Conicalization of the Probabilistic Evolutions for the Ordinary and Forced Van der Pol Equation Under Given Initial Conditions

FATİH HUNUTLU
Marmara University
Department of Mathematics
Göztepe Campus, İstanbul
TÜRKİYE-TURKEY
fatihhunutlu@gmail.com

N.A. BAYKARA
Marmara University
Department of Mathematics
Göztepe Campus, İstanbul
TÜRKİYE-TURKEY
nabaykara@gmail.com

METİN DEMİRALP
İstanbul Technical University
Informatics Institute
Ayazağa Campus-İstanbul
TÜRKİYE-TURKEY
metin.demiralp@gmail.com

Abstract: This abstract deals with Van der Pol and Forced Van der Pol equations. The relation between Probabilistic Evolution and the initial conditions will be investigated. The Space Extension method will be applied to the Van der Pol and Forced Van der Pol equations leading to a second degree structure and the solution will be obtained via triangular structure.

Key–Words: Van der Pol Equation, Forced Van der Pol Equation, Van der Pol Oscillator, Probabilistic Evolution, Ordinary Differential Equation, Initial Condition, Conicalization, Space Extension, Triangularity

1 Introduction

This work considers the probabilistic evolution of the Van der Pol equation [1, 2]. It describes a special type oscillator which was originally proposed by the Dutch electrical engineer and physicist Balthasar Van der Pol (27 January 1889 - 6 October 1959) in his studies on electrical circuits utilizing vacuum tubes. These studies lead to relaxation oscillations which are now known as limit cycles. The studies also revealed one of the first discoveries of deterministic chaos. The Van der Pol equation is used in physics and biological sciences like studies on neurons and geological faults in seismology. It is given by the equation

\[ \frac{d^2 x}{dt^2} - \mu (1 - x^2) \frac{dx}{dt} + x = 0 \]  

(1)

when there is no forcing external agent. In this equation \( x \) denotes the unknown position variable while \( t \) stands for the independent variable time whereas \( \mu \), a scalar parameter, somehow describes nonlinearity and the decaying power of the system.

We can rewrite Van der Pol Equation in the following form

\[ \ddot{x} - \mu (1 - x^2) \dot{x} + x = 0; \quad \mu > 0 \]  

(2)

where we have used the shorthand dot representation for the temporal differentiation. If the \( x^2 \) term is ignored beside 1 in the coefficient of first temporal derivative of \( x \) when \( x \) remains sufficiently small then the system is characterized by an ODE including the decaying term \((-\mu \dot{x})\). Thus, \((x = 0, \dot{x} = 0)\) fixed point in the phase space corresponds to instability. On the contrary, \( x^2 \) dominates 1 in the first order derivative including term by making the decay positive.

In the case where \( \mu \) vanishes the equation turns out to be \( \ddot{x} + x = 0 \) which describes a simple harmonic oscillator whose motion can be explicitly given in the very well-known form \( x(t) = c_1 \cos(t) + c_2 \sin(t) \) where arbitrary \( c_1 \) and \( c_2 \) coefficients can be determined by the imposed initial conditions.

The term \((-\mu (1 - x^2) \dot{x})\) is interpreted as friction or resistance in general. For these cases the factor \((-\mu (1 - x^2))\) is positive. If this factor is negative then “negative resistance” is under consideration.

In fact, the Van der Pol equation can be obtained as a specific approximated form of Rayleigh differential equation which can be given as

\[ \ddot{y} - \mu (1 - \frac{1}{3} \dot{y}^2) \dot{y} + y = 0; \quad \mu > 0 \]  

(3)

when the \( \dot{x}(t) \) in the first order temporal derivative term is replaced by \( x(t) \) (which can be somehow interpreted as an asymptotic relation).

Van der Pol discovered the importance of relaxation oscillations and grasped that they will be a cornerstone of geometric singular perturbation theory. During these studies “Forced Van der Pol Equation” [3, 4] was proposed by adding a periodical inhomogeneity which imposes a forcing action to the system. Its explicit form is given as

\[ \ddot{x}(t) + \mu (x^2 - 1) \dot{x}(t) + x(t) = a \sin(2\pi \nu t) \]  

(4)

where the parameters \( a \) and \( \nu \) are given constants corresponding to the amplitude and frequency of the forcing function. The solution characteristics depend on
these parameters and the chaotic behavior and/or bifurcation phenomenon can be encountered.

