Analysis of Blood Flow in Coronary Arterial Tree Using Mathematical Models

C. CORCIOVĂ, M. TURNEA, F. CORCIOVĂ
1Department Biomedical Science
University of Medicine and Pharmacy „Grigore T. Popa”, Faculty of Medical Bioengineering
2Department Cardiology
Institute of Cardiovascular Diseases
M. Kogalniceanu Street, No 9-13, Iasi
ROMANIA
calin.corciova@bioinginerie.ro http://www.bioinginerie.ro

Abstract: Many people on the world have coronary artery disease. They are diagnosed with coronary artery disease on the basis of classical examinations (like electrocardiograms) and are treated with medication, interventional procedures or open heart surgery. Flow rate can be measured invasively during interventional procedures. To predict the flow rate and the pressure of the coronary arterial system of a patient, computational simulations can be used as an alternative approach. Computational simulations have been proven useful in studying blood flow in the cardiovascular system. Some of the investigations include the study of the hemodynamics of healthy and diseased blood vessels, the design and evaluation of vascular medical devices, planning of vascular surgeries, and the prediction of the outcomes of interventions. In simulations, we studied coronary flow and pressure of normal coronary arteries. Blood flow can be approximated as a Newtonian fluid. We can solve the problem of the incompressible blood flow using the Navier-Stokes equations and the motion of the vessel wall using the dynamics equations. We built a model and studied temporal changes in the heart rate, contractility of the left and right ventricle and the venous compliance. The simulation results could provide valuable data that may affect the process of selecting clinical interventions and may replace certain invasive procedures that are painful, expensive and sometimes unnecessary.

Keywords: simulation, cardiac flow, heart, algorithms, mass conservation, hemodynamics, electrical equivalent circuit, boundary conditions

1 Introduction
Diseases of the human cardiovascular system are one of the main problems in contemporary health care in the industrial countries. These diseases cause the majority of deaths and also often afflict people in their most productive age. In this context different approaches have been used with the aim of providing better understanding and modeling of the arterial tree and heart mechanism in the human cardiovascular system. Several mathematical and electrical models dedicated particularly to simulate the operation of cardiovascular system over the past decade by many scientists such as Beeler and Reuter (1977), Stergiopulos and Young (1992), Nebot et al. (1998), Mukkamala and Cotton (2001), Migliavacca and Pennati (2001), Heldet and Shim (2002), Darovic and Stratton (2002), Urquiza and Blanco (2003), Manoliu (2004), Liang and Liu (2005) and Waite (2006).

Flow rate and pressure of coronary vascular beds can be measured invasively during interventional procedures or open heart surgery. The information obtainable from the medical imaging techniques and the invasive flow and pressure measurement techniques is limited because the image resolutions of the medical imaging techniques are up to small artery levels at most, and flow and pressure measurement techniques are highly invasive and restricted by time and the physical condition of the patient.

This information on the flow rate and pressure of the coronary vascular beds of a patient is crucial to determine treatments. Computational simulations can be used as an alternative approach to predict the flow rate and the pressure of the coronary vascular beds of a patient. Also are useful in studying blood flow in the cardiovascular system [8].

Some of the investigations include the study of the hemodynamics of healthy and diseased blood vessels, the design and evaluation of vascular medical devices, planning of vascular surgeries and the prediction of the outcomes of interventions.
Computational simulations can be categorized into a wide spectrum of methods. The aim of this study is to develop computational methods that accurately simulate the physiological coronary flow. These methods can be used to predict the flow rate and the pressure of the coronary vascular beds of patients in order to improve diagnosis and select the best treatment option. These methods can be also used to study the hemodynamics of the coronary circulation in relation to the arterial system and the heart.

2 Problem Formulation
The liquid motion equations for circulatory system are very complicated. In order to obtain analytical solution following simplifications have been assumed. Blood is non-Newtonian fluid consisting of blood cells and blood plasma. Proportional relation between cells and plasma is determined by hematocrit value \[6\]. The hematocrit value is the most important parameter defining blood viscosity. The hematocrit of normal human blood is about 45% and it relates to blood viscosity about \(4 \times 10^{-3}\) [Pa.s].

Assumption of constant blood viscosity and homogeneity in whole vessels tree is necessary to estimate blood flow through blood vessel trees. According to researches \[1\] during a normal flow in straight arteries blood behaves as a near Newtonian fluid. In real blood vessels system, vessel walls are elastic and can change its diameters. In this way resistance of blood vessel system is regulated. This process is known as auto regulation and corrects nutrition of all cells in human body. Assumption of vessel wall as a rigid pipe with constant diameter for given vessel segment is necessary to application of hydrodynamic equations and analytical calculation of modeled trees.

Blood flow estimation assumes laminar flow for the entire fractal vessel tree. In large arteries systolic aberrations of laminar flow is a result of wave propagation. Turbulent flow is also observed in pathological vessels. In small arteries, which are subject to described research, assumption of laminar flow is correct. Hydrodynamic equations for small arteries give correct results in biological circulatory system.

Flow evaluation starts at terminal vessel and proceed to the root segment. With each terminal vessel equal flow value is associated. Recurrent procedure gives flow parameters evaluation on lower branching level according to mass conservation law and Poiseuille’s law.

 Coronary blood flow is different from the flow in the other parts of the cardiovascular system because coronary flow is strongly influenced by the contraction and relaxation of the ventricles and atria. The ventricular and atrial pressure can be used to approximate the intramyocardial pressure and model the compressive force acting on the coronary vessels caused by the contraction and relaxation of the ventricles and atria throughout the cardiac cycle \[1\]. The ventricular and atrial pressure should be determined by the interactions between the heart and the vascular system. The simplest method is to use lumped parameter models. Lump parameter models are developed on the basis of the analogy between blood flow and electrical current and between blood pressure and electrical voltage. Similarly to electric circuit systems, lumped parameter models can be composed of any of the following: pressure sources, resistances, capacitances, inductances, or diodes \[3\].

![Fig 1 Electrical circuit analog of the intact human pulsatile heart and circulation. Each box encompassing a circuit element denotes a nonlinear element](image)

Each component has a physical meaning. For example, a pressure source can represent a venous pressure reservoir and a diode can represent a mitral valve. The resistance, inductance, and capacitance also have physical meanings, and their values can be derived by integrating Navier - Stokes equations over the cross section with simplifying assumptions. We can represent the complex flow rate and pressure waveforms of the cardiovascular system with a rather simple lumped parameter model.

Another method is one-dimensional equations of blood flow. One-dimensional equations of blood flow are derived from the Navier - Stokes equations with the assumptions that the dominant component of blood flow velocity is along the vessel axis and that the pressure is constant over the cross-section of the vessel. These methods model wave propagation phenomenon and can reproduce realistic flow and pressure waveforms \[2\].
Geometry (length, radius at the entrance to the exit radius) of the vessel are obtained by data derived from measurements on a subject and other properties of the artery (the main wall elasticity and peripheral resistance), but with adjustable parameters, comparable with experimental dates. Simulation of blood flow and pressure in compliant vessels requires three equations. Two equations that ensure conservation of volume and time and an equation that describes the influence of fluid on the vascular wall and its adherence. A typical dish is shaped like a cylinder asymmetrically compliant. Flow velocity inside the cylinder is:

\[ u = \left[u_r(r, x, t), u_x(r, x, t)\right] \tag{1} \]

where \( r \) is the radial coordinate, \( x \) is position along the vessel, \( t \) is time, \( u(r) \) is the radial velocity and \( u_x(x, t) \) is the axial velocity [8].

Vascular wall is thought to be impermeable. So, non-slip conditions are satisfied if the flow velocity in the wall velocity is equal to the wall. The density \( \delta \) and viscosity \( \mu \) are considered constant. Consider \( p(x, t) \) as the fluid pressure and \( p_0 \) (which is constant) diastolic pressure. Suppose that the pressure does not vary greatly along the cross-sectional area of the vessel, so \( p \) can be considered independently of \( r \). We consider \( R(x, t) \) radius of the vessel and \( A(x, t) = \pi R^2(x, t) \) corresponding cross sectional area. The vessel is assumed to be exponentially narrow, so the balance beam is described by the relationship:

\[ r_0(x) = r_0 \left[ \log \frac{r}{r_0} \right] \]  \tag{2}

The mathematical model consists of two parts: the large arteries where blood flow and pressure are predictable at any point along its entire length and small arteries, in which a relationship between flow and pressure conditions involve the external flow connection for small arteries. Blood flow and pressure in systemic arteries (large and small) are determined using asymmetric incompressible Navier - Stokes equations for a Newtonian fluid [9]. Dimensional model is obtained integrating the equations in the cross section area of the vessels. For an asymmetric flow, continuity equation is described by the following formula:

\[ \frac{\partial u_x}{\partial x} + \frac{1}{r} \frac{\partial (ru_r)}{\partial r} = 0 \tag{3} \]

For an asymmetric flow without spiraling motion, consider the time instant equation described the relationship:

\[ \frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial x} + u_x \frac{\partial u_r}{\partial r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} = \frac{\nu}{r} \left( \frac{\partial u_r}{\partial r} \right) \]  \tag{4}

The first three terms represent axial fluid acceleration effect and the other is the sum of all forces acting on the fluid. For this model are pressure and viscosity forces \( (v = \frac{\mu}{\rho} \) represent cinematic viscosity).

Dimensional model is obtained integrating the cross-sectional area function, since \( p \) is considered constant at the level of this area. We get the relations:

\[ 0 = 2\pi \frac{\partial}{\partial x} \left( u_r \right) dr - 2\pi \frac{\partial R}{\partial x} \left( r u_r \right)_R + 2\pi \left( R u_r \right)_R \]

\[ 2\pi \frac{\partial}{\partial t} \left( \frac{1}{r} \frac{\partial r}{\partial t} + 2\pi \left( u_r \frac{\partial u_r}{\partial x} + u_x \frac{\partial u_r}{\partial r} \right) r dr + \frac{A}{\rho} \frac{\partial \rho}{\partial x} = \frac{2\pi}{r} \left( \frac{\partial u_r}{\partial r} \right)_R \]  \tag{5}

We define the flow through the vessel \( q \) by formula

\[ q = 2\pi \left( u_r \right)_0 r dr = A \pi \left( 1 - \frac{\delta}{R} + \sigma (\delta^2) \right) \]

\[ 2\pi \left( u_r \right)_0 r^2 dr = A \pi \left( 1 - \frac{4}{3} \frac{\delta}{R} + \sigma (\delta^2) \right) \]  \tag{6}

where:

\[ u_r(x,t) = \begin{cases} \frac{\pi}{\delta} (R-r) & \text{pentru } r \leq R - \delta \\ \frac{\pi}{\delta} (r-R) & \text{pentru } R - \delta < r \leq R \end{cases} \]

To obtain one - dimensional model:

\[ \frac{\partial q}{\partial x} + \frac{\partial A}{\partial t} = 0 \]  \tag{7}

\[ \frac{\partial q^2}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q^2}{A} \right) + A \frac{\partial \rho}{\partial x} = - \frac{2\pi v R}{\delta} \frac{q}{A} \]  \tag{8}

We have defined the continuity equation and the Navier - Stokes equation in the three directions thus we have four equations. To compute the four unknown variables: the velocity of blood in the three directions and the pressure at that point, we need to solve these four differential equations. We impose the boundary conditions as the known pressure drop across the branch ends of the artery and the second being the no slip condition. If \( a_i \) is the local unit normal to the surface, the condition is expressed as:

\[ a_i v_i = 0 \]  \tag{9}

Viscosity is responsible for the velocity component that is tangential to the wall. The no slip condition is velocity at the wall is given as:

\[ v_1 = 0 \]  \tag{10}

The deformation in the artery results as the pulsating blood flow exerts a pressure on the inner wall of the artery. We assert that the deformation
can be measured only relatively with respect to some reference state in which the cross-sectional plane is considered to be undeformed. We consider the state of the beginning of the pressure pulse in the artery as the undeformed state and neglect the residual strain present in the artery. To model the deformation in the artery we start with the Green’s strain tensor $E_{ij}$ that characterizes the deformation near a point:

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right)$$  (11)

The pressure exerted by the blood on the wall of the artery produces the stress in the artery that causes the artery to deform. To model the stress-strain relationship, we consider the artery and cardiac muscle and to be a hyper elastic material characterized by the potential energy function $\omega(E)$ where we use $E$ to denote the Green strain vector.

3 Results and discussions

We have constructed a comprehensive computational model of the cardiovascular system, which incorporates the patient specific factors and models questions posed by clinicians in order to create an effective algorithm with physiological interpretability. The model is simple enough to offer an insight in the interrelations of the circulatory system between inputs, physiological parameters and outputs.

Our simulation used three physical variables (pressure, flow, compliance) that we need in order to obtain a quantitative description of the circulation, to explain the system of units used by physiologist and to give typical value that occur in the different part of the circulation. Since blood is nearly incompressible, the volume of the blood serves as a convenient measure of the amount of blood in any part of the circulation.

The flow is the volume of blood per unit time passing a point in the circulation. The most important flow in the circulation is the cardiac output, which is defined as the volume of blood pumped per unit time by either side of the heart. The simulation is implemented by the following MATLAB program. We first described the function, so it will be clear what the script is doing when it calls these functions. We built a computer model of the coronary arterial system. It treats the ventricle as a compliant vessel; its compliance is not constant, but instead is variable function of time. The input arguments include the desired parameter values characterizing the human cardiovascular model and its execution, while the outputs are the simulated data all pressures $p(t)$, volumes $Q(t)$, flow rates $q(t)$, ventricular elastances $E(t)$.

An important aspect is how we consider the valves of the heart. When these valves are healthy, they present a low resistance to flow, but they completely block backflow. Valves diseases alter their proprieties, the resistance to forward flow may be increased or the valve may become leaky and allow the backflow. The program can be used to calculate some of the circulatory consequences of the valve malfunction.

![Fig. 2 MATLAB flowchart](image)

![Fig. 3 Model of the left heart and systemic arteries: simulation of the vascular compliance, pressure and flow at rate of 60 beats per minute.](image)

![Fig. 4 Model of the right heart and systemic arteries: simulation of the vascular compliance, pressure and flow at rate of 60 beats per minute.](image)
Observe a simulated parameter change depending on heart rate, major changes to the heart (left and right) to a frequency of 120 beats per minute.

The results presented demonstrate that the physiological values of pressure, flow and compliance can be achieved and illustrates the importance of using physiological conditions similar to those.

We formulate numerical examples in which the elasticity of the artery is considered to have a non-homogeneous profile Fig. 11. In Figure 12 we illustrate the meansquare-error as a function of signal-to-noise ratio. As expected the error decreases...
as SNR increases however even for low SNR we are able to estimate the unknown pressure.

![Fig. 12 Mean-square-error of pressure estimation](image)

**4 Conclusions**

With a billion of arteries in the vascular system, the domain of interest that can be modeled may only include a limited number of vessels. Since the cardiovascular system is pressurized, blood flow and pressure at a certain location depend on the upstream energy available and the downstream blood flow demands or compliance of the vessels.

Modeling blood flow at realistic levels of pressure enabled the development of fluid-structure interaction modeling. Prescribing flow and pressure, even if properly measured with all the information on wall properties, lacks predictive capabilities. The research directions in the area of simulation of medical signals with computing systems give power and flexibility in medical decisions.

**References**


