Representing a System in Terms of Fuzzy Logic

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Abstract: There are several processes, frequently appearing in a system's operation, involving. vagueness and/or uncertainty. In the present paper we develop a general fuzzy model for representing such kind of processes. More explicitly, the main stages of these processes are represented as fuzzy subsets of a set of linguistic labels characterizing the degree of the system's success at the respective stage and the probabilities and possibilities of all possible profiles of the subjects involved are calculated. Examples are also presented from the areas of Education and Management illustrating the use of our model in practice.

Keywords: Fuzzy Sets and Logic, Fuzzy Probabilities and Possibilities, Systems' Theory, Learning, Management.

1. Introduction: Systems and Fuzzy Logic

The word *system* (from Latin *systema,* in turn of the Greek *σύστημα*) in its meaning here has a long history which can be traced back to Plato (*Phillebus)*, Aristotle (*Politics)* and Euclid (*Elements)*. It had meant "total", "crowd" or "union" in even more ancient times, as it derives from the verb *sunistemi*: uniting, putting together. Nowadays, in the most general sense, system means a configuration of parts connected and joined together by a web of relationships.

The first to develop the concept of a system in the natural sciences was the French physicist *Carnot* (1824) who studied thermodynamics. In 1850 the German physicist *Clausius* generalized Carnot's picture to include the concept of the surroundings and began to use the term "working body" when referring to the system. One of the pioneers of the general systems' theory was the biologist *Bertalanffy* [2], while significant development to the concept of the system was done by *Wienner* and *Ashby,* who pioneered the use of mathematics to study systems [10]. Contemporary ideas from systems' theory have grown with diversified areas, exemplified by the works of *Banathy* [1], *Hammond* [3], *Odum* [6] and others.

Currently, applications of the system concept include information and computer science, engineering and physics, social and cognitive sciences, management and economics, strategic thinking, fuzziness and uncertainty, etc. Systems' theory thus serves as a bridge for interdisciplinary dialogue between autonomous areas of study, as well as within the area of systems' science itself.

Most systems share common characteristics including structure, behavior, interconnectivity (the various parts of a system have functional and structural relations to each other), sets of functions, etc. Systems' theory views the world as a complex system of interconnected parts. We scope a system by defining its boundary; this means choosing which entities are inside the system and which are outside - part of the environment. We then take simplified representations (models) of the system in order to understand it and to predict or impact its future behavior.

Systems' modelling is generally a basic principle in engineering and in social sciences. The model is the representation of the system's entities under concern. Hence inclusion to or exclusion from system's context is dependent of the intention of the modeler. Thus, no model of a real system could include all features and/or all entities belonging to it.

They appear often processes in a system's operation characterized by a degree of vagueness and/or uncertainty. In education, for example, during the processes of learning, of problem-solving, of modelling, etc, students' cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teacher's point of view there usually exists an uncertainty about the degree of students' success in each of the stages of the corresponding didactic situation.

There used to be a tradition in science and engineering of turning to *probability theory* when one is faced with a problem in which uncertainty plays a significant role. This transition was justified when there were no alternative tools for dealing with the uncertainty. Today this is no longer the case. *Fuzzy systems*, including fuzzy logic and fuzzy set theory provide a rich and meaningful addition to standard logic. The mathematics generated by these theories is consistent and fuzzy logic is a generalization of classic logic. The applications which may be generated from or adapted to fuzzy logic are wide-ranging and provide the opportunity for modelling under conditions which are inherently imprecisely defined, despite the concerns of classical logicians. Many systems may be modelled, simulated and even replicated with the help of fuzzy systems, not the least of which is human reasoning itself.

The history shows some traces of foundational ideas of fuzzy logic in the philosophical thoughts put forth by *Buddha*, who lived in India during 500 BC. His philosophy was based on the thought that almost everything contains some of its opposite, or in other words that things can be X and not X at the same time. However, it was *Plato* (427-347 BC) who laid the foundation for what would become fuzzy logic, including that there was a third region (beyond True and False), where these opposites "tumbled about". Other, more modern philosophers echoed his sentiments, notably *Hegel*, *Marx* and *Engels*. But it was *Lukasiewicz* [5] who first proposed a systematic alternative to the bi-valued logic of *Aristotle*. In the early 1900's he described the 3-valued logic by adding the term "Possible" and he assigned it a numeric value between 'True" and "False". Later he explored 4 and 5-valued logics and then declared that in principle there was nothing to prevent the derivation of an infinite-valued logic.