2 Probabilistic Evolution for ODEs Under Initial Conditions

The probabilistic evolution is a recently proposed important agent which converts nonlinearities to infinity in the differential equation structure. The purpose is to obtain a linear homogeneous infinite set of ODEs with constant matrix coefficient for a given ODE accompanied by appropriate initial conditions. Here we do not intend to give all details of this new theory, even though we present certain important conceptualities.

To recall certain details of probabilistic evolution we can consider the following one unknown including ODE and the accompanying initial condition for presentation simplicity,

$$\dot{x}(t) = f(x(t)); \quad x(0) = a, \quad t \in [0, \infty) \quad (5)$$
even though equations with more than one unknown can also be treated in a similar manner as well. However the multi unknown case requires the use of multiway arrays or in the recently proposed entities [5, 6] folarrays (folded arrays), folvecs (folded arrays), folmats (folded matrices) and folarrs (folded arrays).

In (5) we have assumed that the descriptive right hand side function \( f \) does not explicitly depend on \( t \). This feature is called autonomy and does not cause any apparent loss of generality since the definition of a new unknown which is equivalent to \( t \) always converts the given nonautonomous (\( t \) dependent \( f \) containing) ODE to set of ODEs with two unknowns and two initial conditions.

The descriptive function \( f \) is assumed to be a function preferably analytic everywhere in the \( x \) complex plane. The \( x \)-complex plane corresponds to Riemann sphere which is centered at the origin of this plane with unit radius. Every point in the complex plane is joined to its north pole by a straight line segment whose intercept defines a sphere point.

All complex plane points outside the Riemann sphere equator corresponds to north hemisphere points while the south hemisphere points correspond to interior points of the equator. All points of the equator are invariant, in other words, they equivalently belong to both the Riemann sphere’s equator and the complex plane. The north pole of the Riemann sphere does not correspond to any point in the \( x \)-complex plane. It corresponds in fact to infinity in the complex plane. By adding this infinity to the complex plane an extended plane is obtained. When we say “\( x \)-complex plane” we mean the complex plane not the “Extended Complex Plane”. The analyticity on the complex plane means that the corresponding function \( f \) is an entire (equivalently) integer function.

By having an entire \( f \) function we can use the Taylor series representation which can be expressed as

$$f(x(t)) = \sum_{j=0}^{\infty} f_j \left(x(t) - x(r)\right)^j \quad (6)$$

It is in fact an infinite linear combination of the following functions

$$x_k(t) \equiv \left(x(t) - x(r)\right)^k, \quad k = 0, 1, 2, \ldots \quad (7)$$

These functions are linearly independent despite the fact that they are functionally dependent. The linear dependence makes them temporally varying basis functions for the space of entire functions. Hence it is quite natural to seek the possibility of constructing a set of ODEs satisfied by them. To this end we can simply differentiate both side of (7) with respect to \( t \). Then we can write

$$\dot{x}_k(t) = k \sum_{j=0}^{\infty} f_j \left(x(t) - x(r)\right)^{k+j-1}$$

$$= k \sum_{j=0}^{\infty} f_j x_{k+j-1}(t) \quad (8)$$

where we have used (5). These equations can be rewritten in the following form which is more amenable to get matrix algebraic structures

$$\dot{x}_0(t) = 0$$

$$\dot{x}_1(t) = \sum_{j=0}^{\infty} f_j x_j(t)$$

$$\dot{x}_k(t) = k \sum_{j=0}^{\infty} f_j x_{k+j-1}(t) \quad k = 2, 3, \ldots \quad (9)$$

Thus we reach at the following

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ f_0 & f_1 & f_2 & f_3 & \cdots \\ 0 & 2f_0 & 2f_1 & 2f_2 & \cdots \\ 0 & 0 & 3f_0 & 3f_1 & \cdots \\ \vdots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \quad (10)$$

If we define

$$x(t) = [x_0(t) \ x_1(t) \ x_2(t) \ \ldots]^T \quad (11)$$
we can get the following more compact form

\[ \dot{x}(t) = Ex(t) \]  

(12)

where

\[
E = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & f_0 & f_1 & f_2 & f_3 & \cdots \\
0 & 2f_0 & 2f_1 & 2f_2 & \cdots \\
0 & 0 & 3f_0 & 3f_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]  

(13)

whose matrix elements are all known constant values.

