Effect of First- and Second-Order Extensions on UFIR Filtering of Nonlinear Models

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Abstract: The trade-off between the first- and second-order extended unbiased finite impulse response filters (EFIR1 and EFIR2, respectively) is examined for suboptimal estimation of nonlinear discrete-time state-space models with additive white noise. An important applied feature of the unbiased FIR filter is that it does not require noise statistics and initial errors. Based upon the tracking problem solved in a horizontal plane for a moving object, we show that in most of the cases, the difference between the EFIR1 and EFIR2 filter outputs appears to be negligible. In a few cases, the second-order approximation can improve a local performance. But it can also deteriorate it in some others or produce mixed effect. We therefore can give no definitive recommendations about practical usefulness of the EFIR2.

Key–Words: Nonlinear state-space model, extended FIR filter, extended Kalman filter

1 Introduction

In applications to nonlinear problems, the linear optimal Kalman filter (KF) is often used in its first-order extended form proposed by Cox [1] and others employing the first-order Taylor series expansion in the presence of additive white Gaussian noise in the process and measurement. The relevant first-order extended Kalman filter (EKF1) has been used extensively in diverse applications such as system state estimation, tracking of moving objects [2], navigation, Global Positioning System (GPS), process control, etc. The second-order extended Kalman filter (EKF2) was proposed and investigated by Athans, Wishner and Bertolini in [3]. With decades, EKF2 has been developed and investigated in details. But even though its ability to reduce errors was clearly demonstrated in [3], still nothing definitive has been said about its performance in general [4].

Degradation of the KF output in real world is often connected with its infinite impulse response (IIR). This was emphasized by Jazwinski in [5] as opposite to the finite impulse response (FIR) filters having limited memory. Later, his conclusion about higher robustness of FIR structures against the unbounded perturbations in systems has been transformed to the theory of receding horizon control. Daum in [6] has also recently reminded that Gauss’s batch LSs often gives accuracy superior to the best available EKF.

It is known that the unbiased FIR estimate converges to the optimal one reminiscent of the Gauss’s batch when the averaging interval of \( N \) points becomes large, \( N \geq 1 \) [7]. The relevant iterative Kalman-like unbiased FIR (UFIR) estimator ignoring noise covariances and initial errors [8] has already proved its efficiently against KF.

Below, we examine the first- and second-order extensions (EFIR1 and EFIR2, respectively) of the UFIR filter. Pursuing the aim, we investigate the trade-off between the EFIR1 and EFIR2 filter outputs based on an example of tracking of a moving object in a horizontal plane.

2 Nonlinear Model

Consider a nonlinear system represented in additive white noise environment with the state and observation equations, respectively,

\[
\begin{align*}
x_n &= f_n(x_{n-1}) + B_n w_n, \quad (1) \\
y_n &= h_n(x_n) + D_n v_n, \quad (2)
\end{align*}
\]

where \( f_n(x_{n-1}) \) and \( h_n(x_n) \) are time-variant nonlinear vector functions, \( x_n \in \mathbb{R}^K, \ y_n \in \mathbb{R}^M, \ B_n \in \mathbb{R}^{K\times P}, \) and \( D_n \in \mathbb{R}^{M\times M}. \) The white noise vectors \( w_n \in \mathbb{R}^P \) and \( v_n \in \mathbb{R}^M \) are supposed to be zero mean, \( E\{w_n\} = 0 \) and \( E\{v_n\} = 0, \) with the covari-
where \( \partial h_n/\partial x_n \) and \( \partial h_n/\partial x_n \) are both Jacobian, \( \varepsilon_n = x_n - \hat{x}_{n|n-1} \) is the prior estimation error, and \( \varepsilon_n = x_n - \hat{x}_{n|n-1} \) is the estimation error. The second-order terms can be represented as [2]

\[
\begin{align*}
\kappa_n &= \sum_{k=1}^{K} e_k^{K T} G_{kn} \varepsilon_{n-1}, \\
\beta_n &= \sum_{m=1}^{M} e_m^{M T} H_{mn} \varepsilon_{n-1},
\end{align*}
\]

where \( G_{kn} = \partial^2 h_n/\partial x_n^2 \big|_{x_{n|n-1}} \) and \( H_{mn} = \partial^2 h_n/\partial x_n^2 \big|_{x_{n|n-1}} \) are both Hessian and \( f_{kn} \) and \( h_{mn} \) are the \( k \)th and \( m \)th components of \( f_n(x_{n-1}) \) and \( h_n(x_n) \), respectively. Also, \( e^K = \mathbb{R}^K \) and \( e^M = \mathbb{R}^M \) are Cartesian basic vectors with the \( k \)th and \( m \)th components unity and all others zeros, respectively.

