A Possible Solution of Trisection Problem

SIAVASH H. SOHRAB
Robert McCormick School of Engineering and Applied Science
Department of Mechanical Engineering
Northwestern University, Evanston, Illinois 60208
s-sohrab@northwestern.edu
http://www.mech.northwestern.edu/web/people/faculty/sohrab.php

Abstract: - A solution of the ancient Greek problem of trisection of an arbitrary angle employing only compass and straightedge and its algebraic proof are presented. It is shown that while Wantzel’s [1] theory of 1837 concerning irreducibility of the cubic \( x^3 - 3x - 1 = 0 \) is correct it does not imply impossibility of trisection of arbitrary angle since rather than a cubic equation the trisection problem is shown to depend on the quadratic equation \( y^2 - 3 + c = 0 \) where c is a constant. The earlier formulation of the problem by Descartes the father of algebraic geometry is also discussed. If one assumes that the ruler and the compass employed in the geometric constructions are in fact Platonic ideal instruments then the trisection solution proposed herein should be exact.

Key-Words: The trisection problem, angle trisection, Wantzel theory, regular polygons, heptagon.

1 Introduction
The classical trisection problem requires trisecting an arbitrary angle employing only a compass and a straightedge or unmarked ruler. The general rules concerning the construction instruments and acceptable solution of the problem are most eloquently described by Dunham [2]

Indeed, Greek geometers performed trisection by introducing auxiliary curves like the quadratrix of Hippias or the spiral of Archimedes, but these curves were not themselves constructible with compass and straightedge and thus violated the rules of the game. It is rather like reaching the top of Everest by helicopter: It achieves the end by an unacceptable means. For a legitimate trisection, only compass and straightedge need apply.

The second rule is that the construction must require only a finite number of steps. There must be an end to it. An “infinite construction,” even if it has trisection as a limiting outcome, is no good. Construction that goes on forever may be the norm for interstate highways, but it is impermissible in geometry.

Finally we must devise a procedure to trisect any angle. Trisection a particular angle, or even a thousand particular angles, is insufficient. If our solution is not general, it is not a solution.”

In this study an unexpectedly simple solution of the ancient Greek trisection problem along with the algebraic proof of its validity will be presented. Historically, it was proven by Wantzel [1] that the trisection of an arbitrary angle by only compass and straightedge is impossible if such a construction requires the existence of rational roots of the cubic equation

\[ x^3 - 3x - 1 = 0 \] (1)

As described by Dunham [2]:

(a) If we can trisect the general angle with compass and straightedge,
(b) Then we can surely trisect a 60° angle,
(c) So, we can find a constructible solution of
\[ x^3 - 3x - 1 = 0 , \]
(d) So, we can find a rational solution for
\[ x^3 - 3x - 1 = 0 , \]
(e) And this rational solution must be either \( c/d = 1 \) or \( c/d = -1 \).

when \( x \) is a rational number denoted by the ratio \( x = c/d \). Since by Wantzel’s [1] proof (1) is irreducible and (d) is not true then one must conclude that (a) cannot be true. The algebraic equation (1) in Wantzel’s theory [1] originates from the trigonometric equation

\[ \cos \theta = 4 \cos^3(\theta/3) - 3 \cos(\theta/3) \] (2)
that when applied to the angle $\theta = \pi/3$ with $x = 2\cos(\theta/3)$ results in (1). A cubic equation of the form

$$x^3 + qx + r = 0 \quad (3)$$

was also employed by Descartes [3] in connection to the trisection problem [4]

“Descartes dealt with the problem of angle trisection by reducing the problem to a third-degree equation and constructing it via intersection of circle and parabola”

Descartes proposed a solution of trisection problem by employment of a parabola, a non-constructible hence transcendental curve, as shown in Fig.1 reproduced from his book of geometry [3]

![Fig.1 Descartes solution of trisection of an arbitrary angle $\theta = NOP$ employing a parabola GAF [3].](image)