The formal solution of (12) can be written as follows

\[ x(t) = e^{tE}x(0) \]  

(14)

where

\[
x(0) = \begin{bmatrix} x_0(0) & x_1(0) & x_2(0) & \cdots \end{bmatrix}^T
\]

\[
= \begin{bmatrix} 1 & (a - x^{(r)}) & (a - x^{(r)})^2 & \cdots \end{bmatrix}^T
\]  

(15)

which is called the “Evolution Matrix” since it defines evolution of the system under consideration. This explains why we use the word “Evolution” although the use of the word “Probabilistic” remains in the dark side. To understand the situation more precisely we need to look at the structure of the initial vector \(x(0)\) given by (15). This vector is composed of the elements which are natural number powers of the same entity, \((a - x^{(r)})\), ordered in ascending powers. This brings limitations to the solution in (14). However, the Probabilistic Equation given by (12) has a broader solution as long as the initial vector is composed of not powers but just general values. In other words, if the initial vector is given as

\[ x(0) = [a_0 \ a_1 \ a_2 \ \ldots]^T \]  

(16)

then the general term can be assumed to be defined through

\[ a_k \equiv \int_{-\infty}^{\infty} d\alpha w(\alpha)\alpha^k, \quad k = 0, 1, 2, \ldots \]  

(17)

due to certain representation theorems. This urges us to write

\[ x_k(t) \equiv \int_{-\infty}^{\infty} d\alpha W(\alpha, t)\alpha^k, \quad k = 0, 1, 2, \ldots \]  

(18)

under the condition

\[ W(\alpha, 0) = w(\alpha) \]  

(19)

which means that the solution vector’s \(k\)th element is a temporarily varying expectation value, the expectation value of \(\alpha^k\) under the weight function \(W(\alpha, t)\). Since the weight function \(W(\alpha, t)\) can be considered as a probability density and it is initialized by the weight \(w(\alpha)\), these expectation values evolve in time regarding the evolution of the probability density. This is the reason why we call the relevant equations “Probabilistic Evolution Equations”. Our single unknown ODE with the given initial condition corresponds to the case where the weight function is the Dirac delta function with a support at \((a - x^{(r)})\).

What we have obtained in (14) is the solution for the probabilistic evolution. The solution vector is in fact composed of natural number powers of the same entity and especially its second element is just \(x(t) - x^{(r)}\). Hence it is possible to get the solution of the originally given single unknown ODE and accompanying initial condition by extracting this element. This action produces

\[ x(t) = e_2^T x(t) + x^{(r)} \]  

(20)

which is the exact solution of the original single unknown ODE under the given initial condition.

The solution in (20) requires the determination of the infinite vector \(x(t)\). However, this is not an easy task as it is seen at the first glance because of the infinity in the structures of the relevant entities. The difficulties arising from the infinities can be overcome by using finite truncations as one of the possibilities. This produces a sequence of approximants and convergence under certain conditions on the \(f\) function and \(a\) parameter exists although we do not intend to give more details.

If we truncate the Probabilistic Evolution Equations from left uppermost square submatrix of \((n+1)\times(n+1)\) type then we can write the following equations

\[
\begin{bmatrix}
\dot{x}_0 \\
\dot{x}_1 \\
\vdots \\
\dot{x}_n
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
f_0 & f_1 & \cdots & f_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & n f_1
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{bmatrix}
\]  

(21)

If we define

\[ x^{(n+1)}(t) = [x_0(t) \ x_1(t) \ \ldots \ x_n(t)]^T \]  

(22)

then we can write

\[ \dot{x}^{(n+1)}(t) = E^{(n+1)} x^{(n+1)}(t) \]  

(23)

and

\[ x^{(n+1)}(t) = e^{E^{(n+1)}t} x^{(n+1)}(0) \]  

(24)
where

$$x^{(n+1)}(0) = \begin{bmatrix} 1 & (a - x^{(r)}) & \cdots & (a - x^{(r)})^n \end{bmatrix}^T.$$  

(25)

The finite exponential matrix above can be evaluated by using the spectral decomposition of the “Truncated Evolution Matrix”. To this end we can write

$$E^{(n+1)} \xi_i^{(n+1)} = \lambda_i \xi_i^{(n+1)}; \quad i = 1, \ldots, n + 1$$  

(26)

where $\lambda_i$ and $\xi_i^{(n+1)}$ stand for the $i$th eigenvalue and corresponding eigenvector, assuming we have all eigenvalues with zero multiplicity. However, the matrix $E^{(n+1)}$ need not be symmetric and this urges us to evaluate both left and right eigenvalues. If we order the right and left eigenvectors of $E^{(n+1)}$ as the columns of the matrix $Q_R$ and the rows of the matrix $Q_L$ such that $Q_L Q_R = I$ where $I$ is the $(n+1) \times (n+1)$ identity matrix, then we can define

$$\Lambda^{(n+1)} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n+1} \end{bmatrix}$$  

(27)

and write

$$e^{t \Lambda^{(n+1)}} = \begin{bmatrix} e^{t \lambda_1} & 0 & \cdots & 0 \\ 0 & e^{t \lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{t \lambda_{n+1}} \end{bmatrix}$$  

(28)

which implies

$$e^{t E^{(n+1)}} = Q^{(n+1)} e^{t \Lambda^{(n+1)}} (Q^{(n+1)})^T.$$  

(29)

This finalizes the construction of the truncation approximants. We shall not get into further details of this issue.

### 3 Conicalization Through Space Extension

If the descriptive functions are multinomials in $x$ variables then the rather simple recursive structures become available for the production of the truncation approximants. Further simplifications can be obtained when the Evolution Matrix becomes triangular. Triangularity and conicality (having at most second degree multinomial descriptive functions) can be made available by using appropriate space extensions (curious readers can look at some related references [7–13]). That is, it is possible to change the structure of the ODEs under consideration at the expense of increase in the number of unknowns. The main goal of this paper is to get conicality in the ODEs for Van der Pol and Forced Van der Pol equations.

Let us reconsider the following ODE

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0 \quad ; \mu > 0$$  

(30)

which can be converted to

$$\dot{x} = \mu \left( x - \frac{1}{3} x^3 - y \right)$$  

$$\dot{y} = \frac{1}{\mu} x$$  

(31)

by using the so-called Lienard transformation

$$y = x - \frac{x^3}{3} - \frac{\dot{x}}{\mu}$$  

(32)

Since there exists a third degree structure at the right hand side of the first equation in (31) we need to use space extension as follows to get conicality. We define

$$u_1 \equiv x^2 \quad \Rightarrow \quad \dot{u}_1 = 2x \dot{x}$$  

(33)

and get

$$\dot{u}_1 = 2\mu \left( u_1 - \frac{1}{3} u_2 - xy \right)$$  

(34)

which gives

$$\dot{x} = \mu \left( x - \frac{1}{3} xu_1 - y \right)$$  

$$\dot{y} = \frac{1}{\mu} x$$  

$$\dot{u}_1 = 2\mu \left( u_1 - \frac{1}{3} u_2 - xy \right)$$  

(35)

where the accompanying initial conditions can be constructed accordingly despite we have not explicitly given here.

Similarly we can consider the Forced Van der Pol Equation which is given by

$$\ddot{x}(t) + \mu \left( x^2 - 1 \right) \dot{x}(t) + x(t) = a \sin(2\pi \nu t)$$  

(36)

where we can use the transforms

$$t \equiv \frac{\tau}{\mu}, \quad \dot{x} \rightarrow \frac{x'}{\mu}, \quad \ddot{x} \rightarrow \frac{x''}{\mu^2}$$  

$$y = \frac{x'}{\mu^2} + \frac{x^3}{3} - x$$  

(37)

by denoting differentiation with respect to $\tau$ with the prime symbol. This gives the following two unknown sets of ODEs

$$\frac{1}{\mu^2} x' = \frac{y - \frac{x^3}{3} + x}{y'} = -x + a \sin(2\pi \nu \mu t)$$  

(38)
where the definitions $\varepsilon \equiv \frac{1}{\mu^2}$ and $\omega \equiv \nu \mu$ parameters and the new independent variable $\theta \equiv \omega t$ convert these equations to the following autonomous set of equations

\[
\begin{align*}
\varepsilon x' &= y + x - \frac{x^3}{3} \\
y' &= -x + a \sin(2\pi\theta) \\
\theta' &= \omega
\end{align*}
\]