For the unbiased estimate we have \( E\{\varepsilon_n\} = 0 \) and then averaging of \( f_n(x_{n-1}) \) gives us the prior estimate

\[
\begin{align*}
\hat{x}_{n|n-1} &= f_n(\hat{x}_{n|n-1}) + 1/2 \kappa_n, \\
\hat{x}_{n|n-1} &= f_n(\hat{x}_{n|n-1}) + 1/2 \beta_n,
\end{align*}
\]

in which, by the cyclic property of the trace operator, the expectation of \( \kappa_n \) can be found as

\[
\bar{\kappa}_n = \sum_{k=1}^{K} e_k^{K T} \text{tr}\{G_k P_{n-1}\},
\]

where \( \text{tr}\{B\} \) is trace of quadratic \( B \) and \( P_n = E\{\varepsilon_n \varepsilon_n^T\} \) is the estimation error covariance.

The expectation of the prior error \( E\{\varepsilon_n\} = E\{x_n - \hat{x}_{n|n-1}\} \) acquires zero components,

\[
E\{\varepsilon_n\} = E\{A_n \varepsilon_{n-1} + 1/2 (\varepsilon - \hat{x}) + B_n w_n\} = 0,
\]

and the average of \( h_n(x_n) \) can be found as

\[
\bar{h}_n(x_n) = h_n(\hat{x}_{n|n-1}) + 1/2 \beta_n,
\]

where the expectation of \( \beta_n \) is

\[
\bar{\beta}_n = \sum_{m=1}^{M} e_m^M \text{tr}\{H_m P_n\},
\]

in which \( P_n = E\{\varepsilon_n \varepsilon_n^T\} \) is the covariance of the prior estimation error.

#### 3 Kalman-Like E FIR Filter

Following the strategy of EKF, the E FIR2 filtering estimate related to the above-extended nonlinear model can be written as

\[
\bar{x}_{l|l-1} = \hat{x}_{l|l-1} + K_l[\bar{y}_l - \bar{h}_l(\hat{x}_{l|l-1})],
\]

where the gain \( K_l = F_l C_l \) ignores noise statistics and initial errors and is defined iteratively [8] via

\[
F_l = [C_l^T C_l + (A_l F_{l-1} A_l^T)^{-1}]^{-1},
\]

for which the initial value \( F_{\alpha - 1}, \alpha \geq m + K \), can be determined following [8].

By (9) and (10), the prior estimate becomes

\[
\hat{x}_{l|l-1} = f_l(\hat{x}_{l-1|l-1}) + 1/2 \sum_{k=1}^{K} e_k^K \text{tr}\{G_k P_{l-1}\}
\]

and, by (11) and (12), the average of \( \bar{h}_l(x_l) \) at \( \hat{x}_{l|l-1} \) can be written as

\[
\bar{h}_l(\hat{x}_{l|l-1}) = h_l(\hat{x}_{l|l-1}) + 1/2 \sum_{m=1}^{M} e_m^M \text{tr}\{H_m P_l\}.
\]

By simple manipulations involving (1), (5) and (13), the prior estimation error can be found to be

\[
P_n = E\{(x_n - \hat{x}_n)(x_n - \hat{x}_n)^T\} = A_n P_{n-1} A_n^T + B_n R_n B_n^T + 1/2 \bar{F}_n \]

