Delay Independence Stabilization of Uncertain Systems and ASC

TAKSHI AMEMIYA
Setsunan University
Dept. of Business Administration and Information
17-8, Ikedanakamachi, Neyagawa, Osaka 572-8508
JAPAN
amemiya@kjo.setsunan.ac.jp

TOMOAKI HASHIMOTO
Osaka University
Department of Systems Innovation
Toyonaka, Osaka, 560-8531
JAPAN
info@thashi.net

Abstract: On the quadratic stabilization of uncertain linear time varying systems by means of linear state variable feedback, Wei introduced the concept of antisymmetric stepwise configuration (ASC) and proved that having this configuration is a necessary and sufficient condition for uncertain linear systems to be quadratically stabilizable by means of a linear state variable feedback. However, because his condition is constructed on the basis of quadratic Lyapunov functions, his method is not applicable if the system contains delays in the state variables. In this report the conditions for the delay independent stabilization so far obtained on the basis of delay differential inequalities is further developed and it is proved that generally to have ASC is also a sufficient condition for the delay independent stabilizability of linear uncertain delayed systems by means of linear state variable feedback.

Key–Words: Stabilization, Delayindependent Stability, Uncertain Systems, ASC

1 Introduction

Delay independent stabilization of the delayed systems provides a fairly simple and useful method to obtain a stabilizing control for uncertain delayed systems. Delay independent stabilization is of course a kind of robust stabilization for systems with delays. However it is so named because it provides a condition, whose results contains no terms of delays. So it means that such condition is applied to delayed systems however large the contained delays may be, so long as they are bounded. The condition of boundedness is necessary only because we cannot consider the infinite delays actually. The other conditions which explicitly depends on delayed terms are called delay dependent stabilization. The authors presented several results [1,2,3] for delay independent stabilization of delayed systems. In the first report all state variables were assumed to be known. This condition was developed to the case that state variables are partly observed in [2]. In [3] necessary and sufficient conditions for the stabilizability of the problem were investigated.

On the other hand, on the quadratic stabilizability of uncertain time varying linear systems, Wei has proved that to have certain special form called antisymmetric stepwise configuration (ASC) is necessary and sufficient for the quadratic stabilizability via state variable feedback for systems containing uncertainties in system parameters.

So far we noticed [3] that the condition we have proved for the delayed system is equivalent to the most basic form of this ASC. However, it does not satisfy the general form of ASC. Here it is shown that our conditions for the delay independent stabilizability can be improved to the one by Wei, showing that ASC constitutes a sufficient condition for the delay independent stabilization of uncertain linear delayed systems. Recently, on the delayed system analysis, delay dependent stability analysis have been presented often. However, these conditions are usually quite complicated and difficult to see whether it is applicable actually. Contrarily, the delay independent approach is very simple and provide fairly simple and good results. Comparison with the results by LMI method was shown in [3].

The paper is organized as follows. First, in the next section, some notations and terminologies are presented. In Section 3, the description of the considered system with some basic assumptions are given. Then, the conditions for stabilization of uncertain systems so far obtained are presented. In Section 5, Wei’s ASC and his results are introduced. In Section 6 the basic mathematical background for our results is presented. In the succeeding section our main theorem is derived on the basis of these theorems. Since the proof of this theorem is very complicated and there are various cases to be considered, we have no space here to describe. Appendices. Section 7 is devoted to an illustrative example, which is given for the help of understanding. Finally, conclusions are given in Section 8.
2 Notations

For $A = (a_{ij}), B = (b_{ij}) \in R^{n \times m}$, every inequality such as $A \succ B$ indicates that it is satisfied component-wise as $a_{ij} > b_{ij}$. For $A = (a_{ij}) \in R^{n \times m}$, matrices $B = (b_{ij}) \in R^{m \times n}$ and $C = (c_{ij}) \in R^{m \times m}$ defined as $b_{ij} = |a_{ij}|$ or $c_{ij} = |a_{ij}|$, $(j \neq i)$, $c_{ij} = a_{ij}$, $(j = i)$, are called the absolute companion matrix of $A$ or quasiabsolute companion matrix of $A$, respectively. Here, notations $|A|$ or $|A|_q$ are used to denote absolute companion matrices or quasiabsolute companion matrix of $A$, respectively. For $A \in R^{n \times m}$, the inequality $A \geq 0$ indicates $A$ is a nonnegative matrix.

