4(E_2 + \alpha' E_5)$ for some automorphism ψ. Applying an automorphism $\psi = \zeta \circ (R_{\theta_3\theta_2\theta_1}, R_{\theta_6\theta_5\theta_4})$, we immediately get that $r_1 = r_2 = 0$, a contradiction. If we apply an automorphism $\psi = (R_{\theta_3\theta_2\theta_1}, R_{\theta_6\theta_5\theta_4})$, then we get

$$\begin{bmatrix} \cos\theta_2\cos\theta_3 & \cos\theta_3\sin\theta_1\sin\theta_2 - \cos\theta_1\sin\theta_3 \\ \cos\theta_2\sin\theta_3 & \cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3 \\ -\sin\theta_2 & \cos\theta_2\sin\theta_1 \\ 0 & \alpha\,(\cos\theta_6\sin\theta_4\sin\theta_5 - \cos\theta_4\sin\theta_6) \\ 0 & \alpha\,(\cos\theta_4\cos\theta_6 + \sin\theta_4\sin\theta_5\sin\theta_6) \\ 0 & \alpha\cos\theta_5\sin\theta_4 \end{bmatrix} = \begin{bmatrix} r_1 & r_3 \\ r_2 & r_4 \\ 0 & 0 \\ 0 & 0 \\ r_2\alpha' & r_4\alpha' \\ 0 & 0 \end{bmatrix}.$$

Therefore $\sin\theta_2 = 0$ and so $\cos\theta_2 = \pm 1$. Hence $\sin\theta_1 = 0$ and $\cos\theta_1 = \pm 1$. Further simplification yields $\alpha = \alpha'$. Therefore, $\Xi^2_{3,\alpha}$ and $\Xi^2_{3,\alpha'}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. For $\Xi^2_{4,\alpha\beta}$ such a computation becomes more involved.

A verification for three-input systems becomes fairly complicated. We do not know whether theorems 12 and 14 provide a classification.

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