By geometric construction based on the trisected angle shown in Fig.1 Descartes arrived at the cubic equation [3]

$$z^3 - 3z + q = 0 \quad (4)$$

where $z = NQ$ and $q = NP$. It is now clear that for $\theta = 60 = \pi/3$ and a circle of unity radius $NO = 1$ one has $NP = q = NO = 1$ and both (4) as well as the trigonometric relation

$$\sin(\theta/2) = 3\sin(\theta/6) - 4\sin^3(\theta/6) \quad (5)$$

with the definition $z = 2\sin(\theta/6)$ lead to

$$z^3 - 3z + 1 = 0 \quad (6)$$

that like (1) has no rational roots. One notes that the chord $NQ$ in Fig.1 of Descartes becomes the unknown $NQ = z = 2\sin(\theta/6)$. In view of the similarity between (1) and (6) the failure of Wantzel [1] to reference the work of Descartes is unfortunate. Finally, if parallel to (5) instead of full angle in (2) one applies the trigonometric identity for half angle

$$\cos(\theta/2) = 4\cos^3(\theta/6) - 3\cos(\theta/6) \quad (7)$$

and considers $\theta = 60 = \pi/3$ with $y = 2\cos(\theta/6)$ one arrives at

$$y^3 - 3y - \sqrt{3} = 0 \quad (8)$$

that also does not possess any rational roots. It is interesting to note that cubic equation of the type

$$x^3 + b^2x = b^2c \quad (9)$$

that was first solved by Omar Khayyam using intersection of conics [5] also reduces to (1) and (6) when $(b = i\sqrt{3} , c = -1/3)$ and $(b = i\sqrt{3} , c = 1/3)$, respectively.

According to Wantzel’s theory of 1837 [1] only rational numbers $x = c/d$ that are roots of algebraic equations are acceptable solution to the trisection problem. This is because the criterion of geometric constructability based on Descartes’s analytic geometry only admits rational operations of addition, subtraction, multiplication, division, and extraction of square roots thus requiring existence of rational roots of polynomials of various degrees [1-8]. Over three decades after Wantzel’s work Hermite (1873) and then Lindemann (1882) respectively proved the existence of transcendental numbers $e$ and $\pi$ [2]. It is known that concerns about basing geometric constructability only on the application of geometrical (algebraic) curves and not mechanical (transcendental) curves were raised by ancient Greek mathematicians as well as Newton [4]. Clearly, the question of rational versus transcendental numbers could be connected to the admissibility of algebraic versus transcendental curves as means of geometric construction just mentioned. Indeed, in view of the fact that trisecting an angle involves $\pi$ and as was emphasized by Gauss transcendental numbers by far are more abundant than rational numbers, it is reasonable to anticipate that trisection problem may involve transcendental numbers that are not solutions of polynomials with integer coefficients. Therefore Wantzel’s proof may not be applicable to the resolution of trisection problem because this problem involves transcendental numbers and not rational numbers. In other words, the assumption that only rational numbers $x = c/d$ should be allowed as acceptable constructible solutions made in steps (c)-
(d) above may not be valid. Such a situation will be somewhat similar to von Neumann’s proof of impossibility of hidden variables in quantum mechanics [9] and the fact that it was later found to be inapplicable to quantum mechanics not because of an error in the theory but rather due to invalid assumptions made in its axiomatic foundation. Because of the overwhelming abundance of transcendental numbers there is no reason to require the root \( x = 1.879385242 \ldots \) of equation (1) to be a rational number.