Nevertheless, it was not until relatively recently that the notion of an infinite-valued logic took hold. In 1965 *Zadeh* published his seminal work "*Fuzzy Sets*" [11] which described the mathematics of fuzzy set theory and by extension fuzzy logic. This theory proposed making the membership function to operate over the range of real numbers [0, 1]. New operations for the calculus of logic were proposed and showed to be in principle at least a generalization of classic logic ([11], [12]).

Despite the fact that both operate over the same numeric range [0, 1], fuzzy set theory is distinct from probability theory. For example, the probabilistic approach yields the natural language statement "there is an 85% chance that Mary is tall", while the fuzzy terminology corresponds to "Mary's degree of membership within the set of tall people is 0,85". The semantic difference is significant: The first view supposes that Mary is or is not tall (still caught in the law of the Excluded Middle); it is just that we only have a 85% chance of knowing in which set she is in. By contrast, fuzzy terminology supposes that Mary is "more or less" tall, or some other term corresponding to the value of 0,85. Another immediately apparent difference is that the summation of probabilities of the single subsets of a universal set must equal 1, while there is no such requirement for membership degrees. The methods of choosing the suitable membership function for each case are usually empiric, based on experiments made on samples of the population that we study. Further distinctions between probability and fuzziness arising from the operations also exist.

A real test of the effectiveness of an approach to uncertainty is the capability to solve problems which involve different facets of uncertainty. Fuzzy logic has a much higher problem-solving capability than standard probability theory. Most importantly, it opens the door to construction of mathematical solutions of computational problems which are stated in a natural language. In contrast, standard probability theory does not have this capability, a fact which is one of its principal limitations.

All the above gave us the impulsion to introduce principles of fuzzy logic to describe in a more effective way a system's operation in situations characterized by a degree of vagueness and/or uncertainty Therefore our target in this paper is to construct a general (fuzzy) model that could be adapted in each particular case in order to represent the corresponding process.

For general facts on fuzzy sets and logic we refer freely to the book of *Klir* and *Folger* [4].

2. The general fuzzy model

Assume that we want to study the behavior of a system's n entities, $n \geq 2$, during a process involving vagueness and/or uncertainty.

Denote by S_i , $i=1,2,3$ the main stages of this process and by *a, b, c, d,* and *e* the linguistic labels of negligible, low, intermediate, high and complete success respectively of a system's entity in each of the S_i*'s*. Set

$$U = \{a, b, c, d, e\}.$$

We are going to attach to each stage S_i a fuzzy subset, A_i of U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of entities that faced negligible, low, intermediate, high and complete success at stage S_i respectively, $i=1,2,3$, we define the *membership function* m_{Ai} for each x in U, as follows:

$$m_{A_i}(x) = \begin{cases} 1, & \text{if } \frac{4n}{5} < n_{ix} \leq n \\ 0{,}75, & \text{if } \frac{3n}{5} < n_{ix} \leq \frac{4n}{5} \\ 0{,}5, & \text{if } \frac{2n}{5} < n_{ix} \leq \frac{3n}{5} \\ 0{,}25, & \text{if } \frac{n}{5} < n_{ix} \leq \frac{2n}{5} \\ 0, & \text{if } 0 \leq n_{ix} \leq \frac{n}{5} \end{cases}$$

Then the fuzzy subset A_i of U corresponding to S_i has the form:

$$A_i = \{(x, m_{Ai}(x)): x \in U\}, i=1, 2, 3.$$

In order to represent all possible *profiles (overall states)* of the system's entities during the corresponding process we consider a *fuzzy relation*, say R, in U^3 of the form

$$R= \{(s, m_R(s)): s=(x, y, z) \in U^3\}.$$

We make the hypothesis that the stages of the process that we study are depended to each other. This means that the degree of system's success in a certain stage depends upon the degree of its success in the previous stages, as it usually happens in practice. Under this hypothesis and in order to determine properly the membership function m_R we give the following definition:

Definition 2.1: A profile $s=(x, y, z)$, with x, y, z in U, is said to be *well ordered* if x corresponds to a degree of success equal or greater than y and y corresponds to a degree of success equal or greater than z.

For example, *(c, c, a)* is a well ordered profile, while *(b, a, c)* is not.

We define now the *membership degree* of a profile s to be

$$m_R(s) = m_{A_1}(x) m_{A_2}(y) m_{A_3}(z)$$

if s is well ordered, and 0 otherwise.

