A Generalized 3-Component Portfolio Selection Model

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Abstract: In this paper we study a portfolio selection problem corresponding to a financial situation characterized by three components: some returns are mathematically described by random variables, others by fuzzy numbers, and a third group of returns by discrete fuzzy variables. The proposed model unifies probabilistic, possibilistic, and credibilistic aspects of portfolio selection. Both Markowitz probabilistic model and a possibilistic portfolio selection model are generalized. A calculation formula for the optimal solution of the portfolio problem and a formula which gives the minimum value of the associated risk are proved.

Key Words: Portfolio optimization, fuzzy numbers, possibility theory, credibility theory

1 Introduction

In this paper we will study a portfolio selection problem corresponding to a more complex financial situation characterized by three components: some returns are mathematically described by random variables, others by fuzzy numbers and others by discrete fuzzy variables. For the first component the probabilistic indicators associated with random variables are used, for the second component possibilistic indicators associated with fuzzy numbers are used and for the third component credibilistic indicators are used.

The papers is organized as follows.

In Section 2 the definition of a fuzzy number, operations with fuzzy numbers and some examples are recalled. Three indicators associated with fuzzy numbers are recalled: expected value, variance, covariance (cf. [2], [3], [4]). They will be used in the next sections to build a mixed portfolio selection model.

In Section 3 the notions of credibility measure, credibilistic expected value associated with a fuzzy number, and a procedure by which a fuzzy discrete variable $\xi$ is associated with a discrete random variable $Z_\xi$ are presented. The credibilistic expected value $E(Z_\xi)$ which allows some credibilistic models to be converted to probabilistic models.

In Section 4 two portfolio selection models are compared: Markowitz’s and a possibilistic one derived from the former. The difference between the two approaches consists in indicators’ interpretation:

- for the first model the return is evaluated by probabilistic mean value; for the second model the return is evaluated by the possibilistic mean value;
- for the first model the risk is evaluated by probabilistic variance, while for the second model the risk is evaluated by possibilistic variance.

Triplet portfolios are introduced in Section 5. Rentability of some assets is mathematically represented by random variables, rentability of other assets by fuzzy numbers and rentability of the latter assets by discrete fuzzy variables. Two types of indicators are associated with a portfolio:

- possibilistic, probabilistic and credibilistic mean values, and a total mean value;
- possibilistic, probabilistic and credibilistic variances, and a total variance.

In Section 6 the three component portfolio selection problem is formulated using these indicators. The main result of the section is the optimal solution of portfolio selection problem and the calculation of the minimum risk value.

2 Indicators of Fuzzy Numbers

In this section we recall the definition of fuzzy numbers [5], [6] and some possibilistic indicators associated with them [2], [3], [4].

Let $X$ be a set of states. A fuzzy subset of $X$ is a function $A : X \rightarrow [0, 1]$. For any state $x \in X$ the real number $A(x)$ is the degree of membership of $x$ to $A$. The support of a fuzzy set $A$ is supp$(A) = \{ x \in X | A(x) > 0 \}$. A fuzzy set $A$ is normal if there exists $x \in X$ such that $A(x) = 1$. 
In the following we consider \( X = \mathbb{R} \).

Let \( A \) be a fuzzy subset of \( \mathbb{R} \) and \( \gamma \in [0, 1] \). The \( \gamma \)-level set of \( A \) is defined by

\[
[A]^{\gamma} = \begin{cases} 
\{ x \in \mathbb{R} | A(x) \geq \gamma \} & \text{if} \; \gamma > 0 \\
cl(\text{supp}(A)) & \text{if} \; \gamma = 0.
\end{cases}
\]

\( cl(\text{supp}(A)) \) is the topological closure of the set \( \text{supp}(A) \subseteq \mathbb{R} \).

\( A \) is called fuzzy convex if \( [A]^{\gamma} \) is a convex subset of \( \mathbb{R} \) for any \( \gamma \in [0, 1] \).

A fuzzy number is a fuzzy set of \( \mathbb{R} \) normal, fuzzy-convex, continuous and with bounded support.

Let \( A \) be a fuzzy number and \( \gamma \in [0, 1] \). Then \( [A]^{\gamma} \) is a closed and convex subset of \( \mathbb{R} \). We denote \( a_1(\gamma) = \min[A]^{\gamma} \) and \( a_2(\gamma) = \max[A]^{\gamma} \). Hence \( [A]^{\gamma} = [a_1(\gamma), a_2(\gamma)] \) for all \( \gamma \in [0, 1] \).

Let \( A, B \) be two fuzzy numbers and \( \lambda \in \mathbb{R} \). We define the functions \( A + B : \mathbb{R} \rightarrow [0, 1] \) and \( \lambda A : \mathbb{R} \rightarrow [0, 1] \) by

\[
(A + B)(z) = \sup \{ A(x) \land B(y) | x + y = z \}; \quad (\lambda A)(z) = \sup \{ A(x) | \lambda x = z \}.
\]

Then \( A + B \) and \( \lambda A \) are fuzzy numbers.

If \( A_1, \ldots, A_n \) are fuzzy numbers and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) then one can consider the fuzzy number \( \sum_{i=1}^n \lambda_i A_i \).

A non–negative and monotone increasing function \( f : [0, 1] \rightarrow \mathbb{R} \) is a weighting function if it satisfies the normality condition \( \int_0^1 f(\gamma) d\gamma = 1 \).

We fix a fuzzy number \( A \) and a weighting function \( f \). Assume that \( [A]^{\gamma} = [a_1(\gamma), a_2(\gamma)] \) for all \( \gamma \in [0, 1] \).

The \( f \)-weighted possibilistic expected value of \( A \) is defined by

\[
E(f, A) = \frac{1}{2} \int_0^1 (a_1(\gamma) + a_2(\gamma)) f(\gamma) d\gamma.
\]

The \( f \)-weighted possibilistic variance of \( A \) is defined by

\[
\text{Var}(f, A) = \frac{1}{2} \int_0^1 [(a_1(\gamma) - E(f, A))^2 + (a_2(\gamma) - E(f, A))^2] f(\gamma) d\gamma.
\]

Assume now that \( A, B \) are two fuzzy numbers such that \( [A]^{\gamma} = [a_1(\gamma), a_2(\gamma)] \) and \( [B]^{\gamma} = [b_1(\gamma), b_2(\gamma)] \) for any \( \gamma \in [0, 1] \). The \( f \)-weighted possibilistic covariance of \( A \) and \( B \) is defined by

\[
\text{Cov}(f, A, B) = \frac{1}{2} \int_0^1 [(a_1(\gamma) - E(f, A)) (b_1(\gamma) - E(f, B)) + (a_2(\gamma) - E(f, A)) (b_2(\gamma) - E(f, B))] f(\gamma) d\gamma.
\]

In the following when we write \( E(f, A) \), \( \text{Var}(f, A) \) and \( \text{Cov}(f, A, B) \), the weighting function \( f \) will be apriori fixed.

**Proposition 2.1** [7] Let \( A_1, \ldots, A_n \) be fuzzy numbers and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \).

1. \( E(f, \sum_{i=1}^n \lambda_i A_i) = \sum_{i=1}^n \lambda_i E(f, A_i) \);
2. If \( \lambda_1, \ldots, \lambda_n \geq 0 \) then \( \text{Var}(f, \sum_{i=1}^n \lambda_i A_i) = \sum_{k,l=1}^n \lambda_k \lambda_l \text{Cov}(f, A_k, A_l) \).

### 3 Credibilistic Indicators

Let \( \Omega \) be a non–empty set of states and \( \mathcal{P}(\Omega) \) its powerset. The elements of \( \mathcal{P}(\Omega) \) are called events.

**Definition 3.1** A credibility measure on \( \Omega \) is a function \( C_r : \mathcal{P}(\Omega) \rightarrow [0, 1] \) with the properties:

1. \( C_r(\emptyset) = 1 \);
2. If \( A, B \in \mathcal{P}(\Omega) \) then \( A \subseteq B \) implies \( C_r(A) \leq C_r(B) \);
3. For any \( A \in \mathcal{P}(\Omega) \), \( C_r(A) + C_r(A^c) = 1 \);
4. For any family \( (A_i)_{i \in I} \) of subsets of \( \Omega \) with the property \( \sup_{i \in I} C_r(A_i) < 1/2 \) the equality \( C_r(\bigcup_{i \in I} A_i) = \sup_{i \in I} C_r(A_i) \) takes place.

