On the Flow of Power-Law Fluids

VÁCLAV KOLÁŘ
Institute of Hydrodynamics
Academy of Sciences of the Czech Republic
Pod Patankou 30/5, 16612 Prague 6
CZECH REPUBLIC
kolar@ih.cas.cz

Abstract: An update on generalized similarity transformation applied to the flow of power-law fluids is presented. The paper surveys both steady and unsteady flow problems: (i) wall jets and (ii) unsteady liquid-film stretching. The latter case assumes that the flow field is similar in time and space. Some practical issues are addressed.

Key-Words: Flow of power-law fluids; Liquid film; Similarity transformation; Unsteady stretching; Wall jets

1 Introduction

The main positive aspect of the flow analysis using similarity transformation is the ability to reduce the equations of motion and to provide explicitly the functional dependence of characteristic flow scales, namely length, velocity, and (if applicable) pressure scales. In many shear-flow problems numerical solutions and sophisticated flow modelling should be preceded by the similarity analysis revealing analytically the role of relevant flow parameters and their physical and geometrical meaning. The present contribution demonstrates the use of generalized similarity transformation for non-Newtonian power-law fluids which are very important from the technological viewpoint. Both steady and unsteady incompressible flow examples are considered below.

Firstly, axisymmetric wall jets with swirl (formed on bodies of revolution) are discussed. Wall jets are often used in mechanical, chemical, and aerospace engineering, frequently for solid surface conditioning associated with heat and/or mass transfer. The flows past axisymmetric bodies or in the stagnation region are frequently investigated through boundary-layer approximation and similarity analysis [1-13]. The complexity of the present wall-jet flow problem is given by the three following aspects [14]: an arbitrary (axisymmetric) body contour, a non-zero swirl component of velocity, and especially the presence of a wall, the inner wall-jet region being significantly affected by the body surface. The steady wall-jet flow is assumed similar in space so as to obtain characteristic length, velocity, and pressure scales.

Secondly, the unsteady liquid-film stretching is considered. Dynamics of viscous flows due to a stretching surface plays an important role in modern technology (polymer industry, rubber technology, metallurgical processes, namely manufacturing of plastic, rubber or metallic sheets, manufacturing of artificial fibres, wires etc.), as well as in bioengineering (dealing with flexible elastic surfaces as membranes and conduits) and in many other practical applications.

Most of the studies on flows due to a stretching surface considered the semi-bounded fluid extending to infinity [15-37]. Some studies have investigated the flow within the finite-thickness liquid film on a stretching surface [38-41]. From the technological viewpoint the finite (relatively thin) liquid film adhering to a stretched elastic surface is represented, for example, by paints or protective coatings in the extrusion or coextrusion processes. Due to unsteady nature of the liquid-film stretching, characteristic scales are sought as time and space dependent functions assuming that the flow field is similar in time and space.

2 Axisymmetric Wall Jets with Swirl

2.1 Problem Formulation & Transformation

The curvilinear coordinate system (x, y, ϕ) with the curvilinear surface coordinate x defined in axial plane according to Fig. 1 is employed. In the present notation $r \equiv r(x)$ denotes a local body radius, ϕ is a polar angle coordinate. Following [14] the swirling wall jets past axisymmetric bodies for power-law fluids are described by the following set of three equations of motion ($\partial / \partial\phi \equiv 0$ with respect to axisymmetry)

ISBN: 978-1-61804-057-2

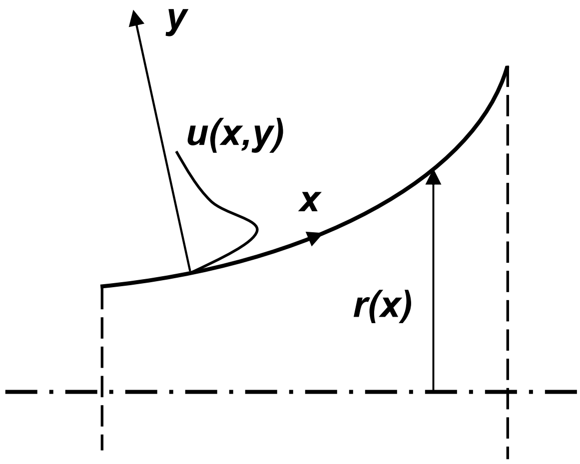

Fig. 1 Curvilinear coordinate system employed (axial cross-section).

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-w^2\cdot\frac{r'(x)}{r(x)}=\frac{1}{\rho}\cdot\frac{\partial \tau_{xy}}{\partial y}, \tag{1}$$

$$w^2\cdot\frac{\left(1-r'^2(x)\right)^{1/2}}{r(x)}=\frac{1}{\rho}\cdot\frac{\partial(\Delta p)}{\partial y}, \tag{2}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+uw\cdot\frac{r'(x)}{r(x)}=\frac{1}{\rho}\cdot\frac{\partial \tau_{\phi y}}{\partial y} \tag{3}$$

where the stress tensor components are given according to power-law model (K, n are power-law model parameters, the given 3D version is valid within the frame of boundary-layer approximations)

$$\tau_{xy}=K\left(\sqrt{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2}\right)^{n-1}\cdot\frac{\partial u}{\partial y}, \tag{4a}$$

$$\tau_{\phi y}=K\left(\sqrt{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2}\right)^{n-1}\cdot\frac{\partial w}{\partial y}, \tag{4b}$$

by the continuity equation

$$\frac{\partial}{\partial x}\left(r(x)\,u\right)+\frac{\partial}{\partial y}\left(r(x)\,v\right)=0, \tag{5}$$

and by the wall-jet boundary conditions for velocity components, stress tensor components, and pressure

$$u(x,0)=0,\quad v(x,0)=0,\quad w(x,0)=0, \tag{6a, b, c}$$

$$\lim_{y\to+\infty}u(x,y)=0,\quad \lim_{y\to+\infty}w(x,y)=0, \tag{7a, b}$$

$$\lim_{y\to+\infty}\tau_{xy}(x,y)=0,\quad \lim_{y\to+\infty}\tau_{\phi y}(x,y)=0, \tag{8a, b}$$

$$\lim_{y\to+\infty}p(x,y)=p_\infty. \tag{9}$$

The application of all boundary conditions is crucial.

The jet width $\delta(x)$ is relatively small so as $\delta(x) << r(x)$, namely the classical assumptions of Boltze for boundary-layer flow on bodies of revolution (according to Schlichting [42]) are adopted as in many previous similarity solutions for power-law fluids [8], [9], [43]-[46]. In addition, divergent shapes are considered, $r'(x)>0$, with non-extreme values of $r''(x)$.

The Glauert-type integral energy equations [14] dealing with the so-called 'flux of exterior momentum flux' (derived from (1), (3), (5)-(8))

$$\frac{\partial}{\partial x}\int_0^\infty \rho\, r(x)\, u\left(\int_y^\infty \rho\, r(x)\, u^2\,\mathrm{d}y\right)\mathrm{d}y=$$

$$=\int_0^\infty \rho\, r(x)\, u\left(\int_y^\infty \rho\, r'(x)\, w^2\,\mathrm{d}y\right)\mathrm{d}y-\int_0^\infty \rho\, r^2(x)\, u\tau_{xy}\,\mathrm{d}y \tag{10}$$

$$\frac{\partial}{\partial x}\int_0^\infty \rho\, r(x)\, u\left(\int_y^\infty \rho\, r^2(x)\, uw\,\mathrm{d}y\right)\mathrm{d}y=$$

$$=-\int_0^\infty \rho\, r^3(x)\, u\tau_{\phi y}\,\mathrm{d}y \tag{11}$$

are needed in the course of the similarity procedure.

The generalized similarity transformation reads

$$\psi(x,y)=A(x)\cdot f(\eta), \tag{12}$$

$$\eta(x,y)=B(x)\cdot y\quad \left(\equiv y/\delta(x)\right), \tag{13}$$

$$w(x,y)=E(x)\cdot h(\eta), \tag{14}$$

$$\tau_{xy}(x,y)=\rho\cdot T_1(x)\cdot T_2(\eta), \tag{15}$$

$$\tau_{\phi y}(x,y)=\rho\cdot T_3(x)\cdot T_4(\eta), \tag{16}$$

$$\Delta p(x,y)\equiv p(x,y)-p_\infty=\rho\cdot P_1(x)\cdot P_2(\eta). \tag{17}$$

The quantity ψ is the stream function, η denotes the similarity variable. Eq. (2) serves only for the determination of the transverse pressure distribution after the determination of the velocity field. The partial similarity results are obtained [14] as follows

(i) for the similarity functions:

$$f'(\eta)\equiv h(\eta)\quad \textit{for all}\ \eta\in[0,+\infty), \tag{18}$$

$$T_2(\eta)\equiv T_4(\eta)\quad (\equiv T(\eta))\quad \textit{for all}\ \eta\in[0,+\infty), \tag{19}$$

$$T'+C_1\cdot ff''+C_2\cdot f'^2=0, \tag{20}$$

(ii) for the (positive) similarity coefficients: $A(x), B(x), E(x), \ldots$ are determined as $A(x;C_1,C_2), B(x;C_1,C_2), E(x;C_1,C_2), \ldots$

(iii) for the spatial flow geometry:

$$\frac{w(x,y)}{u(x,y)}=\frac{\tau_{\phi y}(x,y)}{\tau_{xy}(x,y)}=\frac{e}{\left(r^2(x)-e^2\right)^{1/2}} \tag{21}$$

where e is the swirl parameter (to be discussed later).

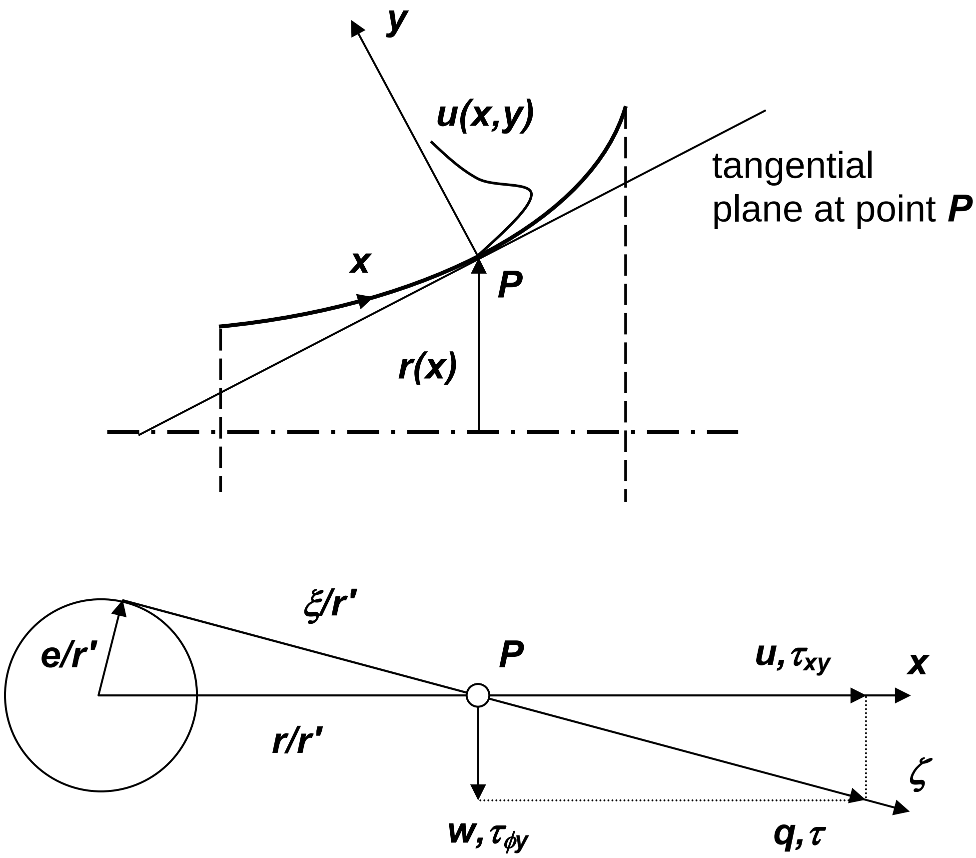

Fig. 2 Axial cross-section and the flow resultants projected onto the tangential plane at the point P.

2.2 Similarity Solution & Discussion

The partial similarity results (stated at the end of the last section) are effectively employed for the transformation of the original 'swirling' problem formulation into the formally 'non-swirling' problem formulation (for detailed analysis see [14]). The results are obtained in terms of the velocity and shear-stress resultants, q and τ respectively, in ζ direction according to Fig. 2. It holds

$$q=\left(u^2+w^2\right)^{1/2}=ru/\xi=rw/e\,, \tag{22}$$

$$\tau=\left(\tau_{xy}^2+\tau_{\phi y}^2\right)^{1/2}=r\tau_{xy}/\xi=r\tau_{\phi y}/e\,, \tag{23}$$

$$\xi\equiv\xi(x)=\left(r^2(x)-e^2\right)^{1/2}. \tag{24}$$

The quantity ζ in Fig. 2 may be considered as the curvilinear surface coordinate following the resulting 'helical' fluid motion past the body surface.

The solution in terms of the original coordinates and the velocity resultant can be summarized as [14]

$$q=D\cdot\left(\frac{K}{\rho F}\cdot D^{2n-1}\right)^{-CF}\cdot$$

$$\cdot\left(\int_{x_0}^{x} r(x)\left(r^2(x)-e^2\right)^{n/2}\,\mathrm{d}x\right)^{-CF}\cdot f(\eta) \tag{25}$$

$$\eta\equiv y/\delta(x) \tag{26}$$

$$\delta(x)=\frac{1}{D}\cdot\left(\frac{K}{\rho F}\cdot D^{2n-1}\right)^{(1+C)F}\cdot\left(r^2(x)-e^2\right)^{-1/2}\cdot$$

$$\cdot\left(\int_{x_0}^{x} r(x)\left(r^2(x)-e^2\right)^{n/2}\,\mathrm{d}x\right)^{(1+C)F} \tag{27}$$

$$D=\left(\frac{\rho}{W}\int_0^\infty\left(f'\right)^{\frac{C+1}{C}}\,\mathrm{d}\eta\right)^{-C} \tag{28}$$

$$W=\int_0^\infty \rho\,\xi\,q^{\frac{C+1}{C}}\,\mathrm{d}y \tag{29}$$

$$F\,\left(\equiv F(n)\right)=\left(n+1+(2n-1)C\right)^{-1} \tag{30}$$

where the constant $C\equiv C(n)$. For a given flow behaviour index n there is a unique value of C obtained by solving the similarity function $f(\eta)$ satisfying the resulting boundary-value problem valid for the power-law model in the form

$$n\cdot\left|f''\right|^{n-1}\cdot f'''+f\,f''+C\cdot f'^2=0\,, \tag{31}$$

$$f(0)=0,\quad f'(0)=0,\quad f'(\infty)=0\,. \tag{32a, b, c}$$

The substitution of the peripheral velocity component w, expressed through (22) and (25), in Eq. (2), yields the transverse pressure distribution, namely the pressure similarity scale $P_1(x)$ and the pressure similarity function $P_2(\eta)$

$$P_1(x) = D\left(\frac{K}{\rho F}\cdot D^{2n-1}\right)^{(1-C)F}\cdot\left(\frac{1-r'^2}{r^2-e^2}\right)^{1/2}\cdot\frac{e^2}{r^3}\cdot \left(\int_{x_0}^{x} r(x)\left(r^2(x)-e^2\right)^{n/2}\mathrm{d}x\right)^{(1-C)F} \quad (33)$$

$$P_2(\eta) = -\int_{\eta}^{\infty} f'^2(\eta)\,\mathrm{d}\eta\,. \quad (34)$$

The jet flow is governed by the explicitly shape-dependent centrifugal and Coriolis forces (see (1)-(3)) what results in the explicitly shape-dependent characteristic scales (25), (27) and (33). The other basic aspect of the given jet flow is the swirl parameter e with the geometrical interpretation according to Fig. 2. In practice this parameter has to do with outflow parameters at the nozzle exit. In the frame of the similarity transformation adopted, the parameter e can be determined as

$$e = Z/W \quad (35)$$

where Z represents, similarly as W given by (29), an integral invariant (constant for a given value of n)

$$Z = \int_0^{\infty} \rho\,\xi\, r w q^{\frac{1}{C}}\,\mathrm{d}y\,. \quad (36)$$

Invariants Z and W may be approximated by outflow parameters at the nozzle exit using a suitable iterative approach (note that both Z and W contain the quantity ξ defined by (24)).

The integral quantity W represents a specific wall-jet flow invariant which can be obtained within the frame of the given similarity transformation only. This quantity which may be understood as a 'generalized flux of momentum flux', serves as a certain substitution for the Glauert-type integral invariant dealing with the 'flux of exterior momentum flux'

$$\int_0^{\infty} \rho\,\xi(x)\,q\left(\int_y^{\infty}\rho\,\xi(x)\,q^2\mathrm{d}y\right)\mathrm{d}y = const \quad (37)$$

which is valid exclusively for the wall-jet flow of a Newtonian fluid, $n = 1$.

For the integral invariant W it is obtained in the limit case $e \to 0$

$$\lim_{e\to 0} W \equiv \lim_{e\to 0}\left(\int_0^{\infty}\rho\,\xi\,q^{\frac{C+1}{C}}\,\mathrm{d}y\right) = \int_0^{\infty}\rho\, r u^{\frac{C+1}{C}}\,\mathrm{d}y\,, \quad (38)$$

i.e. just the integral quantity introduced in [47] for the case of non-swirling wall jets on bodies of revolution for the flow of power-law fluids. The same limiting procedure $e \to 0$ applied to the characteristic scales (25), (27) and (33) provides the results for axisymmetric wall jets without swirl [47].

The next section is to demonstrate the use of 'time-dependent' similarity transformation.

3 Unsteady Liquid-Film Stretching

3.1 Problem Formulation & Transformation

The flow of a thin liquid film of power-law fluids due to the unsteady stretching (or contracting) of an elastic surface along x-axis (stretched at $y = 0$) is described for the plane and radial problems by the following set of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\cdot\frac{\partial \tau_{xy}}{\partial y} \quad (39)$$

$$\frac{\partial}{\partial x}\left(x^{\varepsilon}\, u\right) + \frac{\partial}{\partial y}\left(x^{\varepsilon}\, v\right) = 0 \quad (40)$$

where $\varepsilon = 0$ for the plane problem and $\varepsilon = 1$ for the radial one. The shear stress component is described according to the power-law model by

$$\tau_{xy} = K\left|\partial u/\partial y\right|^{n-1}\partial u/\partial y \quad (41)$$

where K, n are power-law model parameters.

The stretching (or contracting) process is plane symmetric or axisymmetric with respect to the fixed origin at $x = 0$. The corresponding boundary conditions are the no-slip condition on the stretching surface at $y = 0$, and the kinematic and zero shear-stress conditions on the free surface at $y = h(x,t)$ where h denotes the film thickness.

$$u(x,0,t) = U(x,t)\,, \quad (42)$$

$$v(x,0,t) = 0\,, \quad (43)$$

$$v(x,h(x,t),t) = \mathrm{d}h/\mathrm{d}t\,, \quad (44)$$

$$\tau_{xy}(x,h(x,t),t) = 0\,. \quad (45)$$

The distributions of the stretching velocity $U(x,t)$ and initial film thickness $h(x,0)$ are not prescribed and their forms permitting similarity solution are sought and determined within the present analysis using 'time-dependent' similarity transformation. Further, it is assumed that the end effects (due to applied forces) and the gravity effects are negligible as well as that the film surface remains smooth and stable throughout the stretching (or contracting) process.

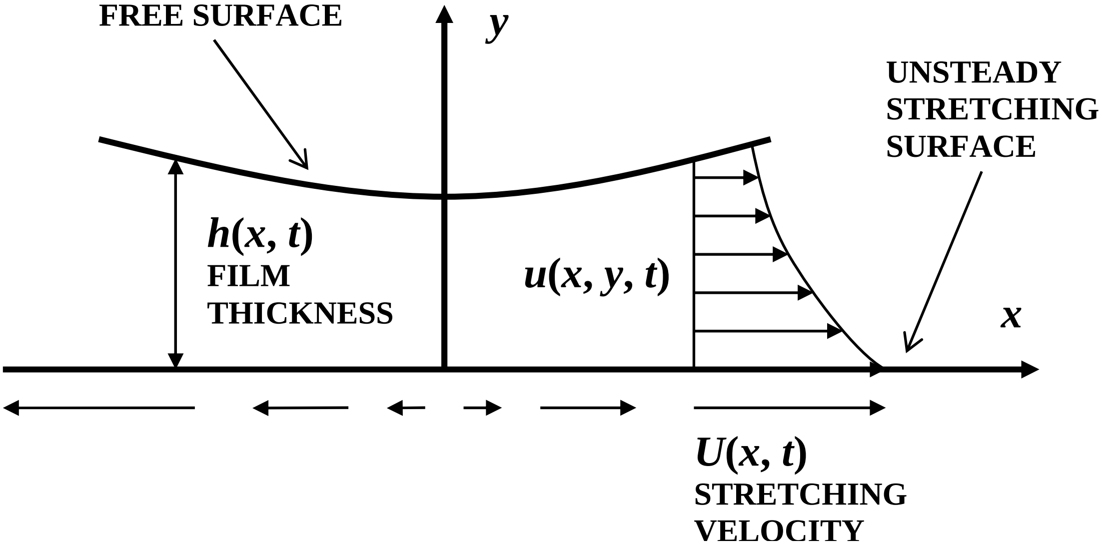

Fig. 3 Flow sketch of the unsteady liquid-film stretching and the coordinate system employed.

The 'time-dependent' similarity transformation of the stream function ψ and the similarity variable η reads

$$\psi(x, y, t) = A(x) \cdot E(t) \cdot f(\eta) \quad (46)$$

$$\eta(x, y, t) = B(x) \cdot F(t) \cdot y \quad (47)$$

where *A*, *B*, *E* and *F* are positive similarity coefficients and *f* is a similarity function.

For the case of plane stretching ($\varepsilon = 0$) and for the power-law model (41) the equation of motion (39) takes — after the similarity transformation and rearrangement — the form

$$\begin{aligned}&\frac{(EF)'}{EF^3} \cdot f' + \frac{F'}{F^3} \cdot \eta f'' + \\ &\quad + (AB)' \cdot \frac{E}{F} \cdot f'^2 - A'B \cdot \frac{E}{F} \cdot f f'' = \\ &\quad = \left(\frac{nK}{\rho}\right) \cdot B^2 \left(AB^2 \cdot EF^2\right)^{n-1} \cdot \left|f''\right|^{n-1} f''' \end{aligned} \quad (48)$$

where the primes indicate differentiation with respect to the arguments. One obtains a similar equation for radial stretching ($\varepsilon = 1$).

In the frame of exact similarity analysis, the strict application of all boundary conditions is a must. The similarity equation (48) reduces for the stretching surface at $y = 0$ or $\eta = 0$, and for the free surface at $y = h(x, t)$ or $\eta = \beta$ (where β denotes the dimensionless film thickness). According to the zero shear-stress condition on the free surface (45) it follows that $f''(\beta) = 0$. Consequently, the restriction $n \geq 1$ (dilatant and Newtonian fluids) must be considered (the denominator on the RHS of Eq. (48) turns out to zero for $n < 1$).

The proper application of boundary conditions provides for the plane problem the following partial similarity results

$$\left(A(x)B(x)\right)' = C_1, \quad (49)$$

$$\frac{\left(E(t)F(t)\right)'}{\left(E(t)F(t)\right)^2} = C_2, \quad (50)$$

$$\frac{K}{\rho}\left(A(x)B(x)\right)^{n-1} B^{n+1}(x) = C_3, \quad (51)$$

$$\left(E(t)F(t)\right)^{n-2} F^{n+1}(t) = C_4. \quad (52)$$

Similar results can be found for radial stretching ($\varepsilon = 1$). From (49)-(52) one obtains the similarity scales *A*, *B*, *E*, *F*.

On the contrary to the required boundary conditions, the initial stretching velocity $U(x, 0)$, better say the admissible stretching velocity $U(x, t)$, and the initial film thickness $h(x, 0)$ are not prescribed and sought so as to permit the similarity solution in the form (46) and (47). It follows directly for the relation between the streamwise velocity and the sought stretching velocity which represents a characteristic velocity scale (plane case)

$$\begin{aligned} u(x, y, t) &= A(x)B(x) \cdot E(t)F(t) \cdot f'(\eta) \\ &= U(x, t) \cdot f'(\eta) \end{aligned} \quad (53)$$

3.2 Similarity Solution & Discussion

The similarity solution answers two main questions: What is the admissible $U(x, t)$ and $h(x, 0)$?

The unsteady stretching velocity $U(x, t)$ is found to be of the form

$$U = \frac{bx}{1 - \alpha t} \qquad (54)$$

where b and α are constants (by introducing $C_2 \equiv \alpha$ and letting, without loss of generality, $C_1 = C_3 \equiv b$), b is positive for a stretching surface and negative for a contracting surface, α is positive for an accelerating surface provided that $t < 1/\alpha$ and negative for a decelerating surface. The stretching velocity is clearly independent of the fluid properties (i.e. power-law model parameters), K and n.

Introducing the relative unsteadiness parameter $S = \alpha/b$, the final similarity equation reads (plane case)

$$n|f''|^{n-1} f''' + \left(\frac{2n}{n+1}\right) f f'' - f'^2 - \\ - S\left[f' + \left(\frac{2-n}{n+1}\right)\eta f''\right] = 0 \qquad (55)$$

The result (54) was first assumed for a Newtonian flow and unsteady liquid-film stretching in [38], with reference to the specific form of transformation of the (non-reduced) unsteady Navier-Stokes equations established in [48] using group theory (see also the very recent review on stretching [49] dealing with the Navier-Stokes equations and a number of relevant references therein).

The problem of plane unsteady liquid-film stretching of non-Newtonian power-law fluids in the frame of boundary-layer approximations and similarity transformation assuming that the stretching process is governed *a priori* by (54) has been studied in [40]. However, the applicability of similarity analysis proposed for $n < 1$ (pseudoplastic fluids) is dubious. Moreover, the application of the kinematic condition at the free surface in [40] is flawed, though it is just the required starting point for the numerical procedure applied to the obtained boundary-value problem dealing with a non-linear third-order ODE.

The time scale $F(t)$ characterizing the film thickness is of the form (same for both stretching geometries)

$$F(t) = (1 - \alpha t)^{(n-2)/(n+1)}. \qquad (56)$$

The admissible initial film thickness found is the same for the plane and radial stretching

$$h(x,0) = \beta/B(x) = \beta\,(K/\rho)^{1/(n+1)}\, b^{(n-2)/(n+1)}\, x^{(n-1)/(n+1)}. \qquad (57)$$

The film thickness itself reads

$$h(x,t) = \beta/(B(x)F(t)) = h(x,0)/(1-\alpha t)^{(n-2)/(n+1)}. \qquad (58)$$

It is clearly non-uniform and, hence, a function of both time and spatial position.

The stretching kinematic condition at the free surface (44) provides the dimensionless flow rate for the plane case

$$f(\beta) = \left(\frac{2-n}{2n}\right) S\,\beta, \qquad (59)$$

or, alternatively, the condition (44) provides for the radial case

$$f(\beta) = \left(\frac{2-n}{3n+1}\right) S\,\beta. \qquad (60)$$

On physical grounds, the relations (58)-(60) imply the usual upper limit for the range of power-law index, namely $n < 2$. It should be recalled that the lower limit for power-law fluids in the present stretching problem has been already established through the requirement $n \geq 1$ indicating the validity for dilatant and Newtonian fluids only.

4 Conclusions

The use of generalized similarity transformation for the scaling of governing equations is surveyed and examined for the flow of power-law fluids. The explicit functional dependence of characteristic flow scales, namely length, velocity, and (if applicable) pressure scales is determined. The analytical approach is based on the assumption that the flow field is similar in space (steady problems) or in both time and space (unsteady problems). From the mathematical viewpoint, the original system of governing PDEs is simplified and reduced to more easily solvable ODE for function(s) of the single similarity variable.

In the present case of non-Newtonian power-law fluids, the similarity transformation has been applied to both steady and unsteady flow problems. In both cases, the strict use of all boundary conditions is prerequisite to obtain correct similarity solution. The analysis of axisymmetric wall jets reveals the analytical dependence of characteristic scales on the shape and swirl parameters. The physical and geometrical meaning of various parameters and flow invariants is specified. The analysis of unsteady liquid-film stretching answers mainly two questions: What is the stretching velocity and initial film thickness permitting similarity solution? By comparing the results (54), (57), and (58), it is obvious that unlike the stretching velocity the film thickness is dependent on the power-law fluid properties, K and n. By comparing the final ODEs for the two examined flow problems, (31) and (55), it follows that the film-stretching ODE (55) is explicitly dependent ─ apart from the flow behaviour index n ─ on the relative unsteadiness parameter S.

Acknowledgement
This work was supported by the Grant Agency of the Academy of Sciences of the Czech Rep. through grant IAA200600801, and by the Academy of Sciences of the Czech Rep. through AV0Z20600510.

References
[1] Bagai, S., Effect of Variable Viscosity on Free Convection Over a Non-Isothermal Axisymmetric Body in a Porous Medium with Internal Heat Generation, *Acta Mechanica*, Vol. 169, No.1-4, , 2004, pp. 187-194.
[2] Bagai, S., Similarity Solutions of Free Convection Boundary Layers over a Body of Arbitrary Shape in a Porous Medium with Internal Heat Generation, *International Communications in Heat and Mass Transfer*, Vol. 30, No. 7, 2003, pp. 997-1003.
[3] Yang, Y.-T., Wang, S.-J., Free Convection Heat Transfer of Non-Newtonian Fluids Over Axisymmetric and Two-Dimensional Bodies of Arbitrary Shape Embedded in a Fluid-Saturated Porous Medium, *International Journal of Heat and Mass Transfer*, Vol. 39, No. 1, 1996, pp. 203-210.
[4] Roy, S., Nath, G., Unsteady Laminar Compressible Swirling Flow with Massive Blowing, *AIAA Journal*, Vol. 30, No. 11, 1992, pp. 2604-2605.
[5] Nakayama, A., Koyama, H., Buoyancy-Induced Flow of Non-Newtonian Fluids over a Non-Isothermal Body of Arbitrary Shape in a Fluid-Saturated Porous Medium, *Applied Scientific Research*, Vol. 48, No. 1, 1991, pp. 55-70.
[6] Nakayama, A., Koyama, H., General Similarity Transformation for Combined Free and Forced-Convection Flows within a Fluid-Saturated Porous Medium, *Journal of Heat Transfer*, Trans. ASME, Vol. 109, No. 4, 1987, pp. 1041-1045.
[7] Chen, J.L.S., Mixed Convective Flow About Slender Bodies of Revolution, *Journal of Heat Transfer*, Trans. ASME, Vol. 109, No. 4, 1987, pp. 1033-1036.
[8] Chang, T.-Ch. A., Jeng, D.R., De Witt, K.J., Natural Convection to Power-Law Fluids from Two-Dimensional or Axisymmetric Bodies of Arbitrary Contour, *International Journal of Heat and Mass Transfer*, Vol. 31, No. 3, 1988, pp. 615-624.
[9] Kim, H.W., Jeng, D.R., De Witt, K.J., Momentum and Heat Transfer in Power-Law Fluid Flow over Two-Dimensional or Axisymmetrical Bodies, *International Journal of Heat and Mass Transfer*, Vol. 26, No. 2, 1983, pp. 245-259.
[10] Mohanty, H.K., Similarity Solutions of Unsteady Laminar Incompressible Boundary Layer Equations for Flow, Heat and Mass Transfer in Non-Newtonian Fluids around Axisymmetric Bodies, *Rheologica Acta*, Vol. 17, No. 4, 1978, pp. 342-352.
[11] Lin, F.N., Chao, B.T., Laminar Free Convection over Two-Dimensional and Axisymmetric Bodies of Arbitrary Contour, *Journal of Heat Transfer*, Trans. ASME, Vol. 96, No. 4, 1974, pp. 435-442.
[12] Hoernel, J.-D., On the Similarity Solutions for a Steady MHD Equation, *Communications in Nonlinear Science and Numerical Simulation*, Vol. 13, No. 7, 2008, pp. 1353-1360.
[13] Takhar, H.S., Nath, G., Similarity Solution of Unsteady Boundary Layer Equations with a Magnetic Field, *Meccanica*, Vol. 32, No. 2, 1997, pp. 157-163.
[14] Kolář, V., Similarity Solution of Axisymmetric Non-Newtonian Wall Jets with Swirl, *Nonlinear Analysis: Real World Applications*, Vol. 12, No. 6, 2011, pp. 3413-3420.
[15] Crane, L.J., Flow past a Stretching Plate, *Zeitschrift für angewandte Mathematik und Physik*, Vol. 21, No. 4, 1970, pp. 645-647.
[16] McCormack, P.D., Crane, L.J., *Physical Fluid Dynamics*, Academic Press, 1973.
[17] Crane, L.J., Boundary Layer Flow due to a Stretching Cylinder, *Zeitschrift für angewandte Mathematik und Physik*, Vol. 26, No. 5, 1975, pp. 619-622.
[18] Siddappa, B., Khapate, B.S., Rivlin-Ericksen Fluid Flow past a Stretching Plate, *Revue Roumaine des Sciences Techniques*, Série de Mécanique Appliquée, Vol. 21, No. 4, 1976, pp. 497-505.
[19] Gupta, P.S., Gupta, A.S., Heat and Mass Transfer on a Stretching Sheet with Suction or Blowing, *Canadian Journal of Chemical Engineering*, Vol. 55, No. 6, 1977, pp. 744-746.
[20] Carragher, P., Crane, L.J., Heat Transfer on a Continuous Stretching Sheet, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 62, No. 10, 1982, pp. 564-565.
[21] Rajagopal, K.R., Na, T.Y., Gupta, A.S., Flow of a Viscoelastic Fluid over a Stretching Sheet, *Rheologica Acta*, Vol. 23, No. 2, 1984, pp. 213-215.
[22] Wang, C.Y., The Three-Dimensional Flow due to a Stretching Flat Surface, *Physics of Fluids*, Vol. 27, No. 8, 1984, pp. 1915-1917.

[23] Wang, C.Y., Fluid Flow due to a Stretching Cylinder, *Physics of Fluids*, Vol. 31, No. 3, 1988, pp. 466-468.

[24] Wang, C.Y., Stretching a Surface in a Rotating Fluid, *Zeitschrift für angewandte Mathematik und Physik*, Vol. 39, No. 2, 1988, pp. 177-185.

[25] Dutta, B.K., Ray, P., Gupta, A.S., Temperature Field in Flow over a Stretching Sheet with Uniform Heat Flux, *International Communications in Heat and Mass Transfer*, Vol. 12, No. 1, 1985, pp. 89-94.

[26] Dutta, B.K., Cooling of a Stretching Sheet in Non-Newtonian Flow, *Zeitschrift für angewandte Mathematik und Physik*, Vol. 38, No. 4, 1987, pp. 642-647.

[27] Dutta, B.K., Heat Transfer from a Stretching Sheet in Viscous Flow with Suction or Blowing, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 68, No. 6, 1988, pp. 231-236.

[28] Dutta, B.K., Gupta, A.S. Cooling of a Stretching Sheet in a Viscous Flow, *Industrial and Engineering Chemistry Research*, Vol. 26, No. 2, 1987, pp. 333-336.

[29] Siddappa, B., Abel, M.S., Viscoelastic Boundary Layer Flow past a Stretching Plate with Suction and Heat Transfer, *Rheologica Acta*, Vol. 25, No. 3, 1986, pp. 319-320.

[30] Devi, C.D.S., Takhar, H.S., Nath, G., Unsteady, Three-Dimensional Boundary Layer Flow due to a Stretching Surface, *International Journal of Heat and Mass Transfer*, Vol. 29, No. 12, 1986, pp. 1996-1999.

[31] Char, M.-I., Chen, C.-K., Temperature Field in Non-Newtonian Flow over a Stretching Plate with Variable Heat Flux, *International Journal of Heat and Mass Transfer*, Vol. 31, No. 5, 1988, pp. 917-921.

[32] Lakshmisha, K.N., Venkateshwaran, S., Nath, G., Three-Dimensional Unsteady Flow with Heat and Mass Transfer over a Continuous Stretching Surface, *Journal of Heat Transfer*, Trans. ASME, Vol. 110, No. 3, 1988, pp. 590-595.

[33] Dandapat, B.S., Gupta, A.S., Flow and Heat Transfer in a Viscoelastic Fluid over a Stretching Sheet, *International Journal of Non-Linear Mechanics*, Vol. 24, No. 3, 1989, pp. 215-219.

[34] Cortell, R., Similarity Solutions for Flow and Heat-Transfer of a Viscoelastic Fluid over a Stretching Sheet, *International Journal of Non-Linear Mechanics*, Vol. 29, No. 2, 1994, pp. 155-161.

[35] Gorla, R.S.R., Dakappagari, V., Pop, I., Stretching a Surface in a Rotating Power-Law Fluid, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 75, No. 11, 1995, pp. 876-878.

[36] Hassanien, I.A., Abdullah, A.A., Gorla, R.S.R., Flow and Heat Transfer in a Power-Law Fluid over a Nonisothermal Stretching Sheet, *Mathematical and Computer Modelling*, Vol. 28, No. 9, 1998, pp. 105-118.

[37] Kumaran, V., Ramanaiah, G., A Note on the Flow over a Stretching Sheet, *Acta Mechanica*, Vol. 116, No. 1-4, 1996, pp. 229-233.

[38] Wang, C.Y., Liquid film on an unsteady stretching surface, *Quarterly of Applied Mathematics*, Vol. 48, No. 4, 1990, pp. 601-610.

[39] Usha, R., Sridharan, R., The Axisymmetrical Motion of a Liquid-Film on an Unsteady Stretching Surface, *Journal of Fluids Engineering*, Trans. ASME, Vol. 117, No. 1, 1995, pp. 81-85.

[40] Andersson, H.I., Aarseth, J.B., Braud, N., Dandapat, B.S., Flow of a Power-Law Fluid Film on an Unsteady Stretching Surface, *Journal of Non-Newtonian Fluid Mechanics*, Vol. 62, No. 1, 1996, pp. 1-8.

[41] Liu, I.-C., Andersson, H.I., Heat Transfer in a Liquid Film on an Unsteady Stretching Sheet, *International Journal of Thermal Sciences*, Vol. 47, No. 6, 2008, pp. 766-772.

[42] Schlichting, H., *Boundary Layer Theory*, McGraw-Hill, New York, 1968.

[43] Lin, F.N., Chern, S.Y., Laminar Boundary-Layer Flow of Non-Newtonian Fluid, *International Journal of Heat and Mass Transfer*, Vol. 22, No. 10, 1979, pp. 1323-1329.

[44] Hayasi, N., Similarity of the Two-Dimensional and Axisymmetric Boundary-Layer Flows for Purely Viscous Non-Newtonian Fluids, *Journal of Fluid Mechanics*, Vol. 23, 1965, pp. 293-303.

[45] Acrivos, A., Shah, M.J., Petersen, E.E., On the Solution of the Two-Dimensional Boundary-Layer Flow Equations for a Non-Newtonian Power Law Fluid, *Chemical Engineering Science*, Vol. 20, No. 2, 1965, pp. 101-105.

[46] Bizzell, G.D., Slattery, J.C., Non-Newtonian Boundary-Layer Flow, *Chemical Engineering Science*, Vol. 17, No. 10, 1962, pp. 777-782.

[47] Filip, P., Kolář, V., Hájek, R., Similarity Prediction of Wall Jets Past Axisymmetric Bodies for Power-Law Fluids, *Acta Mechanica*, Vol. 88, No. 3-4, 1991, pp. 167-173.

[48] Birkhoff, G., *Hydrodynamics: A Study in Logic, Fact and Similitude*, Princeton, 1960.

[49] Wang, C.Y., Review of Similarity Stretching Exact Solutions of the Navier-Stokes Equations, *European Journal of Mechanics B/Fluids*, Vol. 30, No. 5, 2011, pp. 475-479.