In fact, if for example profile *(b, a, c)* possessed a nonzero membership degree, how it could be possible for an object that has failed during the middle stage, to perform satisfactorily at the next stage?

Next, for reasons of brevity, we shall write m_s instead of $m_R(s)$. Then the *probability* p_s of the profile s is defined in a way analogous to crisp data, i.e. by

$$P_s = \frac{m_s}{\sum_{s \in U^3} m_s} .$$

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We define also the *possibility* r_s of s by

$$r_s = \frac{m_s}{\max\{m_s\}},$$

where $max\{m_s\}$ denotes the maximal value of m_s, for all s in U^3. In other words the possibility of s expresses the "relative membership degree" of s with respect to $max\{m_s\}$.

Assume finally that one wants to study the *combined results* of behavior of k different groups of the system's entities, $k\geq 2$, during the same process. For this we introduce the *fuzzy variables* $A_1(t)$, $A_2(t)$ and $A_3(t)$ with $t=1, 2,\ldots, k$. The values of these variables represent fuzzy subsets of U corresponding to the stages of the process for each of the k groups; e.g. $A_1(2)$ represents the fuzzy subset of U corresponding to the first stage of the process for the second group ($t=2$). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the probability $p(s)$ and the possibility $r(s)$ of each profile s with respect to the membership degrees of s for all groups. For this reason we introduce the *pseudo-frequencies*

$$f(s) = \sum_{t=1}^{k} m_s(t)$$

and we define

$$p(s) = \frac{f(s)}{\sum_{s\in U^3} f(s)}$$

and

$$r(s) = \frac{f(s)}{\max\{f(s)\}},$$

where $max\{f(s)\}$ denotes the maximal pseudo-frequency.

Obviously the same method could be applied when one wants to study the combined results of behaviour of a group during k different situations.

3 Applications of the fuzzy model

In earlier papers we have applied the above model (or similar ones) for a more effective description of several situations involving fuzziness and uncertainty, mainly in the areas of Mathematics Education and of Artificial Intelligence (e.g. see [9] and its references). In this section we shall sketch one of these applications for the process of learning and we shall present another example from the area of Management

3.1 The process of learning

The concept of learning is fundamental to the study of human cognitive action and very many theories and models were developed by researchers and educators describing its nature and process.

Voss [9] argues that learning basically consists of successive problem–solving activities, in which the input information is represented of existing knowledge, with the solution occurring when the input is appropriately interpreted. The whole process involves the following stages: *Representation* of the input data, *interpretation* of these data in order to produce the new knowledge, *generalization* of the new knowledge to a variety of situations and *categorization* of the generalized knowledge.

In developing our fuzzy model for the process of learning we considered a group of n students, $n\geq 2$, during the learning process of a subject matter in classroom and we denoted by S_i, $i=1, 2, 3$, the stages of representation/interpretation (as a joined stage), generalization and categorization of the Voss's model. To each of the S_i's we attached a fuzzy subset A_i of U defining the membership function m_{Ai} in terms of the frequencies, i.e. by

$$m_{Ai}(x) = \frac{n_{ix}}{n}$$

for each x in U. Thus we can write

$$A_i = \{(x, \frac{n_{ix}}{n}) : x\in U\}.$$

The development of the fuzzy model for learning follows then the general lines presented in the previous section. For more details and classroom applications of the model see [8].

3.2 An application to Management

An enterprise is willing to evaluate the data of a market's research about the consumers' correspondence for its negotiable products. The level of this correspondence is characterized by the fuzzy linguistic labels of

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a=negligible, b=low, c=moderate, d=high and e=very high respectively.

The research has been made separately for men and women and for 3 different categories of age, namely 18-30, 31-50 and over 50 years old.

The consumers' correspondence for each of the above categories of age can be represented by a fuzzy subset

$$A_i = \{(x, m_{A_i}(x)), x \in U\},\ i=1,2,3$$

of $U = \{a, b, c, d, e\}$.

In order to cover separately men and women, we introduce the fuzzy variables $A_i(t)$, $t=1,2$

Let us assume further that according to the fuzzy data of the market's research we have:

	$A_1(t)$	$A_2(t)$	$A_3(t)$
t = 1	(0,486 / c) + (0,228 / d) + +(0,286 / e)	(0,171/ a) + (0,171/ b) + +(0,4 / c) + (0,257 / d)	(0,343 / a) + +(0,286 / b) + +(0,371/ c)
t = 2	(0,2 / b) + (0,5 / c) + +(0,3 / d)	(0,2 / a) + (0,267 / b) + +(0,533 / c)	(0,4 / a) + +(0,3 / b) + +(0,3 / c)

The fuzzy sets in the above table are written in their symbolic form as a sum, where the elements of U possessing membership degree 0 are omitted.