Another important observation concerning the classical proof (a)-(e) discussed above is that the term \( \cos \theta \) on the left hand side of (2) is treated as a constant by substitution for \( \cos(\theta = \pi/3) = \frac{1}{2} \) while \( x = 2\cos(\theta/3) \) on the right hand side of (2) is treated as the unknown variable that is to be determined. Clearly, such a practice disregards the fact that both the left as well as the right hand sides of the trigonometric equation (2) are in fact functions of the true independent variable of the problem that is the angle \( \theta \). To show how such a practice could influence judgments about constructability one expresses (7) as

\[
\cos(\theta/2) = 4\cos^2(\theta/6) - 3
\]  

(10)

If in the process of geometric construction the ratio on the left hand side of (10) becomes a constant \( \cos(\theta/2)/\cos(\theta/6) = c \) for a given angle \( \theta \) then one arrives at

\[
c = y^2 - 3
\]  

(11)

where \( y = 2\cos(\theta/6) \). Now, as opposed to the cubic equation (1) the quadratic equation (11) only involves extraction of square root thus making trisection of angle constructible. In fact, once the ratio \( \cos(\theta/2)/\cos(\theta/6) \) is determined through construction the sine of the trisected angle can be directly related to this ratio by noting that (7) could also be expressed as

\[
2\cos(\theta/3) = 1 + \frac{\cos(\theta/2)}{\cos(\theta/6)}
\]  

(12)

that results in \( x = 1+ c \). Therefore, even though Wantzel [1] proof of irreducibility of the cubic equation (1) is correct, it does not imply impossibility of constructing angle trisection since the later problem depends on the quadratic equation (11) as is further described in the following.

2 A Solution of Trisection Problem

Before discussing the trisection problem however, let us first examine two different geometric configurations that one encounters in the construction of regular polygons shown schematically in Figs.2a and 2b that were helpful in arriving at the suggested resolution of trisection problem to be discussed in the sequel. In Fig.2a the edges of an arbitrary angle AG are located at points of tangency A' and G' of the small circles R, with origins at the vertices of regular polygons. On the other hand, in Fig.2b the edges of the angle GF are located at the position of origins of small circles at points G and F.

Fig.2 Two configurations of dividing finite angles into regular polygons: (a) Integral number of full circles (t-o-t-o-t-o-t-o-t) versus (b) Two half circles at the boundaries (o-t-o-t-o-t-o-t-o).

Also, as seen in these figures the angle between the chord and the sector boundaries are perpendicular in Fig.2a but slanted at the origins of small circles in Fig.2b. Moreover, in Fig.2a the origin points B, D, and F on the radial lines identified as “o” are on the large circle R, while the tangency points A’, C’, and G’ on the radial lines identified as “t” are not located on the large circle. Finally, one notes that while the angles of all the sectors are identical the two segments DC’ and C’B are aligned and form the
straight chord line $DB$ but the last section $BA'$ is not on this same line as shown in Fig.2a. This asymmetry or misalignment between the segments $DB$ and $BA'$ of the consecutive chords just described plays a significant role in the trisection problem.

We now present a solution to the classical problem of trisection of an arbitrary angle by only employing compass and straightedge schematically shown in Fig.3a.

(2) Draw the tangents to the small circles from $O'$ thus constructing an arbitrary trisected angle $\alpha = \angle M'O'N'$ that intersects the legs of the angle $\theta = \angle MON$ at points $G$ and $A$. Draw the trisecting lines $O'P'$ and $O'Q'$ for angle $\alpha$.

(3) Draw a circle with radius $OA = OG = R_o$ with origin at point $O$. Connect the intersection of this circle with the lines $O'P'$ and $O'Q'$ at points $C$ and $E$ to the origin $O$ to get the trisecting lines $OP$ and $OQ$ for angle $\theta$.

(4) We have now trisected the angle $\theta = \angle MON$ since $\angle GOE = \angle EOC = \angle COA = \theta/3$.

In the particular example shown in Fig.3a the arbitrary trisecting angle is $\alpha = \angle M'O'N' = 36^\circ$ and the angle to be trisected is $\theta = \angle MON = 75^\circ$.

The most important point is that trisection of an angle that in general represents a sector of a circle is being accomplished by employing again circles that act as “atomic” angular units. This is in harmony with the classical practice that employs linear line elements for the trisection of a given linear length [2]. Thus, the small circles play the role of angular elements to be used for trisection of angles. It appears that it is precisely this practice of using angular units to trisect angles that helps in avoiding the problem of rational versus transcendental numbers discussed above. This is because both the large circle $R_o$ and the element or the small circle $R_i$ involve the transcendental number $\pi$. To further clarify this point, one notes that once any angle is trisected one arrives at a hierarchy of sets of small circles at various radii as schematically shown in Fig.3b. Hence, at outer radius $R_o$ we have three circles with the inner radii $R_{i1}$, similarly at $R_{o2}$ we have three small circles with radii $R_i$ as shown in Fig3b. This hierarchy of circles continues ad infinitum for both smaller and larger scales corresponding to $R_o$ approaching either zero or infinity.

As another example of the application of the method discussed in steps (0)-(4) above, we consider the trisecting angle $\alpha = 24^\circ$ that intersects the angle $\theta = 69^\circ$ as shown in Fig.4a. As will be shown in the following, the actual size of the trisecting angle is not significant parallel to the length of the trisecting line $AG'$ employed in the trisection of a given length $AG$ as shown in Fig.4b.

---

**Fig.3** (a) Trisecting an arbitrary angle $\theta$ into $\angle GOE = \angle EOC = \angle COA = \theta/3$ employing only compass and straightedge. (b) Hierarchies of trisecting circles.

(0) Given an arbitrary angle $\theta = \angle MON$ with compass and straightedge construct the bisector of $\angle MON$ and obtain the line $OD$.

(1) On the extension of line $OD$ draw a circle with origin at point $O'$ at arbitrary radius $R_o$ and draw a small circle with arbitrary radius $R_i$ and origin $D'$. With compass mark equal arc lengths $DE' = DC'$ to obtain the points $F'$ and $B'$. Draw two more small circles with radius $R_i$ and origins at $F'$ and $B'$ points.
Fig. 4 (a) Trisecting angle \( \theta = 69^\circ \) into \( \angle \) GOE = \( \angle \) EOC = \( \angle \) COA = \( \theta / 3 = 23^\circ \) by employing the arbitrary trisecting angle \( \alpha = 24^\circ \). (b) Trisection of the length AG by employing an arbitrary trisecting line AG'.

The three small trisecting circles in Fig. 4a actually divide the angles (\( \alpha = 24^\circ \), \( \theta = 69^\circ \)) into six equal pieces hence producing six identical chords of (\( \alpha/6 = 4^\circ \), \( \theta/6 = 11.5^\circ \)) size on the circles (R_1, R_2) such that every two adjacent sectors form the trisected sector (\( \alpha/3 = 8^\circ \), \( \theta/3 = 23^\circ \)). Further examples of the application of the above method to trisect various angles while employing the same trisecting angle \( \alpha = 60^\circ \) are shown in Figs. 5a-f.

Fig. 5. Trisection of angles \( \theta = \) (a) 78\(^\circ\) (b) 84\(^\circ\) (c) 90\(^\circ\) (d) 96\(^\circ\) (e) 117\(^\circ\) (f) 150\(^\circ\) employing the arbitrary trisecting angle \( \alpha = 60^\circ \). To show that the actual size of the trisecting angle is not significant the example in Fig. 5(d) is repeated with another trisecting angle \( \alpha = 48^\circ \) as shown in Fig. 6.

Fig. 6. Trisection of angle \( \theta = 96^\circ \) accomplished by arbitrary trisecting angle \( \alpha = 48^\circ \).

Fig. 7 Arbitrary angle \( \theta \) trisected by two different trisecting angles at two arbitrary positions O' and O''.

Finally, the position of the point O' is not significant and another point such as O'' will result in the same trisection of the angle \( \theta \) as shown in Fig. 7.
3 Algebraic Proof of the Solution

Before discussing the algebraic proof of the proposed solution it will be shown how such a solution could be reconciled with the impossibility proof of Wantzel [1-8]. According to the results in Figs.2-7, the problem of division of angles into smaller segments involves the interplay between an outer large circle defining the sector of the angle and an inner small circle defining the sectors of the subdivided angles. As an example for the angle \( \alpha = \angle GOA \) shown in Fig.8a the outer radius is \( R_{o1} = OG \) and the inner radius is \( R_{i1} = EH' \).

Fig.8a Outer and inner radii \( R_{o1} = OG \) and \( R_{i1} = EH' \) for angle \( \alpha \).

One may now consider the trisecting angle \( \alpha \) and write

\[
\sin(\alpha / 2) = GJ / OG = GJ / R_{o1} \quad (13a)
\]

and

\[
\sin(\alpha / 6) = EH' / O'E' = R_{i1} / R_{o1} \quad (13b)
\]

leading to

\[
\frac{\sin(\alpha / 2)}{\sin(\alpha / 6)} = \frac{GJ}{R_{i1}} = b \quad (14)
\]

where \( b \) is a constant. Also, the trigonometric relation (5) gives

\[
\frac{\sin(\alpha / 2)}{\sin(\alpha / 6)} = 3 - 4\sin^2(\alpha / 6) \quad (15)
\]

such that from (14) and (15) one has

\[
2\sin(\alpha / 6) = \sqrt{3 - b} \quad (16)
\]

Similarly, for the arbitrary angle \( \theta \) in Fig.8b one identifies the outer and inner radii \( R_{o2} = OG \) and \( R_{i2} = EH \) and writes

\[
\sin(\theta / 2) = GJ / OG = GJ / R_{o2} \quad (17a)
\]

and

\[
\sin(\theta / 6) = EH / OE = R_{i2} / R_{o2} \quad (17b)
\]

leading to

\[
\frac{\sin(\theta / 2)}{\sin(\theta / 6)} = \frac{GJ}{R_{i2}} = d \quad (18)
\]

and by procedures parallel to (14)-(16) one arrives at

\[
2\sin(\theta / 6) = \sqrt{3 - d} \quad (19)
\]

It is thus clear that the solution steps (0)-(4) allow for trisection of arbitrary angle \( \theta \) (Figs.3-8) because the trisecting angle \( \alpha \) provides the desired constant \( d \) needed to trisect the angle \( \theta \). Therefore, the pairs of triangles \( \triangle O'GJ, O'G' \) and \( \triangle O'J, O'E'H' \) respectively corresponding to the angles \( \alpha/2, \alpha/6 \) and \( \theta/2, \theta/6 \) by (14) and (18) provide the constants \( b \) and \( d \) and due to (16) and (19) only require square root operation for the determination of sine of the desired trisected angles \( \alpha/6 \) and \( \theta/6 \). Hence, the formation of the ratio in (18) involving both the right and the left hand sides of (5) leads to constructible square root operation in (19) and thereby avoids the constraint imposed by the irreducibility proof of Wantzel [1].

To arrive at the algebraic proof that the angle \( \theta \) has indeed been trisected one first notes that on any circle with origin at \( O \) and arbitrary radii \( R_{o1} = O'G \), \( R_{o2} = O'G' \), \( R_{o3} = O'G'' \), \( R_{o4} = O'G''' \), \( R_{o5} = O'G'''' \), \( R_{o6} = O'G''''' \) one can draw three chords \( \{G'E', E'C', C'A\}, \{G''E'', E''C'', C''A''\}, \{G'''E''', E'''C''', C'''A'''\}, \ldots \) that are parallel to each other and trisect angle \( \alpha \) as shown in Fig.9a.
Fig.9a Chords: \((GE', E'C', CA)\); \((G'E'' E'C'', C''A'')\) at arbitrary radii \(R_{o1} = OG\), \(R_o = OG''\) for trisecting angle \(\alpha\).

Fig.9b Chords \((IE, CQ)\) on circle with radius \(R_{o3} = O'I\) for angle \(\alpha\) rotated about points \(E, C\) to form chords \((GE, CA)\) on radius \(R_{o2} = OG\) for angle \(\theta\).