where the \((uv)\)th component of \( \bar{F}_n \) is specialized with

\[
\bar{F}_{(uv)n} = \text{tr}\{G_{uv} P_{n-1} G_{uv} P_{n-1}\}.
\]
The estimation error $P_n$ can be represented with
\[
P_n = E\{(x_n - \hat{x}_n)(x_n - \hat{x}_n)^T\} = (I - K_n C_n)P_n^-(I - K_n C_n)^T + K_n D_n Q_n D_n^T K_n^T + \frac{1}{2}(\bar{f}_n C_n^T K_n^T + K_n C_n \bar{f}_n) + \frac{1}{2}K_n \hat{h}_n K_n^T - \frac{1}{2}(\bar{M}_n K_n^T + K_n \bar{M}_n^T),
\]
where the $(rg)$th component of $\hat{h}_n$ is computed as
\[
\hat{h}_{(rg)n} = \text{tr}\left[H_{rn} P_n^H H_{gn} P_n^-\right] \quad (20)
\]
and the $(ur)$th one of $\bar{M}_n$ is determined by
\[
\bar{M}_{(ur)n} = \text{tr}[G_{un} P_{n-1} H_{rn} P_{n-1}^-] + \sum_{q=1}^{K} \sum_{l=1}^{K} H_{rn} \times \text{tr}[G_{un} P_{n-1} G_{qn} P_{n-1}^- G_{ln} P_{n-1}^-]. \quad (21)
\]

### 3.1 The EFIR2 Filtering Algorithm

Given (1) and (2) with the mutually uncorrelated additive white noise sources, $w_n$ and $v_n$, the EFIR2 filtering algorithms is thus as follows
\[
x_{n|l} = \hat{x}_{n|l-1} + F_l C_l [y_l - \bar{h}_l(\hat{x}_{n|l-1})], \quad (22)
\]
where
\[
\hat{x}_{n|l-1} = f_l(\hat{x}_{n-1|l-1}) + \frac{1}{2} \sum_{k=1}^{K} e_k^T \text{tr}\left\{G_l P_{l-1}\right\}, \quad (23)
\]
\[
\bar{h}_l(\hat{x}_{n|l-1}) = h_l(\hat{x}_{n|l-1}) + \frac{1}{2} \sum_{m=1}^{M} e_m^T \text{tr}\left\{H_l P_{l-1}\right\}, \quad (24)
\]
\[
F_l = [C_l^T C_l + (A_l F_{l-1} A_l^T)^{-1}]^{-1}, \quad (25)
\]
and $P_{l-1}$ and $F_l$ are given by (17) and (19), respectively. The initial error $P_{0-1}$ can be put to zero and $F_{a-1}$ and $\hat{x}_{n-1|a-1}$ computed as in [8]. A variable $l$ ranges from $\alpha \geq m + K$ to $n$ and the true estimate is taken at $l = n$.

### 3.2 The EFIR1 Filtering Algorithm

The EFIR1 algorithm appears from EFIR2 by neglecting the second-order terms. That leads to
\[
\hat{x}_{n|l} = f_l(\hat{x}_{n-1|l-1}) + F_l C_l [y_l - \bar{h}_l(\hat{x}_{n|l-1})], \quad (26)
\]
where
\[
F_l = [C_l^T C_l + (A_l F_{l-1} A_l^T)^{-1}]^{-1}. \quad (27)
\]

Figure 1: Object tracking with the EKF1 and EFIR1 filter. Two DMSs are located at $(0,0)$ and $(0,50)$. Measurement is provided at 1000 discrete points with step $\tau = 0.1$ s.

and $F_{a-1}$ and $\hat{x}_{a-1|a-1}$ are computed following [8]. It follows that EFIR1 needs only $N$, $K$, and $\alpha \geq n - N + 1 + K$ to start computing and updating all of the matrices. To estimate $P_n$, one needs $R_n$, $Q_n$, and $P_{a-1}$. As well as in the case of EFIR2, an iterative variable $l$ ranges here for each $n$ as $\alpha \leq l \leq n$ and the true $\hat{x}_{n|n}$ and $P_n$ are taken at $l = n$.

### 4 Critical Evaluation of the EFIR1 and EFIR2 Estimates

Below, we investigate the EFIR1 and EFIR2 filtering estimates based on a typical example of tracking of a moving object. For a comparison, we also exploit the EKF1 and EKF2 algorithms taken from [4].

Two distance measurement stations (DMSs) are located at $(0,0)$ and $(0,50)$ as shown in Fig. 1 and it is supposed that a object and the DMSs are all in a horizontal plane. Each DMS transmits a pulse that reflects from the object and returns back to DMS. The transit time is interpreted in terms of distance $d_1$ or $d_2$. We suppose that an object has four states ($K = 4$): $x_{1n}$ is the coordinate $x$; $x_{2n}$ velocity along $x$; $x_{3n}$ $\geq 0$ coordinate $y$; and $x_{4n}$ velocity along $y$. The behavior is thus modeled with (1), in which $f_n(x_{n-1}) = A x_{n-1}$, $B_n = I$, $x_n = [x_{1n} x_{2n} x_{3n} x_{4n}]^T$, and
\[
A = \begin{bmatrix} 1 & \tau & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (28)
\]

White noise $w_n = \begin{bmatrix} 0 & w_{2n} & 0 & w_{4n} \end{bmatrix}^T$ is zero mean with
the variances \( \sigma^2_{w} = \sigma^2_{n} = \sigma^2_{w} \) and covariance

\[
R = \begin{bmatrix}
\frac{\tau^2}{3} & \frac{\tau}{2} & 0 & 0 \\
\frac{\tau}{2} & 1 & 0 & 0 \\
0 & 0 & \frac{\tau^2}{2} & \frac{\tau}{2} \\
0 & 0 & \frac{\tau^2}{2} & 1 \\
\end{bmatrix}
\]

(29)

The zero mean white measurement noise \( v_n = [v_{1n} \ v_{2n}] \) has the variance \( \sigma^2_{v} = \sigma^2_{v1} = \sigma^2_{v2} \) and covariance \( Q = \begin{bmatrix}
\sigma^2_{v} & 0 \\
0 & \sigma^2_{v} \\
\end{bmatrix} \). The observation equation (2) is specialized with

\[
h_n(x_n) = \begin{bmatrix}
\sqrt{x_n + y_n} \\
\sqrt{(a - x_n)^2 + y_n^2}
\end{bmatrix}
\]

(30)

Measurement has been conducted at 1000 points with step \( \tau = 0.1 \) s, \( \sigma_{w} = 0.01 \) m, and \( \sigma_{v} = 0.2 \) m. To estimate \( x_{s|s} \) at the initial point \( s = a - 1 \) with \( a = m + K = m + 4 \), projections \( s_{xn} \) and \( s_{yn} \) of the measurement to \( x \) and \( y \) were formed as, respectively,

\[
s_{xn} = \frac{1}{2a}[y_{1n}^2 - y_{2n}^2 + a],
\]

(31)

\[
s_{yn} = \sqrt{y_{1n}^2 - s_{xn}^2}.
\]

(32)

### 4.1 First- vs. Second-Order Estimates

To learn the difference between the estimates provided with EFIR1, EFIR2, EKF1, and EKF2, the process was multiply generated and filtered. Observing the outputs, the only conclusion coming to mind was that made by Simon in [4]: nothing definitive can be said about the performance of the second-order extended filters. Indeed, most of the runs have revealed the same trajectories for the first- and second-order filters as shown in Fig. 1 for EFIR1 and EKF1. The relevant filtering errors are sketched in Fig. 2 for an ideal case of fully known both the model and noise.

Just in a few cases, different behaviors were observed as shown in Fig. 3 and Fig. 4. This certainly does not allow one to make a preference in favor of a certain filter. In fact, both EFIR2 and EKF2 are able to improve the performance (Fig. 3a and Fig. 4a) in a way similar to that demonstrated in [3]. But they may also deteriorate it (Fig. 3b and Fig. 4b) or produce mixed effects (Fig. 3d and Fig. 4d).

### 5 Conclusion

In this paper, we have shown experimentally that the EFIR2 and EKF2 estimators have no definitive advantages against the EFIR1 and EKF1 ones, at least in the tracking problem considered as an example of applications. In most of the runs, the first- and second-order estimates have traced in general along the same trajectories. In some cases, EKF2 and EFIR2 improved the local performance, but they also deteriorated it in some other ones. Since the EFIR1 filter does not require the noise statistics and initial errors, it may be more preferable than EKF1.

### References:


Figure 3: Effect of the second-order approximation on the extended UFIR filter errors: (a) performance improvement, (b) performance deterioration, (c) local mixed effect, and (d) prolonged mixed effect.
Figure 4: Effect of the second-order approximation on the extended KF errors: (a) performance improvement, (b) performance deterioration, (c) prolonged mixed effect, and (d) prolonged mixed effect.