Observe that the fuzzy data in the above table are normalized, i.e. we have that

$$\sum_{s \in U^3} m_{A_i}(x) = 1,\ i = 1,2,3 \quad (1)$$

The overall states (profiles) of the fuzzy system that we study are in 1-1 correspondence with the elements of U^3, e.g. the element (c, c, a) corresponds to the state where the consumer's correspondence for the products of the enterprise is moderate for the ages 18-30 and 31-50 and negligible for the ages over the 50 years etc.

In calculating frequencies of the profiles for fuzzy data the membership degrees pertaining to states of individual variables must be properly aggregated into membership degrees of the profiles. For this consider the fuzzy relation

$$R = \{(s, m_R(s)) : s=(x,\psi,z) \in U^3\}$$

with membership function

$$m_R(s) = m_{A_1}(x) m_{A_2}(\psi) m_{A_3}(z)$$

for each s in U^3. The above definition of m_R is suitable in our case, since the stages of the corresponding process are independent to each other. Using (1) we get that

$$\sum_{s \in U^3} m_R(s) = 1 \quad (2).$$

In order to simplify our notation we shall write next m_s instead of $m_R(s)$.

Table 1: Profiles with non zero pseudo-frequencies

A_1	A_2	A_3	$\mu_s(1)$	$\mu_s(2)$	N(s)	p(s)	r(s)
b	b	b	0	0.016	0.016	0.008	0.092
b	a	b	0	0.012	0.012	0.006	0.069
b	c	b	0	0.032	0.032	0.016	0.184
b	b	a	0	0.021	0.021	0.010	0.121
b	b	c	0	0.016	0.016	0.008	0.092
b	a	a	0	0.016	0.016	0.008	0.092
b	a	c	0	0.012	0.012	0.006	0.069
b	c	a	0	0.042	0.042	0.021	0.241
b	c	c	0	0.032	0.032	0.016	0.184
c	c	c	0.072	0.080	0.152	0.076	0.874
c	a	c	0.082	0.030	0.112	0.056	0.644
c	b	c	0.031	0.040	0.071	0.036	0.408
c	d	c	0.046	0	0.046	0.023	0.264
c	c	a	0.067	0.107	0.174	0.087	1
c	c	b	0.056	0.008	0.064	0.032	0.368
c	a	a	0.028	0,040	0.068	0.034	0.391
c	a	b	0.024	0.030	0.054	0.027	0.310
c	b	a	0.028	0.053	0.081	0.040	0.466
c	b	b	0.024	0.040	0.064	0.032	0.368
c	d	a	0.043	0	0.043	0.022	0.247
c	d	b	0.036	0	0.036	0.018	0.207
d	d	a	0.020	0	0.020	0.010	0.115
d	d	b	0.017	0	0.017	0.008	0.098
d	d	c	0.022	0	0.022	0.011	0.126
d	a	a	0.013	0.024	0.037	0.018	0.213
d	a	b	0.011	0.018	0.029	0.014	0.167
d	a	c	0.015	0.018	0.033	0.016	0.190
d	b	a	0.013	0.032	0.045	0.022	0.259
d	b	b	0.011	0.024	0.035	0.018	0.201
d	b	c	0.014	0.024	0.038	0.019	0.218
d	c	a	0.031	0.064	0.095	0,048	0.546
d	c	b	0.026	0.048	0.074	0.037	0.425
d	c	c	0.034	0.048	0.082	0.041	0.471
e	a	a	0.017	0	0.017	0.008	0.098
e	a	b	0.014	0	0.014	0.007	0.080
e	a	c	0,018	0	0.018	0.009	0.103
e	b	a	0.017	0	0.017	0.008	0.098
e	b	b	0.014	0	0.014	0.007	0.080
e	b	c	0.018	0	0.018	0.009	0.103
e	c	a	0.039	0	0.039	0.020	0.224
e	c	b	0.033	0	0.033	0.016	0.190
e	c	c	0.042	0	0.042	0.021	0.241
e	d	a	0.025	0	0.025	0.012	0.144
e	d	b	0.021	0	0.021	0.010	0.121
e	d	c	0.027	0	0.027	0.014	0.155

The pseudo-frequency f(s) of the appearance of the profile s(t) is given by the sum

$$f(s) = m_s(1) + m_s(2)$$

and the probability $p(s)$ is given by

$$p(s) = \frac{f(s)}{\sum_{s \in U^3} f(s)}.$$

From (2) we get that $\sum_{s \in U^3} f(s)(s) = 2$,

therefore we finally have that $p(s) = \frac{f(s)}{2}$.