Hence, the chords \((IE, EC, CQ)\) of Fig.8b have the same standard shape as any arbitrary set of trisecting chords \((G'E'', E'C'', C''A'')\) shown in Fig.9a. Based on this correspondence, a new circle with radius \(R_{o3} = O'E\) and with origin at \(O'\) is drawn that passes through the points E and C as shown in Fig. 9b. It is important to note that the circles with radii \(R_{o3}\) and \(R_{o2}\) that respectively correspond to the angles \(\alpha\) and \(\theta\) share the central chord EC and the former circle intersects the legs of the angle \(\alpha\) at points I and Q. Hence, the points \((G, E, C, A)\) on \(R_{o3}\) circle and \((I, E, C, Q)\) on \(R_{o2}\) circle allow simultaneous trisection of the angles \(\theta\) and \(\alpha\). However, in Fig 9b as opposed to Fig.9a, the two different angles \((\theta, \alpha)\) share the same points \((G, A)\) rather than different points \((G, A)\) and \((I, Q)\). Hence, the chords \((EI, CQ)\) corresponding to the angle \(\alpha\) when rotated about the points E and C result in the new chords \((GE, CA)\) now corresponding to the arbitrary angle \(\theta\) as shown in Fig.9b.

The connections between the trisecting angle \(\alpha\) and its chords on circle \(R_{o3}\) and the arbitrary angle \(\theta\) and its chords on circle \(R_{o2}\) are more clearly revealed in Figs.10-12. The equalities \(IE = EC = CQ\) hold exactly because of the initial construction of the trisecting angle \(\alpha\) as described above. Also, because the central chord EC is common to both circles \(R_{o3}\) and \(R_{o2}\), simultaneous validity of \(IE = EC = CQ\) and \(GE = EC = CA\) will constitute the proof of trisection of the angle \(\theta\). To arrive at such a proof, one first notes the central importance of the point E since it provides the chord EC needed to trisect the angle \(\theta\). One therefore starts by drawing the chord GE and the line OL perpendicular to GE bisecting the angle GOE and extends this line to cross the line OF’ at point F. For now the point F is identified as the crossing point of \(O'F'\) and OF and the fact that it is located on the circle \(R_{o2}\) will be proved a posteriori. It is clear that there exists a unique point at the crossing of \(O'F'\) and OF lines that simultaneously bisects of the angles GOE and GOE. As a result, one has the equalities \(FI = FE = FG = FE'\) such that the points G, I, E, E’ will all locate on a circle with origin at F as shown in Fig.10.

Fig.10 Equalities \(FI = FE = FG = FE'\) leading to points \(G, I, E, E'\) on a circle with origin at F.

Fig.11 The enlarged view of the region GIE'E' of Fig.10 and the circle at origin F.

To facilitate observations the region GIE'E’ of Fig.10 with the circle at origin F have been enlarged and shown in Fig.11. Connecting points I and E to the origin F and extending these lines results in the lines IN and EM that are respectively perpendicular to the mid points N and M of the chords EE’ and GI as shown in Fig.11. Therefore, the equality of the

Applied Mathematics in Electrical and Computer Engineering

ISBN: 978-1-61804-064-0

283
angles $\angle IEM = \angle MEG = \alpha/6$ makes GEI an isosceles triangle thus establishing the equality of the chords $EI = EG$ and by symmetry one has $CQ = CA$ as shown in Fig.10. Therefore, by the known equality $IE = EC = CQ$ based on trisecting angle $\alpha$ one arrives at $EG = EC = CA = IE = CQ$. Furthermore, the result $\angle IEM = \angle MEG = \angle EIN = \angle NIE' = \alpha/6$ leads to $EL = EK$ (Fig.12). Because it is known that $EK = EH$ due to constructed trisecting angle $\alpha$, one arrives at the equalities of the half chords $EK = EL = EH$ and hence the equality of the angles $\angle GOF = \angle FOE = \angle EOD = \theta/6$ that concludes the algebraic proof of the trisection of the arbitrary angle $\theta$ (Fig.10).

$$\angle GOE = \angle EOC = \angle COA = \theta/3 \quad (20)$$

Also, the equality of angles $\angle GOF = \angle FOE = \angle EOD = \theta/6$ and the triangles $OFE = OED$ and $EFL = EDH$ (Fig.12) result in $OD = OF$ thus proving that the point $F$ locates on the circle with origin at $O$ and radius $R_o$.

Since by constructions the chords (IE, EC, CQ) trisect the angle $\alpha$ exactly and the connection between the angles $\alpha$ and $\theta$ is through exact algebraic relations corresponding to geometric construction as shown in Fig.12.

Fig.12 Further details about connections between the trisecting angle $\alpha$ versus the trisected arbitrary angle $\theta$.

the exactness of trisection of the arbitrary angle $\theta$ should be on the same footing as that of the initial trisecting angle $\alpha$. This is in the same spirit as the exactness of trisection of a linear length by construction through employment of a trisecting line shown in Fig.4b. Therefore, if one assumes that the ruler and the compass employed in the geometric constructions are in fact Platonic ideal instruments, then the trisection procedure proposed herein should be exact. It is reasonable to anticipate that the precise meaning of “exactness” just mentioned will be related to the problem of continuum [10] requiring elaborate mathematics of normalization [11] and internal set theory [12] and scale invariant definition of hierarchies of infinitesimals [13].

The procedures outlined in steps (0)-(4) above may be used to construct regular polygons of any size such as hexagon and nonagon (9-sides) as shown in Figs.13a and 13b. Also, as shown in Fig.13c one can apply the method shown in Fig.3a to trisect the angle $\theta = \pi /3 = 60^\circ$ that has played an important role in the history of the trisection problem.

Fig.13 Inner and outer circles of regular polygons: (a) $60^\circ$ (b) $40^\circ$ (c) $20^\circ$ sectors (d) Division of angle $\theta = 60^\circ$ into seven equal parts allowing construction of regular heptagon.

The resulting regular polygon of 18-sides having $20^\circ$ angles per sector shown in Fig.13c could be converted to a conjugate regular 20-sided polygon with $18^\circ$ angles per sector. Also, applying the steps (0)-(4) but now drawing seven instead of three small circles with arbitrary radius $R_s$ on the arc of arbitrary radius $R_i$ allows the construction of heptagon through division of an arbitrary angle, say $\theta = 60^\circ$, into seven equal parts as shown in Fig.13d. In other words, a regular hexagon containing seven small circles per sector could be viewed as a regular heptagon with six small circles per sector.

For regular polygons, the circumferential distances along outer radius $R_o$ are fitted with integral, i.e. “quantum”, numbers of small inner circles of radius $R_i$ (Fig.13). For $n$-sided regular
polygon with the sector angle $\theta = (360/n)$ there is a unique relation between radii of the outer versus the inner circles because

$$\sin(\theta/2) = \frac{\sin(180/n)}{R_i/R_o} \tag{21}$$

Further implications of the connections between adjacent scales of inner versus outer radii ($R_o$, $R_{\beta-1}$) and angles ($\theta_{\beta}$, $\theta_{\beta-1}$) require future examinations. It is reasonable to suspect that for the analysis of continuum [10] one may need to apply Gauss’s error function to define a “measure” for normalization [11] of both linear as well as angular coordinates [13] in order to achieve higher degrees of symmetry between these coordinates.

4 Concluding Remarks
The unexpectedly simple solution shown in Fig.3a fully justifies the intuition of all Trisectors [14, 15] amongst both professional and amateur mathematicians since Wantzel’s paper of 1837 who believed that the solution of the trisection problem may indeed be possible.

Acknowledgements: The author expresses his deep gratitude to Professor G. ’t Hooft and Professor A. Odlyzko for their helpful and constructive comments on an earlier version of this Note.

References: