Abstract: In this paper, we consider a family of Benjamin-Bona-Mahony equation with strong nonlinear dispersive which is of considerable interest in mathematical physics. We determine the subclass of equations which are self-adjoint.

Key–Words: Symmetries, Partial differential equation, self-adjointness, conservation laws

1 Introduction

Benjamin et al [2] proposed the regularised long wave (RLW) equation, or Benjamin-Bona-Mahony equation (BBM),

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

as an alternative model to the Korteweg–de Vries equation for the long wave motion in nonlinear dispersive systems. These authors argued that both equations are valid at the same level of approximation, but that BBM equation does have some advantages over the KdV equation from the computational mathematics viewpoint.

In order to understand the role of nonlinear dispersion in the formation of patterns in an undular bore, Yalong [20] introduced and studied a family of BBM-like equations with nonlinear dispersion, $B(m,n)$ equations

$$u_t + (u^m)_x - (u^n)_{xxt} = 0, \quad m, n > 1.$$

In [20], the exact solitary-wave solutions with compact support and exact special solutions with solitary patterns of the equations were derived.

In [16] introduced the family of BBM equation with strong nonlinear dispersive $B(m,n)$ equation:

$$u_t + u_x + a (u^m)_x + (u^n)_{xxt} = 0,$$

by using an algebraic method the authors obtained solitary pattern solutions. The case $n = 1$ and $m = 2$ corresponds to the BBM equation, [2]. This equation is an alternative to the Korteweg-de Vries (KdV) equation and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. The BBM equation is not only convenient for shallow water waves but also for hydromagnetic and acoustic waves and therefore it has some advantages compared with the KdV equation.

Clarkson [9] showed that the similarity reduction of the equation (1) for $m = 3$, $n = 1$ and $a = \frac{1}{3}$, obtained by using the classical Lie group method reduces the partial differential equation (PDE) to an ordinary differential equation (ODE) of Painlevé type; whereas the PDE doesn’t possess the Painlevé property for PDEs as defined by Weiss et al [7]. The author proved that the only non-constant similarity reductions of this equation obtainable either using the classical Lie method or the direct method, due to Clarkson and Kruskal [10], are the travelling wave solutions.

In this paper we study the Lie symmetries of equation

$$u_t + bu_x + a (u^m)_x + (u^n)_{xxt} = 0,$$

where $a, b$ are constants and $m$ or $n \neq 1$, by using the Lie method of infinitesimals. We determine, for equation (1), the subclasses of equations which are self-adjoint.

2 Lie Symmetries

To apply the classical method to Eq. (1) we consider the one-parameter Lie group of infinitesimal transformations in $(x, t, u)$ given by

$$x^* = x + \epsilon \xi(x, t, u) + O(\epsilon^2),$$
$$t^* = t + \epsilon \tau(x, t, u) + O(\epsilon^2),$$
$$u^* = u + \epsilon \eta(x, t, u) + O(\epsilon^2),$$

where $\epsilon$ is a small parameter, $\xi, \tau, \eta$ are the infinitesimal generators of the Lie group, and $O(\epsilon^2)$ represents terms of order $\epsilon^2$ or higher. The infinitesimal generators are determined by the invariance of the equation with respect to the transformations $x^*$, $t^*$, and $u^*$. The invariance conditions are obtained by requiring that the equation remains unchanged under the infinitesimal transformations. These conditions lead to a system of determining equations for the infinitesimal generators. Solving this system provides the Lie symmetries of the equation.

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where $\epsilon$ is the group parameter. We require that this transformation leaves invariant the set of solutions of (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x, t, u), \tau(x, t, u)$ and $\eta(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u.$$  

(2)

Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (2) leads to a set of twenty six determining equations. Solving this system we obtain $\xi = \xi(x), \tau = \tau(t)$ and $\eta = \frac{\alpha(x, t)}{t^{n-1}} - \frac{k_1}{2n} u + \frac{\xi_x}{2n} u$ where $\xi, \tau$ and $\alpha$ are related by the following conditions:

$$\xi_{xxx} n^2 u^{2n} + k_1 n u^{n+1} + 3 \xi_x n u^{n+1} - k_1 u^{n+1} + 2 \alpha n^2 u + 2 \alpha n u = 0,$$

$$a \xi_{xxm} m u^{m+n} + 2 a \alpha_x n m u + b \xi_{xx} u^{n+1} + 2 \alpha n u + 2 \alpha n u = 0,$$

$$-a k_1 m^2 u^{m+n} + a \xi_x m^2 u^{m+n} + a k_1 n m u^{m+n} + 2 a n^2 u + 2 \alpha n u = 0,$$

$$2 a \alpha n^2 m u^{m+n} + a \xi_x n m u^{m+n} + 2 a n^2 u + 2 a n u = 0,$$

$$+ b \xi_x n u^{n+1} - k_1 u^{n+1} + b \xi_x u^{n+1} - 2 \alpha b n^2 u + 2 \alpha b n u = 0.$$

The solutions of this system depend on the parameters of Eq. (1). If $a$ and $b$ are arbitrary constants, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t.$$

For $\lambda \mathbf{v}_1 + \mathbf{v}_2$ the similarity variables and similarity solution are:

$$z = x - \lambda t,$$

$$u = h(z)$$  

(3)

where $h(z)$ satisfies

$$\lambda (h^n)^m + \lambda h' - a m h^{n-1} h' - b h' = 0.$$

This equation, after integrating once with respect to $z$, can be reduced to

$$\lambda (h^n)^m = a h^n + (b - \lambda) h + k_1,$$  

(4)

where $k_1$ is an integrating constant.

The cases for which Eq.(1) with $b \neq 0$ have extra symmetries have been studied by Bruzón and Gandarias in [7].

### 3 Determination of self-adjointness equations

Given (1+1)-dimensional evolution equation of order $n$, $F \equiv F(x, u, u^{(1)}(x), \ldots, u^{(n)}(x)) = 0$, where $x = (x, t)$ are independent variables, $u = u(x)$ is a dependent variable and $u^{(l)}(x)$ denotes the set of all the partial derivatives of order $l$ of $u$; a conservation law is of the form

$$D_t \rho + D_x J = 0,$$

where $\rho$ is the conserved density, $J$ is the associated flux, $D_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{N} \frac{\partial J}{\partial u_k} u^{(k+1)} x$, $N$ is the order of $J$, and $D_t \rho = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_k} D_x^k u$, with $M$ the order of $\rho$.

In [13] Ibragimov introduced a new theorem. The theorem is valid for any system of differential equations where the number of equations is equal to the number of dependent variables. The new theorem does not require existence of a Lagrangian and this theorem is based on a concept of an adjoint equation for non-linear equations.

Given

$$F = u_t + bu_x + a (u^m)_x + (u^n)_{xxx},$$

the adjoint equation $F^* = 0$ is defined

$$F^* \equiv \delta \frac{\delta}{\delta u} (vF) = 0,$$

where $v = v(x, t)$ is a new dependent variable and the variational derivative is

$$\delta \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \left( \frac{\partial}{\partial u_i} \right) D_j D_k \left( \frac{\partial}{\partial u_{ijk}} \right) + \cdots$$

We obtain

$$F^* \equiv -b n u^{n-1} v_{xxx} - a u^m v_x - l u^{l-1} v_t.$$  

(5)

Setting $v = u$,

$$F^* \equiv -b n u^{n-1} u_{xxx} - a u^m u_x - l u^{l-1} u_t.$$

Comparing $F^*$ with $F$ we obtain that $F^* = \lambda F$ if $\lambda = -1$ and $n = 1$ and, consequently, we get the following result:

**Proposition.** Equation $F \equiv u_t + bu_x + a (u^m)_x + (u^n)_{xxx} = 0$ is self-adjoint if $n = 1$, i.e. when it has the following form

$$F = u_t + bu_x + a (u^m)_x + u_{xxx}.$$
4 Conclusions

We have considered classical symmetries of a $B(m, n)$ equation. The concept of self-adjoint equation was introduced by NH Ibragimov in [12, 13]. In this paper we found the general classes of the self equations (1).

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