Finally the possibility $r(s)$ of the appearance of $s(t)$ is given by

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$$r(s) = \frac{f(s)}{\max\{f(s)\}}$$

In Table 1 we calculate the probabilities and possibilities of the profiles having nonzero pseudo-frequencies. For example, for the profile $s=(c, c, a)$ we have that

$m_s(1)=m_{A_1}(c)m_{A_2}(c)m_{A_3}(\alpha)=$ $=0,486.0,4.0,343=0,067$, and $m_s(2)= 0,5.0,533.0,4 = 0,107$ Therefore $f(s)=0,174$. It turns out that the above profile has the greatest probability of appearance $p(s)=0,174/2= 0,087$ or $8,7\%$, while its possibility is 1.

5. Conclusions and remarks

In this paper we developed a general fuzzy model for representing several processes in a system's operation involving fuzziness and/or uncertainty. An application of the above model for the process of learning a subject matter by students in the classroom, presented in detail in an earlier paper, was sketched and also another example was presented from the area of Management.

Our fuzzy model, apart from quantitative information, also gives a realistic *qualitative view* of the corresponding process through the study of all possible profiles of the subjects involved. Another of its advantages is that it gives the opportunity for a combined study of results of two or more groups (or systems) during the same situation, or alternatively for a combined study of results of the same group (or system) during two or more different situations.

We must finally underline the importance of use of *stochastic methods (Markov chain models)* as an alternative approach for the same purposes (for example see [9] and its references. Nevertheless Markov models, although easier sometimes to be applied in practice by a non expert (e.g. the teacher), apart from the quantitative information that they provide - e.g. measures for the problem-solving or model-building abilities of student groups, short and long-run forecasts (probabilities) for the evolution of various phenomena, etc- they are self restricted in describing only the *ideal behaviour* of the subjects involved in the process that they represent. Therefore one could claim that the fuzzy model presented in this paper could be more useful for a deeper study of the corresponding real process, because, apart from the quantitative information, it provides also the possibility of a realistic qualitative analysis of the problems involved.

References

[1] Banathy, B. , *Designing Social Systems in Changing World,* Plenum, New York, 1996

[2] Bertalanffy, L. von, An Outline of General System Theory, *British Journal for the Philosophy of Science*, Vol. 1 (No 2), 1950.

[3] Hammond, D. , *The Science of Synthesis*, University of Colorado Press, Colorado, 2003

[4] Klir, G. J. and Folger, T. A., *Fuzzy Sets, Uncertainty and Information*, Prentice Hall Int., London, 1988.

[5] Lejewski, C. , Jan Lukasiewicz, *Encyclopedia of Philosophy,* Vol. 5, 414-417, MacMillan, New York, 1967

[6] Odum, H., *Ecological and General Systems: An introduction to systems ecology*, Colorado University Press, Colorado, 1994.

[7] Voskoglou, M. Gr. , Transitions across levels in the process of learning: A fuzzy model, *International Journal of Modelling and Application,*(University of Blumenau, Brazil), 1, 37- 44, 2009.

[8] Voskoglou, M. Gr. , *Stochastic and Fuzzy Models in Mathematics Education, Artificial Intelligence and Management*, Lambert Academic Publishing, Saarbrucken, Germany, 2011 (look at http://amzn.com/3846528218).

[9] J. F. Voss, Learning and transfer in subject learning: A problem solving model, *International Journal of Educational Research*, 11, 607-622, 1987

[10] Wiener, N. & Ashby, R., *Cybernetics: Or the Control and Communication in the Animal and the Machine*, Librairie Herrman & Cie, Paris and MIT Press, Cambridge, 1948

[11] Zadeh, L. A. , Fuzzy sets, *Information and Control,* 8, 338-353, 1965.

[12] Zadeh, L. A. , Fuzzy algorithms, *Information and Control,* 12, 94-102, 1968.