A real nonsingular matrix $D = (d_{ij}) \in R^{n \times n}$ is called an $M$-matrix, if it satisfies all the following conditions:
(i) all off-diagonal elements satisfy $d_{ij} \leq 0$, $(i \neq j)$, (ii) the inverse of $D$ satisfies $D^{-1} \geq 0$.
The set of all $M$-matrices is denoted as $\mathcal{M}$.

Let $[a, b]$ be an interval in $R$. The sets of all $R^n$ continuous or piecewise continuous functions with domain $[a, b]$ are denoted by $C^m[a, b]$ or $D^m[a, b]$, respectively.

3 System Description

The system considered in this paper is given as follows,

$$
\dot{x}(t) = A^0 x(t) + \Delta A^1(t) x(t) + \sum_{i=1}^{m} \Delta A^{2i}(t) \dot{x}(t - \tau_i(t)) + (b + \Delta b) u(t)
$$

(1)

where $x \in R^n$ and $t \in [t_0, \infty)$. The solution of (1) with initial curve $\phi \in D^n[t_0 - \tau_0, t_0]$ is denoted as $x(t, \phi)$. $u \in R$ is a control variable and $A \in R^{n \times n}$, $b \in R^n$ are constant. $\tau_i : R \rightarrow R$ is a piecewise continuous function and is assumed to be bounded, i.e. for a constant $\tau_0 \in R$ it satisfies,

$$
0 \leq \tau_i(t) \leq \tau_0, \quad i = 1, \ldots, n, \quad \text{for } t \in R.
$$

(2)

The upper bound $\tau_0$ in (2) is not necessarily assumed to be known and may be arbitrarily large. It does not affect the stability condition. That why this condition is called delay independent. The concept of ‘delay independent’ already found in the literature.

$\Delta A^1, \Delta A^{2i} \in R^{n \times n}, i = 1, \ldots, m, \Delta b$ denote uncertain parts of system parameters. All elements of them are piecewise continuous functions of $t$. These matrices satisfy for constant $n \times n$ matrices $\Delta A^{10}, \Delta A^{2i0}$, and $\Delta b^0 \in R^n$ and for $t \geq t_0$,

$$
|\Delta A^1| \leq \Delta A^{10}, \quad |\Delta A^{2i}| \leq \Delta A^{2i0}.
$$

(3.1)

$$
|\Delta b| \leq \Delta b^0
$$

(3.2)

On $\Delta b^0$ it is assumed that if some element $\Delta b_i = 0$ for $t \geq 0$ then $\Delta b^0 = 0$.

On the system parameters $A + \Delta A^*$ and input coefficients $b + \Delta b \in R^n$, the following assumption is introduced.

**Assumption 1** If delays are all zero and all uncertain parameters are constant, the system is controllable, whatever values these uncertain parameters may take satisfying the restrictions.

**Definition 1** The system (2) is called robustly stabi-

lizable if it can be made asymptotically stable inde-

pendently of uncertain coefficients satisfying the re-

strictions, by constructing certain control $u$. Spe-

cally, if $u$ can be constructed as linear memoryless

state variable feedback such that

$$
u = c^T x
$$

(4)

by choosing proper constant coefficients $c \in R^n$, the

system is called stabilizable via linear state variable

feedback. It is also assumed that all state variables

are directly accessible. Here the system is called glo-

ally asymptotically stable if every solution of it con-

verges asymptotically to $x = 0$ whatever initial curve

$\phi \in D^n[t_0 - \tau_0, t_0]$ it may start from.

**Definition 2** (Delay independent stabilizability, DIS)

If the system (2) is robustly stabilizable by the condi-

tion, which does not depend explicitly on the delayed

term, the system is called delay independently stabil-

izable.

The following is the problem considered in this paper.

**Problem** What conditions must system parameters satisfy for the system to be DIS, via linear state variable feedback (4), however large the upperbound of uncertainties may be, provided they are known.

As the most basic system, the following assumption is introduced.

**Assumption 2** The pair $(A^0, b)$ of the nominal system is a controllable pair and is in the controllable canonical form and $A^0$ is given as

$$
A^0 = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots \\
0 & \cdots & 1 & 0
\end{pmatrix}, \quad b = \begin{pmatrix} \\
0 \\
\vdots \\
1
\end{pmatrix}.
$$

4 Uncertain Coefficients and ASC(Antisymmetric Stepwise Configuration)

Here the configuration of the uncertain coefficients to be considered is defined. For this purpose a set of ma-
trices with regards to the uncertain parameters $\Delta A^1$, $\Delta A^{2i}$ is introduced.

**Definition 3** For an integer $k$ satisfying $0 \leq k \leq n$, let $\Omega(k) = \{G = (g_{ij}) \in R^{n \times n}\}$ be a set of all matrices with the following properties:

(i) If $1 \leq i \leq n - k$ then $g_{ij} = 0$, for $j \leq i + 1$ and $j \geq 2n - 2k - i + 1$.

(ii) If $n - k + 1 \leq i \leq n$ then $g_{ij} = 0$, for $j \leq n - 2k - i + 1$ and $j \geq i + 1$.

For the delay independent stabilization of this uncertain system, the following theorem has been obtained.

**Theorem 4** In case $\Delta A^1 \in \Omega(k)$, $\Delta A^{2i} \in \Omega(k)$, $i = 1, \ldots, m$ for certain common $k$, the system system (2) is DIS by a constant linear state variable feedback (4).

Here the condition by Wei[4] and the ASC are presented for comparison sake.

Consider the system with no delays.

$$\dot{x}(t) = \bar{A}^0 x(t) + \Delta A^1(t) x(t) + \bar{b}u(t). \hspace{1cm} (5)$$

where $\bar{A}^0 \in R^{n \times n}$ and $\bar{b} \in R^n$ are defined as

$$\bar{A}^0 = \begin{pmatrix} 0 & \theta & 0 & \cdots \\ \vdots & \ddots & \vdots & \cdots \\ 0 & \cdots & 0 & \theta \\ \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here $\theta$s are all sign fixed uncertainties.

**Definition 5** An uncertain system is called quadratically stable if there exists a positive definite lyapunov function of quadratic form $V = x^T P x$ such that the time derivative of this lyapunov function is negative definite or semidefinite along the solution of the equation however the uncertainties may change.

The following assumption is introduced.

**Assumption 3** All $(i, i + 1)$ elements of $\Delta A^1$ are 0.

We here call such form of system ($\bar{A}^0 + \Delta A^1, b + \Delta b$) as standard form.

**Definition 6** A system is called quadratically stabilizable via linear state variable feedback if there exists a linear feedback (4) such that the resulting system is quadratically stable.

**Definition 7** An $(n+1) \times (n+1)$ real matrix is called to have an antisymmetric stepwise configuration if it has the following properties:

(i) It is in the standard form.

(ii) Let $m_{kp}$ and $m_{uv}$ be $(k, p)$ and $(u, v)$ element of matrix $M$ respectively. In this case, if $p \geq k + 2$ and $m_{kp} \neq 0$ for all $q$, then $m_{uv} = 0$ for all $u \geq v$, $u \leq p - 1$ and $v \leq k + 1$ and for any $q$.

$$\dot{x}(t) = C x(t) + C^1 x(t) + \sum_{i=1}^{m} C^{2i} x(t - \tau_i(t)) \hspace{1cm} (6)$$

defined for $t \geq t_0$, with initial curve $\phi \in D^{n}[t_0 - \tau_0, t_0]$. Here $C, C^1, C^{2i}$ are all $n \times n$ real matrices. $C^1, C^{2i}$ may be time varying. All elements of them are assumed to be piecewise continuous and
bounded, i.e. for given constant nonnegative matrices $C^{10}, C^{2i} \in R^{n \times n}, i = 1, \ldots, m$ they satisfy $|C^1(t)| \leq C^{10}, |C^{2i}(t)| \leq C^{20}$, for $t > t_0$. Each delay $\tau_i$ is assumed to satisfy (2).

The following theorem has been proven.

**Theorem 10** ([11]) Assume

$$(-|C|_q - C^{10} - \sum_{i=1}^{m} C^{2i0}) \in M. \quad (7)$$

Then, every solution of (10) converges uniformly and exponentially in the large and also independently of delays, to the equilibrium point $x = 0$.

Note that the above condition does not depend on the bound $\tau_0$ of delays, this is why we call it the delay independence. On the evaluation of (7), the following propositions [4] is utilized in the subsequent. Let $A, B \in R^{n \times n}$ be constant matrices, satisfying $A \geq B$.

**Proposition 1** For any $K \in R^{n \times n}$, if $(K-A) \in M$, and if all off-diagonal elements of $(K-B)$ are nonpositive, then $(K-B) \in M$.

Substituting (4) into (2) and then introducing the variable transformation

$$v = T^{-1}x \quad (8)$$

then the equation (2) can be transformed into,

$$\dot{v}(t) = T^{-1}(A^0 + bC)Tv(t) + T^{-1}\Delta A^1Tv(t) + T^{-1}\Delta bC[Tv(t) - \tau_i(t)]. \quad (9)$$

Owing to the controllability assumption, it is possible to choose $c \in R^n$ so that all the eigenvalues of $(A^0 + bC)$ are real, negative and distinct. Let $c$ be defined as such. And let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of $(A^0 + bC)$ respectively. Define $T$ as

$$T = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{pmatrix} \quad (10)$$

Then we obtain

$$T^{-1}(A^0 + bC)T = \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

Define $\hat{C}, \hat{C}^{11}, \hat{C}^{12}, \hat{C}^{2i}$ as

$$\hat{C} = \Lambda$$

$$\hat{C}^{11} = T^{-1}(\Delta A^1)T$$

$$\hat{C}^{12} = T^{-1}(\Delta bC)T$$

$$\hat{C}^{2i} = T^{-1}\Delta A^{2i}T$$

$$\hat{C}^{2i} = T^{-1}\Delta A^{2i}T$$

Now the system (10) can be written as

$$\dot{v}(t) = \hat{C}v(t) + \hat{C}^{11}v(t) + \hat{C}^{12}v(t) + \sum_{i=1}^{n} \hat{C}^{2i}(t-\tau_i(t)). \quad (12)$$

Then we obtain

$$|\hat{C}|_q = |A|_q \quad (13.1)$$

$$|\hat{C}^{11}| \leq |T^{-1}|(\Delta A^{10})|T| \quad (13.2)$$

$$|\hat{C}^{12}| \leq |T^{-1}|(|\Delta bC|)|T| \quad (13.3)$$

$$\sum_{i=1}^{n} |\hat{C}^{2i}| \leq |T^{-1}|\Delta A^{20}|T| \quad (13.4)$$

$\Delta A^{20}$ in (13.4) is given by $\sum_{i=1}^{n} \Delta A^{2i0} = \Delta A^{20}$.

Let $\Delta A^{30}$ be defined as

$$\Delta A^{30} = \Delta A^{10} + \Delta b|C'| + \Delta A^{20}, \quad (14)$$

Define $P^1, P^2$ and $P^0$ as

$$\left\{ \begin{array}{l} P^1 = -|\hat{C}|_q \\ P^2 = -|T^{-1}|\Delta A^{30}|T| \\ P^0 = P^1 + P^2. \end{array} \right. \quad (15)$$

Then, owing to Theorem 10, we obtain,

**Proposition 2** If there exist $T$ which assure

$$P^0 \in M, \quad (17)$$

then, every solution of the system (12) converges to the equilibrium point.

To show the existence of $T$ to assure the equation (17), a notation for a class of functions is again introduced here, following the previous paper[1, 2]. Let $\xi(\sigma) \in C^1$ and let $m \in R$ be a constant. If $\xi$ satisfies the conditions,

$$|\xi(\sigma)|_{\sigma^m} < \infty, \quad |\xi(\sigma)|_{\sigma^{m-a}} \rightarrow \infty \quad \text{as} \ |\sigma| \rightarrow \infty,$$

for any constant $a > 0 \in R$, then $\xi$ is called a function of order $m$ and is written as $\text{Ord}(\xi) = m$. The set of all $C^1$ functions of order $m$ is denoted as $O(m)$. It should be noted that $m$ can be a negative number.

### 6 Main Results

In this section it is shown that our previously obtained condition can be developed to the general form of ASC, on the basis of the mathematical background just given above. It is stated in the following theorem.

Following Wei, we here introduce the extended matrix $Q$, constructed by space coefficients such that,

$$Q = \begin{pmatrix} \Delta A^+ & \Delta b \\ 0 & \Delta b \end{pmatrix}. \quad (18)$$
Theorem 11 (Main Theorem) Assume thus obtained \( Q \) satisfy \( Q \in \Omega^0_{\text{ASC}}(k) \), for fixed \( k \), then the uncertain systems (2) is DIS via linear state variable feedback (4), where \( * \) in (18) indicates 1 or 2.1.

The proof of the above theorem depends crucially on the way to choose eigen values of the nominal system. Here it is shown as rules for choosing them. Before presenting the rules, it must be made sure that all eigen values are assumed to be real, negative and distinct. It is possible by the assumption of the controllability of the nominal system.

Let \( \sigma \) be a negative number. The theorem is proved by assigning the order of each eigen values as a function of \( \sigma \).

Whereas for the proof of Theorem so far proved on the case \( \Delta A^* \in \Omega(k) \), these eigen values were to be of order only 1 or -1, just two kinds of eigen values have been needed, a more sophisticated method must be chosen to improve the previous results.

6.1 Edge Points and Rules for Choosing Eigenvalues

Let \((p_1, q_1), \ldots, (p_i, q_i), p_i > p_{i+1}, q_i < q_{i+1}, p_i + 1 < q_i, q_k \leq n + 1 \) be uncertain elements in ASC, consisting the corner positions of ASC. That means all \((p_j, q_j + s)\) elements are zero for all \( s > 0 \) and all \((p_j + r, q_j)\) elements are zero for all \( q_i - p_i - 1 > r > 0 \). These points are called edge points in this paper. For the proof, eigen values of the nominal systems must be chosen depending on these points. Since the rules for choosing eigen values are slightly different on the shape of ASC, to describe this property, define \( q_0 \) and \( h \) as,

\[
q_0 = (p_1 + 2), \quad h = q_1 - q_0 = q_1 - p_1 - 2.
\]

The way to choose eigen values and therefore the proof should be separated into two cases according to this \( h \). They are cases of \( h > 1 \) and \( h = 1 \).

I. Case \( h > 1 \)

(Step 1) Choose \( q_1 - q_0 \) eigen values of order \( 1 \).

(Step 2) Choose \( p_1 - p_2 \) eigen values of order \(-(q_1 - q_0)\)

(Step 3) Assume in the previous steps \( r_i \) eigen values of order \( s_i \) have been chosen, \( i = 1, \ldots, k - 1 \), then the order of the next eigen values are given as \( s_k = -\sum_i(r_is_i + 1) \) if \( k \) is even, that means \( s_k \) is negative, and \( q_k \neq n + 1 \). If \( k \) is odd \( s_k = -\sum_i(r_is_i) + 1 \).

In the next cases, \( q_k = n + 1 \) and \( k \) is even, \( s_k \) should be \( s_k = -\sum_i(r_is_i) - 1 \). In the process if the calculated value \( s_k \) satisfies \(|s_k| < |s_{k-2}|\), then \( s_k \) must selected as \( s_k = s_{k-2} \).

As for the numbers of these eigen values, they should be chosen as \( r_{2k} = p_{k-1} - p_k \), for negative order ones or \( r_{2k+1} = q_{k+1} - q_k \) for positive order ones. If \( k \) is the last one and \( q_k \leq n \) then \( q_{k+1} \) should be given as \( q_{k+1} = n + 1 \).

(Step 4) Go to Step 3 until there exist no edge points. The last eigen value to be chosen is positive order one unless \( q_k = n + 1 \).

2. Case \( h = 1 \)

(Step 1) Choose \( p_1 - p_2 \) eigen values of order \(-1\).

(Step 2) Choose \( q_2 - q_1 \) eigen values of order \( (q_1 - (p_1)) \)

(Step 3) Assume in the previous steps \( r_i \) eigen values of order \( s_i \) have been chosen, \( i = 1, \ldots, k - 1 \), then the order of the next eigen values are given as \( s_k = -\sum_i(r_is_i + 1) \) for odd \( k \) and \( s_k = -\sum_i(r_is_i) \) for even \( k \). If odd \( s_{2k-1} \) becomes the last eigen value, that is \( q_k = n + 1 \) \( s_{2k-1} \) should be moreover replaced by \( s_{2k-1} - 1 \) If the calculated value \( s_k \) satisfies \(|s_k| < |s_{k-2}|\), then \( s_k \) must selected as \( s_k = s_{k-2} \).

Whereas the numbers of these eigen values should be \( r_{2k} = p_{k-1} - p_k \), for negative order eigen values or \( r_{2k+1} = q_{k+1} - q_k \) for positive order eigen values. If \( k \) is the last one \( q_{k+1} \) should be given as \( q_{k+1} = n + 1 \). However, if \( q_k = n + 1 \), the the eigen value which should be chosen last must have negative order and the number of this eigen value \( r_{2k-1} \) must be increased by one to the ordinary value and \( r_{2k-3} \) must be replaced by \( r_{2k-3} - 1 \).

(Step 4) Go to Step 3 until there exist no edge points. The last eigen value to be chosen is positive order one unless \( q_k = n + 1 \).

7 Examples

7.1 Example 1

To help the understanding, the following illustrative example is considered. Let \( n = 9 \) and the uncertain matrix \( \Delta A^* \) is given as in Figure 2. In this case \((p_i, q_i)\) are given as,

\[
(p_1, q_1) = (2, 7), \quad (p_2, q_2) = (1, 9),
\]

For this system the previous method is not applicable. By using the above given decision rule, eigenvalues are given as follows.

(Step 1) 3 eigen value of order 1

(Step 2) 1 eigen value of order -3

(Step 3-1) 2 eigen values of order 1

(Step 3-2) 2 eigen values of order -3

(Step 3-3) 1 eigen values of order 5

By using the above given rules, the following \( P^0 \) is obtained. Note that every number in the matrix indicates the order of the functions of \( \sigma \) and necessary.
restrictions to be an M-matrix on the signs of all diagonal or off diagonal elements are assured to be satisfied. It is clear that this matrix becomes an M-matrix for sufficient large $\sigma$.

$$P^0 \simeq \begin{pmatrix}
-3 & -4 & -4 & 9 & 9 & 9 & 9 & 9 & 40 \\
-4 & -3 & -4 & 9 & 9 & 9 & 9 & 9 & 40 \\
-4 & -4 & -3 & 9 & 9 & 9 & 9 & 9 & 40 \\
-12 & -12 & -12 & 1 & 0 & 0 & 0 & 0 & 28 \\
-12 & -12 & -12 & 0 & 1 & 0 & 0 & 0 & 28 \\
-12 & -12 & -12 & 0 & 0 & 1 & 0 & 0 & 28 \\
-12 & -12 & -12 & 0 & 0 & 0 & 1 & 0 & 28 \\
\end{pmatrix}$$

$\Delta A^*$ eigen values of order -1

$\Delta A^*$ eigen values of order -3

$\Delta A^*$ eigen values of order 2

$\Delta A^*$ eigen values of order 7

In this case $P^0$ can be shown as

$$P^0 \simeq \begin{pmatrix}
-3 & -6 & 0 & 0 & 17 & 17 & 52 & 52 \\
-6 & -3 & 0 & 0 & 17 & 17 & 52 & 52 \\
-10 & -10 & -1 & -2 & 13 & 13 & 48 & 48 \\
-10 & -10 & -2 & -1 & 13 & 13 & 48 & 48 \\
-19 & -19 & -13 & -13 & 2 & 1 & 36 & 36 \\
-19 & -19 & -13 & -13 & 1 & 2 & 36 & 36 \\
-40 & -40 & -43 & -43 & -29 & -29 & 7 & 6 \\
-49 & -49 & -43 & -43 & -29 & -29 & 6 & 7 \\
\end{pmatrix}$$

It is clear that this matrix is again an M-matrix, provided that the additional conditions on the signs of elements are satisfied.

### 7.2 Example 2

Let $n = 8$ and assume

$$(p_1, q_1) = (4, 6), \quad (p_2, q_2) = (2, 8),$$

This is the example for the case. The uncertain matrix of this case $\Delta A^*$ is given as in Figure 3. For this system the following eigenvalues should be selected.

**Step 1** eigen value of order -1

**Step 2** 2 eigen values of order 2

**Step 3** 2 eigen values of order -3

**Step 3-3** 2 eigen values of order 7

In this case $P^0$ can be shown as

$$P^0 \simeq \begin{pmatrix}
0 & 0 & * & * & * & * & * & * & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & * & 0 & 0 \\
0 & 0 & * & * & * & * & * & 0 & 0 \\
0 & 0 & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \leftarrow n - k$$

**Figure 2**: Matrix $\Delta A^* \in \Omega(k)_{ASC}^0$

$$\Delta A^* \in \Omega(k)_{ASC}^0$$

$$\Delta A^* \in \Omega(k)_{ASC}^0$$

**References**:


