An example of symbolic computation of Lyapunov quantities in Maple

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Abstract: In the present paper a realization of a classical method for Lyapunov quantities computation in Maple is considered.

Key–Words: Lyapunov quantity, focus values, symbolic computation, small-amplitude limit cycles, Maple, Hilbert 16th problem

1 Introduction


The development of methods for computation and analysis of Lyapunov quantities was encouraged by as applied engineering problems (such as the study of oscillations excitation and boundaries of domain of stability [3, 4]), as purely mathematical problems (such as center–focus problem, the cyclicity problem, analysis of dynamical systems stability, and famous Hilbert 16th problem [5–8]). Bautin’s perturbation method for Lyapunov quantities [9] turns out to be the only effective method for investigation of small-amplitude nested limit cycles [10–14](a simplest example of hidden oscillations [15–22]), which can not be obtained numerically.

While symbolic expressions for the first and second Lyapunov quantities were obtained in the 40-50-s of the last century by Bautin [9] and Serebryakova [23] respectively, computation of the expressions for consequent Lyapunov quantities in general form became possible only after the appearance of modern computers and computational tools (see, e.g., [24, 25]).

At present there exist a few methods for computation of Lyapunov quantities and their computer realizations, which permit one to find Lyapunov quantities in the form of symbolic expressions depending on the coefficients of the system. These methods are differed by the complexity of realization of algorithms, a space, in which the computation is made, and the compactness of obtained symbolic expressions (see, e.g., [26–33]).

In this paper classical Lyapunov method and its realization in Maple are considered.

2 Lyapunov quantities

Following [31], introduce Lyapunov quantities. Consider sufficiently smooth two-dimensional system

\[
\begin{align*}
\dot{x} &= f_{10} x + f_{01} y + f(x, y), \\
\dot{y} &= g_{10} x + g_{01} y + g(x, y),
\end{align*}
\]  

where in an open neighborhood $U$ of the point $(x, y) = (0, 0)$ with radius $R_U$, the right-hand side of system has continuous partial derivatives of the $n$-th order:

\[
f(\cdot, \cdot), g(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \in C^n(U)
\]  

and the following representation

\[
\begin{align*}
f(x, y) &= \sum_{k+j=2}^{n} f_{kj} x^k y^j + o((|x| + |y|)^n) \\
&= f_n(x, y) + o((|x| + |y|)^n), \\
g(x, y) &= \sum_{k+j=2}^{n} g_{kj} x^k y^j + o((|x| + |y|)^n) \\
&= g_n(x, y) + o((|x| + |y|)^n)
\end{align*}
\]  

is satisfied.
Let the matrix of the first approximation of system has two purely imaginary eigenvalues. In this case, without loss of generality, it can be assumed that \( f_{10} = 0, \ f_{01} = -1, \ g_{10} = 1, \ g_{01} = 0 \). Therefore we can consider the following system

\[
\begin{align*}
\dot{x} &= -y + f(x, y), \\
\dot{y} &= x + g(x, y).
\end{align*}
\]

(4)

For the study of influence of nonlinear terms \( f(x, y) \) and \( g(x, y) \) on the behavior of trajectories of system (4) in small neighborhood of stationary point one can consider, following the method of Poincare, the intersection of trajectory of system (4) with the straight line \( x = 0 \). At time \( t = 0 \) the trajectory \((x(t, h), y(t, h))\) is started from the point \((0, h)\) (\( h \) is sufficiently small)

\[
(x(0, h), y(0, h)) = (0, h).
\]

(5)

Denote by \( T(h) \) return time of trajectory, which is a time between two successive intersections of trajectory with the straight line \( x = 0 \). Then

\[
x(T(h), h) = 0
\]

(6)

and \( y(T(h), h) \) can sequentially be approximated by a series in terms of powers of \( h \):

\[
y(T(h), h) = h + \tilde{L}_2 h^2 + \tilde{L}_3 h^3 + ...
\]

(7)

Here the first nonzero coefficient \( \tilde{L}_m \) is called Lyapunov quantity. It determines stability or instability of stationary point and describes winding/unwinding of trajectory. One can show (see, e.g., [2]) that the first nonzero coefficient have necessarily odd number \( m = (2k + 1) \). The value \( \tilde{L}_{2k+1} \) is called \( k \)-th Lyapunov quantity

\[
\tilde{L}_k = \tilde{L}_{2k+1}
\]

and the equilibrium is a weak focus of \( k \)-th order.

In the case of real part of eigenvalues of the first approximation matrix the Lyapunov quantity is considered similarly. In this case it is introduced the notion of zero Lyapunov quantity \( \tilde{L}_0 = \tilde{L}_1 \) such that

\[
y(T(h), h) = (1 + \tilde{L}_1)h + o(h).
\]

Note that \( \tilde{L}_1 \) describes exponential increase of solutions of system, being caused by real parts of eigenvalues (similarly to Lyapunov exponents or characteristic exponents, see [34]).

3 Classical method for Lyapunov quantities calculation

By transformation to polar coordinates system (4) takes the form

\[
\begin{align*}
\dot{r} &= f(r \cos \theta, r \sin \theta) \cos \theta + g(r \cos \theta, r \sin \theta) \sin \theta \\
\dot{\theta} &= 1 - f(r \cos \theta, r \sin \theta) \sin \theta + g(r \cos \theta, r \sin \theta) \cos \theta.
\end{align*}
\]

We note that the functions \( f(r \cos \theta, r \sin \theta) \) and \( g(r \cos \theta, r \sin \theta) \) contain only terms whose order in \( r \) is not less than two. Thus, we can divide the first equation by \( r \) for \( r \neq 0 \) and then extend the result by continuity to \( r = 0 \). For sufficiently small \( r \) we have \( \dot{\theta} \neq 0 \) and \( \theta \) increases (rotation is counter-clockwise) as time increases (\( \dot{\theta} > 0 \)). So, one can consider

\[
\frac{dr}{d\theta} \left( 1 - \frac{f(r \cos \theta, r \sin \theta) \sin \theta}{r} + \frac{g(r \cos \theta, r \sin \theta) \cos \theta}{r} \right) = R(r, \theta).
\]

By (2) the function \( R(r, \theta) \) is a sufficiently smooth periodic function of \( \theta \) with period \( 2\pi \) for sufficiently small \( r \) and \( R(r, \theta) = o(r) \). Therefore, function \( R(r, \theta) \) can be expressed as a finite sum of degrees of \( r \) plus a reminder.

Then it follows from (2) that

\[
R(r, \theta) = \sum_{k=1}^{n} R_{rk} (\theta) r^k + o(r^n),
\]

(8)

\[
R_{rk} (\theta) = \frac{1}{k!} \frac{\partial^k R(\eta, \theta)}{\partial \eta^k} |_{\eta=0}.
\]

Now \( R(r, \theta) \) can be represented in the following form

\[
\frac{dr}{d\theta} = R(r, \theta) = r R_1 (\theta) + ... + r^n R_n (\theta) + o(r^n),
\]

(9)

where \( R_i \) is a periodic function of \( \theta \) of period \( 2\pi \).

Taking into account the form of right-hand side of (8), we will consider solution \( r(\phi, h) \) \((r(0, h) = h)\) in the following form

\[
\begin{align*}
r(\phi, h) &= h + \tilde{r}_{h^2}(\phi) h^2 + \tilde{r}_{h^3}(\phi) h^3 + ... \\
&+ \tilde{r}_{h^n}(\phi) h^n + o(h^n).
\end{align*}
\]

(10)

Inserting (10) into (9) and analyzing expressions at the same powers of \( h \), we obtain equations for determining the \( \tilde{r}_{h^k}(\phi) \) one by one

\[
\begin{align*}
\frac{d\tilde{r}_{h^2}(\phi)}{d\phi} &= \tilde{R}_{h^2}(\phi), \\
\frac{d\tilde{r}_{h^3}(\phi)}{d\phi} &= \tilde{R}_{h^3}(\phi) + 2 \tilde{R}_{h^2}(\phi) \tilde{R}_{h^2}(\phi), \\
&\ldots \\
\frac{d\tilde{r}_{h^n}(\phi)}{d\phi} &= \tilde{R}_{h^n}(\phi) + 2 \tilde{R}_{h^2}(\phi) \tilde{R}_{h^{n-1}}(\phi) + ... 
\end{align*}
\]

(11)
Considering \( r(\phi, h) \) at \( \phi = 2\pi \)

\[
r(2\pi, h) = h + \bar{r}_2 h^2 + \bar{r}_3 h^3 + \ldots, \quad \bar{r}_k = \bar{r}_k(2\pi),
\]

where \( \bar{r}_k \) are called focus values. If \( \bar{r}_2 = \ldots = \bar{r}_{2m} = 0 \) then \( \bar{r}_{2m+1} \) equals to \( m-th \) Lyapunov quantity.

Further an example of realization of this method in Maple is considered.

## 4 Realization in Maple

```maple
restart;
m := 2; # number of Lyapunov quantities
n := 2*m+1; # system degree
# def of functions f and g
f(x,y) := 0;
g(x,y) := 0;
for k from 2 to n do
  i := k - 1;
  f(x,y) := f(x,y) + (1+i)*x^i*y^i;
  g(x,y) := g(x,y) + (1+i)*x^i*y^i;
end;
# transformation to the polar coordinates
f(r,theta) := subs(x=r*cos(theta), f(x,y));
g(r,theta) := subs(x=r*cos(theta), g(x,y));
R(theta,r) := R(theta,r) + R[i](theta) * r^i;
for i from 1 to m do
  R[i](theta) := subs(r=0, diff(dr/dt, [r$i])/i!);
end:
R(theta,r) := 0:
f(theta,r[0]) := 0:
R(theta,r) := R(theta,r) + R[i](theta) * r^i;
E:=diff(f(theta,r[0]),theta)-subs(r=f(theta,r[0]), diff(dr/dt, [r[i]]))/i!;
r[i](theta) := simplify(S- subs(theta=0,S)+ic[i]);
for i from 1 to m do
  l[i] := simplify(subs(theta=2*Pi,
    u[2*Pi+i](theta));
g[i+2,1] := solve(l[i], g[i+2,1]);
end;
```

### Acknowledgements:
This work was partly supported by Grants programm of the President of RF, Ministry of Education and Science of RF, RFBR and Saint-Petersburg State University.

### References:


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