(39)

where the first equation has again third degree multinomiality and urges us to use space extension. We write

\[
\begin{align*}
u_1 &\equiv x^2 \Rightarrow \nu_1' = 2xx' \\
\nu_1' &= \frac{2}{\varepsilon} \left(xy + \nu_1 - \frac{\nu_1^2}{3}\right)
\end{align*}
\]

and obtain the following conicality in $x$, $y$ and $u_1$ unknowns

\[
\begin{align*}
x' &= \frac{1}{\varepsilon} \left(y + x - \frac{xu_1}{3}\right) \\
y' &= -x + a \sin(2\pi\theta) \\
\theta' &= \omega \\
u_1' &= \frac{2}{\varepsilon} \left(xy + \nu_1 - \frac{\nu_1^2}{3}\right)
\end{align*}
\]

(42)

which excludes the $\theta$ unknown from conicality because of the sinusoidal structure which is far beyond multinomiality. Hence we need to extend the space one more time by defining a new unknown equivalent to the sine term. However this produces cosine term under the temporal differentiation. This means that not only the sine term but also the cosine term should be used to create two new unknowns for extending the space. Thus we define

\[
\begin{align*}
u_2(t) &\equiv \sin(2\pi\theta), \\
u_3(t) &\equiv \cos(2\pi\theta)
\end{align*}
\]

(43)

which converts the equations in (42) to the following ones

\[
\begin{align*}
x' &= \frac{1}{\varepsilon} \left(y + x - \frac{xu_1}{3}\right) \\
y' &= -x + au_2(t) \\
\theta' &= \omega \\
u_1' &= \frac{2}{\varepsilon} \left(xy + \nu_1 - \frac{\nu_1^2}{3}\right) \\
u_2' &= 2\omega \pi \nu_3(t) \\
u_3' &= -2\omega \pi \nu_2(t).
\end{align*}
\]

(44)

where the initial conditions which have not been given explicitly can be constructed accordingly.

Here the space extensions are not unique since it is possible to use other definitions for extension. The equations are rather simple here. In the cases where more unknowns and more complicated structures are considered, the most convenient space extension may not be found so directly like it was done here. On the other hand all our applications and experimentations send the signal that there should be a minimal or optimum space extension even though we have not focused on this issue and we do not intend to do so. Above applications can be extended to the cases where the forcing function may have more complicated structures. Since the sine function has quite a simple structure under differentiation operation the space extension here could be held at rather a simple level.

4 The Importance of Triangularity

As can be noticed easily the Evolution Matrix $E$ is in an upper Hessenberg form as long as $f_0$ does not vanish. $f_0$ is the value of the descriptive function $f$ at the Taylor expansion point $x^{(r)}$. If $f$ has at least one zero then taking $x^{(r)}$ as that zero value makes $f_0$ vanish and therefore puts the Evolution Matrix to triangular form. However, the descriptive function $f$ may not have any zeros, then $f_0$ can never vanish for any choice for $x^{(r)}$. However, it is always possible to extend the space appropriately to get triangularity in higher dimensions.

The truncation approximants can be rather easily constructed for the triangular cases. However, triangular but not sparse matrices, in other words, triangularly full matrices complicate the analysis. However, the space extension method works for almost all types of practically encountered cases to get a set of ODEs with conically multinomial descriptive functions. The conicality becomes very important when it is joined with triangularity since the truncations can be constructed by using first order backward recursions. We do not intend to get into further details of this issue here.

5 Concluding Remarks

This work has been designed to use the space extension method to get conicality in Van der Pol and Forced Van der Pol equations. Extending the space may facilitate many different type analyses. Amongst them we can refer to, for example, the kernel space methods even though it is for a rather diverse field, data processing. The purpose here has not been to construct truncation approximants and to investigate their convergence properties. These issues will be left for future works.
The space extension by defining as if new unknowns in terms of the existing ones carries the problem to a broader class of problems by simplifying the function structures. Here we do not conjecture that the resulting ODE forms are unique. Some other type definitions may take us to different evolution matrices. The uniqueness issue can be an interesting issue for future works although we have not emphasized it here.

As can be noticed immediately, the use of space extension to get the conicality has provided us with triangularity at the same time. This might not be the case even though it has been accomplished here rather easily. Hence we are now at a quite fruitful position to construct the triangular truncations. Their convergence will be very possibly dependent on the specific characters of the descriptive functions.

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