Let \( C_r : \mathcal{P}(\mathbb{R}) \rightarrow [0, 1] \) be a credibility measure on the set \( \mathbb{R} \) of real numbers and \( \xi : \mathbb{R} \rightarrow \mathbb{R} \) be a fuzzy variable. We consider the function \( \mu : \mathbb{R} \rightarrow [0, 1] \) defined by

\[
\mu(x) = (2C_r(\xi = x)) \land 1, x \in \mathbb{R}.
\]

\( \mu \) is called the membership function of the fuzzy variable \( \xi \) w.r.t. the credibility measure \( C_r \).

Let \( C_r : \mathcal{P}(\mathbb{R}) \rightarrow [0, 1] \) be a credibility measure, \( \xi : \mathbb{R} \rightarrow \mathbb{R} \) a fuzzy variable and \( \mu : \mathbb{R} \rightarrow [0, 1] \) the membership function associated with \( \xi \).
Definition 3.2 [12, 13] The credibilistic expected value $Q(\xi)$ of the fuzzy variable $\xi$ (w.r.t. credibility measure $Cr$) is defined by

$$Q(\xi) = \int_0^\infty Cr(\xi \geq r)dr - \int_0^\infty Cr(\xi \leq r)dr \quad (8)$$

provided that the two integrals are finite.

Let $\xi$ be a discrete fuzzy variable whose membership function $\mu$ has the form

$$\mu(x) = \begin{cases} \mu_1 & \text{if } x = a_1 \\ \mu_2 & \text{if } x = a_2 \\ \cdots \cdots \\ \mu_n & \text{if } x = a_n. \end{cases} \quad (9)$$

We assume that $a_1 < a_2 < \ldots < a_n$ and we make the writing convention

$$\xi = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ \mu_1 & \mu_2 & \ldots & \mu_n \end{bmatrix}. \quad (10)$$

Next the simple fuzzy variable $\mu$ will be represented by table (10).

We consider the following real numbers:

$$p_i = \frac{1}{2} \left( \sqrt{\mu_j - \sqrt{i-1}} \right) + \frac{1}{2} \left( \sqrt{\mu_j - \sqrt{n+1}} \right) \quad (11)$$

for $i = 1, \ldots, n$, where $\mu_0 = \mu_{n+1} = 0$.

Proposition 3.3 [13] The numbers $p_1, \ldots, p_n$ verify the following two conditions:

- $p_i \geq 0$ for any $i = 1, \ldots, n$;
- $\sum_{i=1}^n p_i = 1$.

By Proposition 3.3 we can consider the discrete random variable

$$Z_\xi : \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ p_1 & p_2 & \ldots & p_n \end{bmatrix}. \quad (12)$$

Proposition 3.4 [13] Let $\xi$ be the simple fuzzy variable given by (10). Then the credibilistic expected value $Q(\xi)$ coincides with the probabilistic expected value $M(Z_\xi)$.

4 Approaches to Portfolio Selection

In this section we present two approaches to Markowitz probabilistic portfolio selection problem [14, 15] and a possibilistic model ([10, 11]).

4.1 Probabilistic Portfolio Selection Model

We consider $m$ assets $j = 1, \ldots, m$. Assume that the returns of the $m$ assets are random variables $X_1, \ldots, X_m$. The following two elements are known:

- probabilistic mean returns $\mu_j = M(X_j), j = 1, \ldots, m$;
- probabilistic covariances $\sigma_{st} = Cov(X_s, X_t), s, t = 1, \ldots, m$.

A portfolio is a vector $(y_1, \ldots, y_m) \in \mathbb{R}^m$ with

$$\sum_{j=1}^m y_j = 1 \text{ and } y_j \geq 0 \text{ for any } j = 1, \ldots, m.$$ 

The return of the portfolio $(y_1, \ldots, y_m)$ is the random variable $\sum_{j=1}^m y_j X_j$. With each portfolio one associates:

- mean return $M(\sum_{j=1}^m y_j X_j) = \sum_{j=1}^m y_j \mu_j$;
- probabilistic variance $Var(\sum_{j=1}^m y_j X_j) = \sum_{s,t=1}^m y_s y_t \sigma_{st}$.

The probabilistic portfolio selection problem is:

$$\begin{cases} \min \frac{1}{2} \sum_{s,t=1}^m y_s y_t \sigma_{st} \\ \sum_{j=1}^m y_j = p; \sum_{j=1}^m y_j = 1 \\ y_j \geq 0, j = 1, \ldots, m. \end{cases} \quad (13)$$

4.2 Possibilistic Portfolio Selection Model

One considers $n$ assets $i = 1, \ldots, n$. Assume that the returns of the $n$ assets are represented by the fuzzy numbers $A_1, \ldots, A_n$. The following elements are known:

- possibilistic mean returns $\gamma_i = E(f,A_i)^1, i = 1, \ldots, n$;
- possibilistic covariances $\delta_{kl} = Cov(f,A_k,A_l), k, l = 1, \ldots, n$.

In this case a portfolio is a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. The return of the portfolio $(x_1, \ldots, x_n)$ is the fuzzy number $\sum_{i=1}^n x_i A_i$. One associates with each portfolio:

- possibilistic mean return $E(f, \sum_{i=1}^n x_i A_i) = \sum_{i=1}^n x_i \gamma_i$.

\(^1\)According to the convention from Section 2, we assume that the weighting function $f$ is apriori fixed.
The possibilistic portfolio selection problem is:

\[
\begin{align*}
\min & \quad \frac{1}{2} \sum_{k,l=1}^{n} x_k x_l \delta_{kl} \\
& \sum_{i=1}^{n} x_i \gamma_i = \delta; \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0, \quad i = 1, \ldots, n.
\end{align*}
\]

(14)

5 Triple Portfolios

In this section we introduce the triple portfolio and its indicators. These notions correspond to a financial situation in which some assets are modeled by fuzzy numbers, others by random variables and others by discrete fuzzy variables.

One considers \(n + m + w\) assets. We make the following assumptions:

- The returns of the first \(n\) assets are fuzzy numbers \(A_1, \ldots, A_n\).
- The returns of the other \(m\) assets are random variables \(X_1, \ldots, X_m\).
- The returns of the \(w\) assets are discrete fuzzy variables \(\xi_1, \ldots, \xi_w\): \(\xi_u = \begin{bmatrix} a_{u1} & a_{u2} & \cdots & a_{uw} \\ \nu_{u1} & \nu_{u2} & \cdots & \nu_{uw} \end{bmatrix}\), \(u = 1, \ldots, w\).

By Section 3 the discrete fuzzy variables \(\xi_1, \ldots, \xi_w\) are associated with discrete random variables \(Z_1 = Z_{\xi_1}, \ldots, Z_w = Z_{\xi_w}\): \(Z_u = \begin{bmatrix} a_{u1} & a_{u2} & \cdots & a_{uw} \\ p_{u1} & p_{u2} & \cdots & p_{uw} \end{bmatrix}\), \(u = 1, \ldots, w\) in which probabilities \(p_{u1}, \ldots, p_{uw}\) are computed by formula (11) of section 3.

We know the following elements:

- Possibilistic mean returns \(\gamma_i = E(f, A_i), \quad i = 1, \ldots, n\);
- Possibilistic covariances \(\delta_{kl} = Cov(f, A_k, A_l), \quad k, l = 1, \ldots, n\);
- Probabilistic mean returns \(\mu_j = M(X_j), \quad j = 1, \ldots, m\);
- Probabilistic covariances \(\sigma_{st} = Cov(X_s, X_t), \quad s, t = 1, \ldots, m\);
- Credibilistic mean returns \(\theta_u = Q(\xi_u) = M(Z_u), \quad u = 1, \ldots, w\);
- Credibilistic covariances \(\theta_{uv} = Cov(Z_u, Z_v), \quad u, v = 1, \ldots, w\).

A triple portfolio has the form

\[(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_w) \in \mathbb{R}^{n+m+w},\]

where \(\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{u=1}^{w} z_u = 1\) and \(x_i \geq 0, \quad y_j \geq 0, \quad z_u \geq 0\) for any \(i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad u = 1, \ldots, w\). The real numbers \(x_1, \ldots, x_n\) represent the investment proportions of the first \(n\) assets, \(y_1, \ldots, y_m\) represent the investment proportions of the other \(m\) assets and \(z_1, \ldots, z_w\) represent the investment proportions of the \(w\) assets.

We consider a triple portfolio \((x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_w)\).

Now we define the following indicators of portfolio \((x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_w)\):

- Possibilistic mean return \(\gamma_p = \sum_{i=1}^{n} x_i E(f, A_i)\);
- Probabilistic mean return \(\mu_p = \sum_{j=1}^{m} y_j M(X_j)\);
- Credibilistic mean return \(\theta_p = \sum_{u=1}^{w} z_u Q(\xi_u) = \sum_{u=1}^{w} z_u M(Z_u)\);
- Portfolio’s (total) mean return \(R_p = \gamma_p + \mu_p + \theta_p = \sum_{i=1}^{n} x_i E(f, A_i) + \sum_{j=1}^{m} y_j M(X_j) + \sum_{u=1}^{w} z_u Q(\xi_u)\).

We further define:

- Portfolio’s possibilistic variance \(\delta_p = \sum_{k,l=1}^{n} x_k x_l \delta_{kl}\);
- Portfolio’s probabilistic variance \(\sigma_p = \sum_{s,t=1}^{m} y_s y_t \sigma_{st}\);
- Portfolio’s credibilistic variance \(\theta_p = \sum_{u,v=1}^{w} z_u z_v \theta_{uv}\);
- Portfolio’s (total) variance \(Var_p = \delta_p + \sigma_p + \theta_p = \sum_{k,l=1}^{n} x_k x_l \delta_{kl} + \sum_{s,t=1}^{m} y_s y_t \sigma_{st} + \sum_{u,v=1}^{w} z_u z_v \theta_{uv}\).

\(Var_p\) is a risk indicator associated with the portfolio \((x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_w)\). It comprises the possibilistic risk component \(\delta_p\), the probabilistic risk component \(\sigma_p\) and the credibilistic risk component \(\theta_p\) of the portfolio.
6 Triple Portfolio Selection Problem

In this section we will establish the triple portfolio selection problem form, we will compute its optimal solution and the value of the associated minimum risk. The approach that we give to this problem subscribes to the ideas of [1], [14], [15], etc. We keep the notations from the previous section.

The triple portfolio problem has the form:

\[
\begin{aligned}
\min \frac{1}{2} & \left[ \sum_{k,l=1}^{n} x_k x_l \delta_{kl} + \sum_{s,t=1}^{m} y_s y_t \sigma_{st} \\
& + \sum_{u,v=1}^{w} z_u z_v \theta_{uv} \right] \\
\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{u=1}^{w} z_u &= 1 \\
x_i, y_j, z_u &\geq 0 \\
i, j, u &= 1, \ldots, n, m, w = 1, \ldots, w.
\end{aligned}
\]  

(15)

To solve the triple portfolio selection problem means to find a portfolio \((x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_w)\) of minimum risk which ensures a possibilistic mean return \(\rho_1\), a probabilistic mean return \(\rho_2\) and a credibilistic mean return \(\rho_3\).

We denote:

\[
x = (x_1, \ldots, x_n)^T; \quad y = (y_1, \ldots, y_m)^T; \quad z = (z_1, \ldots, z_w)^T;
\]

\[
\gamma = (\gamma_1, \ldots, \gamma_n)^T; \quad \mu = (\mu_1, \ldots, \mu_m)^T;
\]

\[
\beta = (\beta_1, \ldots, \beta_w)^T;
\]

\[
e_n = \begin{pmatrix} 1, \ldots, 1 \end{pmatrix}^T; \quad e_m = \begin{pmatrix} 1, \ldots, 1 \end{pmatrix}^T; \quad e_w = \begin{pmatrix} 1, \ldots, 1 \end{pmatrix}^T;
\]

\[
\Omega_1 = (\delta_{kl})_{k,l=1,\ldots,n}; \quad \Omega_2 = (\sigma_{st})_{s,t=1,\ldots,m}; \quad \Omega_3 = (\theta_{uv})_{u,v=1,\ldots,w}.
\]

With these notations problem (15) is written in matrix form:

\[
\begin{aligned}
\min \frac{1}{2} & \left[ x^T \Omega_1 x + y^T \Omega_2 y + z^T \Omega_3 z \right] \\
x^T \gamma &= \rho_1, y^T \mu = \rho_2, z^T \beta = \rho_3 \\
x^T e_n + y^T e_m + z^T e_w &= 1 \\
x \geq 0, y \geq 0, z \geq 0.
\end{aligned}
\]

(16)

If \(n = w = 0\) then Markowitz model is obtained from (16); if \(m = w = 0\) then (16) is exactly the possibilistic portfolio selection model presented in Section 4.2. It follows that the triple model (16) extends both Markowitz probabilistic model and the possibilistic model mentioned. If \(w = 0\) then we obtain the mixed model from [8], and if \(n = m = 0\) we obtain the credibilistic model from [9].

Next we aim to find the optimal solution of (16). The Lagrangian of the problem (16) is

\[
L = \frac{1}{2} \sum_{k,l=1}^{n} x_k x_l \delta_{kl} + \frac{1}{2} \sum_{s,t=1}^{m} y_s y_t \sigma_{st} + \frac{1}{2} \sum_{u,v=1}^{w} z_u z_v \theta_{uv} - \lambda_1 (\sum_{k=1}^{n} x_k \gamma_k - \rho_1) - \lambda_2 (\sum_{s=1}^{m} y_s \mu_s - \rho_2) - \lambda_3 (\sum_{u=1}^{w} z_u \beta_u - \rho_3).
\]

The first–order conditions are:

\[
\partial L \partial \lambda_1 = 0, \ldots, \partial L \partial \lambda_n = 0; \quad \partial L \partial \lambda_m = 0, \ldots, \partial L \partial \lambda_m = 0; \quad \partial L \partial \lambda_w = 0, \ldots, \partial L \partial \lambda_w = 0.
\]

From the first \(n + m + w\) conditions from above the following equations are obtained:

\[
\begin{aligned}
\sum_{i=1}^{n} \delta_{ki} x_i - \lambda_1 \gamma_k - \lambda_4 &= 0, k = 1, \ldots, n; \\
\sum_{s=1}^{m} \sigma_{st} y_t - \lambda_2 \mu_s - \lambda_4 &= 0, s = 1, \ldots, m; \\
\sum_{u=1}^{w} \theta_{uv} z_u - \lambda_3 \beta_u - \lambda_4 &= 0, u = 1, \ldots, w.
\end{aligned}
\]

(17) (18) (19)

Conditions (17), (18), and (19) can be written:

\[
\begin{aligned}
\Omega_1 x - \lambda_1 \gamma - \lambda_4 e_n &= 0; \\
\Omega_2 y - \lambda_2 \mu - \lambda_4 e_m &= 0; \\
\Omega_3 z - \lambda_3 \beta - \lambda_4 e_w &= 0.
\end{aligned}
\]

(20) (21) (22)

Assume that the matrices \(\Omega_1, \Omega_2, \Omega_3\) are invertible. Then we obtain from (20), (21), and (22):

\[
\begin{aligned}
x &= \lambda_1 \Omega_1^{-1} \gamma + \lambda_4 \Omega_1^{-1} e_n; \\
y &= \lambda_2 \Omega_2^{-1} \mu + \lambda_4 \Omega_2^{-1} e_m; \\
z &= \lambda_3 \Omega_3^{-1} \beta + \lambda_4 \Omega_3^{-1} e_w.
\end{aligned}
\]

(23) (24) (25)

From (23), (24), and (25) one obtains:

\[
\begin{aligned}
x^T &= \lambda_1 \gamma^T \Omega_1^{-1} + \lambda_4 e_n^T \Omega_1^{-1}; \\
y^T &= \lambda_2 \mu^T \Omega_2^{-1} + \lambda_4 e_m^T \Omega_2^{-1}; \\
z^T &= \lambda_3 \beta^T \Omega_3^{-1} + \lambda_4 e_w^T \Omega_3^{-1}.
\end{aligned}
\]

(26) (27) (28)

By (16) we have \(x^T \gamma = \rho_1, y^T \mu = \rho_2, z^T \beta = \rho_3\) and \(x^T e_n + y^T e_m + z^T e_w = 1\). In these identities
replacing $x^T, y^T, z^T$ with their values from (26), (27), and (28) one obtains:

$$
\begin{align*}
\lambda_1&\gamma^T\Omega^{-1}_1\gamma + \lambda_2\mu^T\Omega^{-1}_2\mu = \rho_1 \\
\lambda_3&\beta^T\Omega^{-1}_2\beta + \lambda_4\varepsilon^T\Omega^{-1}_3\varepsilon = \rho_2 \\
\lambda_1&\gamma^T\Omega^{-1}_1\gamma + \lambda_2\mu^T\Omega^{-1}_2\mu + \lambda_3\beta^T\Omega^{-1}_2\beta + \lambda_4\varepsilon^T\Omega^{-1}_3\varepsilon = \rho_3
\end{align*}
$$

We denote $A = \gamma^T\Omega^{-1}_1\gamma, B = \mu^T\Omega^{-1}_2\mu, C = \beta^T\Omega^{-1}_2\beta, D = \varepsilon^T\Omega^{-1}_3\varepsilon$. By (16) we have

$$
\begin{align*}
\lambda_1&\gamma^T\Omega^{-1}_1\gamma + \lambda_2\mu^T\Omega^{-1}_2\mu + \lambda_3\beta^T\Omega^{-1}_2\beta + \lambda_4\varepsilon^T\Omega^{-1}_3\varepsilon = \rho_3
\end{align*}
$$

Replacing in (29) we obtain the system of equations in $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$
\begin{align*}
&A\lambda_1 + D\lambda_3 = \rho_1 \\
&B\lambda_2 + E\lambda_4 = \rho_2 \\
&C\lambda_3 + F\lambda_4 = \rho_3 \\
&D\lambda_1 + E\lambda_2 + F\lambda_3 + G\lambda_4 = 1
\end{align*}
$$

When the system of equations (30) is compatible one can determine the values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ which replaced in (26)-(28) give the values of $x, y, z$. If $x \geq 0, y \geq 0, z \geq 0$ then $(x, y, z)$ is an optimal portfolio.

We intend to compute the variance $\text{Var}_P$ of the triple portfolio $(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_w)$ resulted as a solution of system (30).

One notices that $\beta = x^T\Omega_1x, \sigma_P = y^T\Omega_2y$ and $\theta_P = z^T\Omega_3z$. By (23)-(25), we have

$$
\begin{align*}
x^T\Omega_1x &= \lambda_1x^T\gamma + \lambda_4x^Te_m; \\
y^T\Omega_2y &= \lambda_2y^T\mu + \lambda_3y^Te_m; \\
z^T\Omega_3z &= \lambda_3z^T\beta + \lambda_4z^Te_w.
\end{align*}
$$

Therefore $\text{Var}_P = \beta_P + \sigma_P + \theta_P = x^T\Omega_1x + y^T\Omega_2y + z^T\Omega_3z = \lambda_1x^T\gamma + \lambda_2y^T\mu + \lambda_3z^T\beta.$ By (16) we have $x^T\gamma = \rho_1, y^T\mu = \rho_2, z^T\beta = \rho_3$. Therefore, $\text{Var}_P$, the value of the minimum risk of a portfolio which assures a probabilistic mean return $\rho_1$, a probabilistic mean return $\rho_2$ and a credibility mean return $\rho_3$ is

$$
\text{Var}_P = \lambda_1\rho_1 + \lambda_2\rho_2 + \lambda_3\rho_3 + \lambda_4.
$$

7 Conclusion

The three component portfolio selection problem treated in this paper corresponds to a complex financial risk management situation, where the expected return rates are represented by three types of risk indicators: probabilistic fuzzy numbers, probabilistic random variables and credibilistic discrete fuzzy variables. The risk from our portfolio model is evaluated by summing the three types of risk. We notice that the three components of the model are considered independent. Development of a general model with interdependences remains an open problem.

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References: