A cooperative game theory approach for the equal profit and risk allocation

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Abstract— This paper focuses on the decentralized systems that use the revenue-sharing and cost-sharing contracts as coordination mechanisms. We consider a grand-coalition N with finite players, who agree to cooperate by undertaking part of the system's cost individually, while the remaining costs C and revenues R are shared properly, in order to allocate the system's profits and risks equally among them. We use cooperative game theory, in order to examine the possible coalitions of players and to estimate the finite set of solutions, with which the system's profits and risks are allocated equally among all players. Specifically, each system's solution consists of a pair of vectors r, c Є $\mathbb{R}^N$, with which all player profits are normally distributed with equal mean values and variances. Moreover, we introduce a code that can be used for the computation of the precise number of possible solutions.

Keywords— coalitions of players, cooperative game theory, profit and risk allocation, revenue-cost-sharing

I. Introduction

Over the last decades, the development and exploitation of new products and the implementation of projects is achieved through contractual agreements, where at least two individual players cooperate. In most cases, players form a grand-coalition under a decentralized scheme with several decision makers and follow a revenue-sharing or/and cost-sharing mechanism. Specifically, these mechanisms are developed in order to coordinate all the cooperative parties, so as both their individual objectives and the grand-coalition's performance can be optimized [1]-[2]. Taking into consideration that the individual players should develop a fair revenue-cost-sharing mechanism that will be accepted by all the grand-coalition's members, this mechanism has to measure the risk that is allocated to each player. Generally, in such multi-person situations, where individual decision makers cooperate and the total outcome is influenced by each player's outcome, game theory can be effectively applied [3]. Herein, we examine the correlation of both revenue and cost-sharing mechanisms for a grand-coalition, in which the players agree to undertake parts of the cost individually, while the remaining costs and the system's revenues are shared properly, in order to get all players equal profits under an equal risk allocation scheme. We develop a basic model and we use cooperative game theory, in order to examine the possible coalitions of players and to estimate the system's solutions, with which the grand-coalition's profits and risks are allocated equally among all players. The rest of this paper, which is closely related to the cooperative game theory and the quantitative risk analysis, is organized as follows. The review of the literature is presented in section II and the basic model is presented in sections III and IV, while useful conclusions and the future research issues are discussed in the last section V.

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II. Literature Review

In the literature reviewed, several papers examine the revenue-sharing and cost-sharing contracts as coordination mechanisms [4]-[5]-[6]-[7], while [8] propose the equilibrium payment scheme for the revenue-sharing agreements. However, these mechanisms should use risk as driver, as mentioned in [9], while [10] indicate that firms collaborate in order to have an efficient risk-sharing, as approximately 96% of the US ventures include the risk-sharing between partners. Furthermore, a formula that increases the financial sustainability of partnerships in Greece is developed in [11] and preference to risk-sharing between partners due to financial constraints is demonstrated in [12]. In [13], it is suggested that the shared profit among players should be proportional to their investment and risk taking, while the applications of cooperative game theory to the supply chain's management, focusing on the profit allocation and stability, are surveyed in [3]. Generally, game theory is applied in a finite set of players $N = \{1,2,3,\ldots,n\}$, namely grand-coalition. Moreover, any subset in which this set can be divided is usually called a coalition [14], and any coalition with just one player is called a singleton coalition [15]. A cooperative game is a pair (N, u) where u is the characteristic function representing the collective payoff for a set of players that form a coalition [16]. The game's solution is a vector x Є $\mathbb{R}^N$ representing the allocation of the total profit to each player. A formal solution for the cooperative bargaining process was first introduced by Nash [17], namely Nash-bargaining solution, which consists of an axiomatic derivation of the solution for a bargaining game between two players, who have perfect information [18] and examine to cooperate and share the profits. The solution satisfies a set of axioms that is

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symmetry, Pareto-optimality and feasibility, i.e. identical players receive identical profit allocations, any change to a different allocation that makes at least one player better off will make at least one of the other players worse off, and the sums of the players' allocations do not exceed the total pie. Additionally, the solution is preserved under linear transformations and is independent of irrelevant alternatives.

III. The Basic Model

We focus on a decentralized system with a finite set of players $N = \{1,2,3,\ldots,n\}$ that is the grand-coalition. These players agree to cooperate by undertaking part of the system's cost individually, i.e. the costs c_1, c_2, c_3, …., and c_n, are undertaken by players 1,2,3,…., and n, respectively. Furthermore, the grand-coalition's remaining costs C and revenues R are shared between all players, through a revenue-cost-sharing mechanism. Let P_i denote the profit allocated to each player. A complete list of the notations used in this paper is presented in Table 1.

Obviously, the revenue-cost-sharing mechanism has to be feasible and individually rational, i.e. the sum of the players' allocations does not exceed the total pie and each player gets at least as much as what it could obtain through the non-cooperative option:

$$\mathrm{R}_i \in (0,1) \text{ and } \sum_{i=1}^{n} \mathrm{R}_i = \sum_{i=1}^{n} \mathrm{C}_i = 1 \tag{1}$$

$$\mathrm{P}_i = R\,(\mathrm{R}_i) - C\,(\mathrm{C}_i) - c_i > 0\,,\quad \forall i \in N \tag{2}$$

We assume that there is full information among players and we examine the case where the grand-coalition's profits should be shared equally and be proportional to each player's investment and risk taking [13] – [19]. In order to define the system's risks and to allocate them with fairness among players, the costs and revenues to be shared are normally distributed with specific mean value μ and variance σ^2. That is, a different probability density function Π is assigned in the grand-coalition's revenues: $\Pi_R\,(\mu_R,\ \sigma_R^2)$ and the shared costs: $\Pi_C\,(\mu_C,\ \sigma_C^2)$. According to these distributions, we get the following proposition.

Proposition 1. *The grand-coalition's profits and risks are allocated equally when all players' profits get equal probability distribution functions, satisfying* (3):

$$\begin{aligned} &\mu_1 = \mu_2 = \mu_3 = \ldots = \mu_n \\ &\sigma_1 = \sigma_2 = \sigma_3 = \ldots = \sigma_n \end{aligned} \tag{3}$$

Proof of Proposition 1. Taking into consideration that both the grand-coalition's revenues R and shared costs C are normally distributed, the profits P_1, P_2, P_3 ,…., and P_n , which are allocated to players 1,2,3,…., and n, respectively, are calculated through normal density functions:

TABLE I
LIST OF NOTATIONS

Symbol	Description	
N	Finite set of players (grand-coalition)	$N = \{1,2,3,\ldots,n\}$
c_i	Cost undertaken by player i	$i \in N$, i.e. $i = 1,2,\ldots,n$
R	Grand-coalition's revenues	(to be divided in all players)
C	Grand-coalition's remaining cost	(to be divided in all players)
R_i	Revenue-sharing ratio of player i	$\mathrm{R}_i \in (0,1)$, $\sum_{i=1}^{n} \mathrm{R}_i = 1$
C_i	Cost-sharing ratio of player i	$\sum_{i=1}^{n} \mathrm{C}_i = 1$
P_i	Profit allocated to player i	$\mathrm{P}_i = R\mathrm{R}_i - C\mathrm{C}_i - c_i,\ \forall i \in \mathrm{N}$
Π	Probability density functions: R, C	$\Pi_R(\mu_R,\sigma_R^2)$, $\Pi_C(\mu_C,\sigma_C^2)$
P_i	Profit's probability distribution function	$P_i = (\Pi_R \mathrm{R}_i, \Pi_C \mathrm{C}_i, c_i)$ $\forall i \in N$
μ_i	Expected profit	$\mu_i = \mu_R \mathrm{R}_i - \mu_C \mathrm{C}_i - c_i$, $\forall i \in N$
Confidence Intervals	probability : $(\mu_i - \sigma_i \le \mathrm{P}_i \le \mu_i + \sigma_i) \approx 0.6827$ probability : $(\mu_i - 2\sigma_i \le \mathrm{P}_i \le \mu_i + 2\sigma_i) \approx 0.9545$ probability : $(\mu_i - 3\sigma_i \le \mathrm{P}_i \le \mu_i + 3\sigma_i) \approx 0.9973$	
Equal profit and risk allocation	$P_1(\mu_1,\sigma_1^2) \equiv P_2(\mu_2,\sigma_2^2) \equiv \ldots \equiv P_n(\mu_n,\sigma_n^2) \Leftrightarrow$ $\mu_1 = \mu_2 = \ldots = \mu_n$, $\sigma_1 = \sigma_2 = \ldots = \sigma_n$	
$s(n)$	Number of possible solutions	$j = 1,2,3,\ldots,s(n)$
S	Finite set of possible solutions	$S = \{1,2,3,\ldots,s(n)\}$
r, c ∈ $\mathbb{R}^N$	Solutions for equal profit, risk allocation (pairs of vectors)	$\mathrm{r}^j = (\mathrm{R}_1^j, \mathrm{R}_2^j, \ldots, \mathrm{R}_n^j)$, $\mathrm{c}^j = (\mathrm{C}_1^j, \mathrm{C}_2^j, \ldots, \mathrm{C}_n^j)$, $\forall j \in S$

$$P_1(\mu_1,\sigma_1^2) = \left(1/\sqrt{2\pi}\,\sigma_1\right) e^{-\frac{(P_1-\mu_1)^2}{2\sigma_1^2}},$$

$$P_2(\mu_2,\sigma_2^2) = \left(1/\sqrt{2\pi}\sigma_2\right) e^{-\frac{(P_2-\mu_2)^2}{2\sigma_2^2}}, \ldots\ldots,$$

$$P_n(\mu_n,\sigma_n^2) = \left(1/\sqrt{2\pi}\,\sigma_n\right) e^{-\frac{(P_n-\mu_n)^2}{2\sigma_n^2}}$$ for player $i = 1,2,..$, and n, respectively.

From these functions, the profits and risks are allocated equally when: $P_1(\mu_1,\sigma_1^2) \equiv P_2(\mu_2,\sigma_2^2) \equiv \ldots \equiv P_n(\mu_n,\sigma_n^2) \Leftrightarrow$ $\mu_1 = \mu_2 = \ldots = \mu_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_n$

IV. Equal Profit and Risk Allocation Among All Players

In this section we use insights from the cooperative game theory, in order to estimate the possible solutions for the players' revenue-sharing and cost-sharing ratios, which define the equal profit and risk allocation among them. Initially, we examine the case with two players and further the cases where the grand-coalition consists of $n > 2$ players.

A. Grand-coalition with two-players

In cases where the grand-coalition consists of two players, i.e. $N = \{1,2\}$, the system's profits and risks are allocated equally when (3) is satisfied: $P_1(\mu_1,\sigma_1^2) \equiv P_2(\mu_2,\sigma_2^2) \Leftrightarrow \mu_1 = \mu_2, \sigma_1 = \sigma_2$. Moreover, we derive Theorem 1.

Theorem 1. *There is a unique solution:* $(R_1^*, R_2^*, C_1^*, C_2^*)$ *, with which the system's profits and risks are allocated equally among two players.*

Proof of Theorem 1. The probability distribution functions of the players' 1 and 2 profits are given by: $P_1 = \Pi_R R_1 - \Pi_C C_1 - c_1$ and $P_2 = \Pi_R R_2 - \Pi_C C_2 - c_2$.

According to Proposition 1, the mean values and the profit values in the same confidence interval, e.g. the $(\mu \pm 2\sigma)$ should be equal.

Due to the fact that $R_1 + R_2 = 1$ and $C_1 + C_2 = 1$ from (1), where $R_1, R_2 \in (0,1)$, we define that C_1, C_2 can take negative or higher than 1 values, with respect to (1) and (2), so as to ensure that there is at least one solution $(R_1^*, R_2^*, C_1^*, C_2^*)$, with which $\mu_1 = \mu_2, \sigma_1 = \sigma_2$. Furthermore, in order to prove the uniqueness of this solution, we suppose that there is at least one more solution, denoted by $(R_1^{**}, R_2^{**}, C_1^{**}, C_2^{**})$, which also satisfies (3). Particularly, at least one of $R_1^* \neq R_1^{**}$, or $R_2^* \neq R_2^{**}$, or $C_1^* \neq C_1^{**}$, or $C_2^* \neq C_2^{**}$, while in both cases there is:

$$R_1^* + R_2^* = R_1^{**} + R_2^{**} = C_1^* + C_2^* = C_1^{**} + C_2^{**} = 1 \tag{4}$$

We consider that both $(R_1^*, R_2^*, C_1^*, C_2^*)$, $(R_1^{**}, R_2^{**}, C_1^{**}, C_2^{**})$ solutions satisfy (5), (7) and (6), (8) respectively:

$$\mu_1 = \mu_2 \Leftrightarrow \mu_R R_1^* - \mu_C C_1^* - c_1 = \mu_R R_2^* - \mu_C C_2^* - c_2 \tag{5}$$

$$\mu_1 = \mu_2 \Leftrightarrow \mu_R R_1^{**} - \mu_C C_1^{**} - c_1 = \mu_R R_2^{**} - \mu_C C_2^{**} - c_2 \tag{6}$$

$$\mu_1 \pm 2\sigma_1 = \mu_2 \pm 2\sigma_2 \Leftrightarrow \Pi_R R_1^* - \Pi_C C_1^* - c_1 = \Pi_R R_2^* - \Pi_C C_2^* - c_2 \tag{7}$$

$$\begin{gathered} \mu_1 \pm 2\sigma_1 = \mu_2 \pm 2\sigma_2 \Leftrightarrow \\ \Pi_R R_1^{**} - \Pi_C C_1^{**} - c_1 = \Pi_R R_2^{**} - \Pi_C C_2^{**} - c_2 \end{gathered} \tag{8}$$

From (5) minus (6) we get:

$$\begin{gathered} (5) - (6) \Leftrightarrow \\ \mu_R\left(R_1^* - R_2^* - R_1^{**} + R_2^{**}\right) - \mu_C\left(C_1^* - C_2^* + C_2^{**} - C_1^{**}\right) = 0 \end{gathered} \tag{9}$$

and from (7) minus (8):

$$\begin{gathered} (7) - (8) \Leftrightarrow \\ \Pi_R\left(R_1^* - R_2^* - R_1^{**} + R_2^{**}\right) - \Pi_C\left(C_1^* - C_2^* + C_2^{**} - C_1^{**}\right) = 0 \end{gathered} \tag{10}$$

There is $\mu_R > \mu_C, \Pi_R > \Pi_C$ and thus both the parentheses in (9) as well as in (10) equal zero:

$$\begin{matrix} R_1^* - R_2^* - R_1^{**} + R_2^{**} = 0 \\ C_1^* - C_2^* - C_1^{**} + C_2^{**} = 0 \end{matrix} \overset{(4)}{\Leftrightarrow} \begin{matrix} 1 - R_2^* - R_2^* - 1 + R_2^{**} + R_2^{**} = 0 \\ 1 - C_2^* - C_2^* - 1 + C_2^{**} + C_2^{**} = 0 \end{matrix} \Leftrightarrow$$

$$\begin{matrix} R_2^{**} = R_2^* \\ C_2^{**} = C_2^* \end{matrix} \overset{(4)}{\Leftrightarrow} \begin{matrix} R_1^{**} = R_1^* \\ C_1^{**} = C_1^* \end{matrix}$$

Therefore, the second solution is equal to the first: $R_1^* = R_1^{**}, R_2^* = R_2^{**}, C_1^* = C_1^{**}, C_2^* = C_2^{**}$ and there is a unique solution: $(R_1^*, R_2^*, C_1^*, C_2^*)$ for the equal profit and risk allocation among players 1 and 2.

B. Grand-coalition with n > 2 players

In order to find the solution/s that allocates the grand-coalition's profits and risks equally among all players, we use a cooperative game theory approach. Specifically, the grand-coalition is divided in two coalitions, namely: $N_A = \{1,2,\ldots,h\}$ $N_B = \{h+1,h+2,\ldots,n\}$, with: $1 \leq h < n$. Due to (2), there is no coalition that can be profitably blocked by any coalition of players. Hence, there is no constraint considered for the division of the players, i.e. any player can be placed either in the N_A or in the N_B coalition. However, there is: $R_A + R_B = C_A + C_B = 1$.

Further, we derive Theorems 2 and 3.

Theorem 2. *For each pair of non-empty coalitions N_A, N_B, that the grand-coalition N can be divided, there is a unique solution:* $(R_A^*, R_B^*, C_A^*, C_B^*)$*, with which the system's profits and risks are allocated equally among all players.*

Proof of Theorem 2. In order to demonstrate that the grand-coalition's profits and risks are allocated equally, the mean values and the profit values in the confidence intervals: $(\mu \pm \sigma)$, $(\mu \pm 2\sigma)$ and $(\mu \pm 3\sigma)$ should be equal for all players. We mention that these confidence intervals include the profit values for each player with 68.27%, 95.45% and 99.73% probability, respectively. However, the probability distribution functions of the N_A, N_B coalitions, are given by:

$$P_A = \Pi_R R_A - \Pi_C C_A - \sum_{a=1}^{h} c_a \tag{11}$$

$$P_B = \Pi_R R_B - \Pi_C C_B - \sum_{b=h+1}^{n} c_b \tag{12}$$

According to Proposition 1, the mean values and the profit values in the same confidence interval, e.g. the $(\mu \pm \sigma)$, or the $(\mu \pm 2\sigma)$, or the $(\mu \pm 3\sigma)$, should be equal. Similarly with the proof of Theorem 1, we suppose that there are two solutions: $\left(R^*_A, R^*_B, C^*_A, C^*_B\right)$ and $\left(R^{**}_A, R^{**}_B, C^{**}_A, C^{**}_B\right)$, which both satisfy (13) and (14):

$$\mu_A = h(\mu_i),\ \ \mu_B = (n-h)(\mu_i) \Leftrightarrow \mu_A = \frac{h}{n-h}\mu_B \ \Leftrightarrow$$

$$(\mu_R R^*_A - \mu_C C^*_A - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\mu_R R^*_B - \mu_C C^*_B - \sum_{b=h+1}^{n} c_b) =$$
$$(\mu_R R^{**}_A - \mu_C C^{**}_A - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\mu_R R^{**}_B - \mu_C C^{**}_B - \sum_{b=h+1}^{n} c_b) = 0 \tag{13}$$

$$(\Pi_R R^*_A - \Pi_C C^*_A - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\Pi_R R^*_B - \Pi_C C^*_B - \sum_{b=h+1}^{n} c_b) =$$
$$(\Pi_R R^{**}_A - \Pi_C C^{**}_A - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\Pi_R R^{**}_B - \Pi_C C^{**}_B - \sum_{b=h+1}^{n} c_b) = 0 \tag{14}$$

Following the same concept with the proof of Theorem 1, we solve (13), (14) and we get: $R^*_A = R^{**}_A, R^*_B = R^{**}_B$, $C^*_A = C^{**}_A, C^*_B = C^{**}_B$. Thus, there is a unique solution: $\left(R^*_A, R^*_B, C^*_A, C^*_B\right)$ for each pair of non-empty coalitions N_A, N_B, that the grand-coalition N can be divided, with which the system's profits and risks are allocated equally among all players.

Theorem 3. *The number of possible solutions s(n), with which the system's profits and risks are allocated equally among all players, is equal to the combinations of players divided in pairs of non-empty coalitions iteratively, until all are divided in singleton coalitions:* $\{\{1\},\{2\},\ldots,\{n\}\}$. *Specifically, the number of possible solutions:* $s(n)$, $\forall n \geq 2$ *, is given from* (15) *and* (16), *whether n is odd or even number respectively:*

$$s(n) = \frac{n!}{(n-1)!}s(n-1) + \sum_{k=2}^{\frac{n-1}{2}} \frac{n!}{(n-k)!(k)!}s(n-k)s(k) \tag{15}$$

$$s(n) = \frac{n!}{(n-1)!}s(n-1) + \sum_{k=2}^{\frac{n}{2}-1} \frac{n!}{(n-k)!(k)!}s(n-k)s(k) + \frac{n!}{(\frac{n}{2}!)^2}(s(\frac{n}{2}))^2(\frac{1}{2}) \tag{16}$$

Proof of Theorem 3. According to Theorem 2, for each pair of non-empty coalitions N_A, N_B, that the grand-coalition N can be divided, there is a unique solution: $\left(R^*_A, R^*_B, C^*_A, C^*_B\right)$. However, if we consider the further division of the N_A coalition that includes h players in another pair of coalitions, namely N_{AA} and N_{AB}, where N_{AA} consists of t players and N_{AB} consists of h-t players, then according to Theorem 1, there is a unique solution: $\left(R^*_{AA}, R^*_{AB}, C^*_{AA}, C^*_{AB}\right)$, while the rest players n-h of the N_B coalition can also be divided further in two other coalitions, namely N_{BA}, N_{BB}, and with the same concept there is another unique solution: $\left(R^*_{BA}, R^*_{BB}, C^*_{BA}, C^*_{BB}\right)$. We consider the iterative divisions in pairs of coalitions, until all the n players of the grand-coalition are divided in singleton coalitions: $\{\{1\},\{2\},\ldots,\{n\}\}$. Taking into account that for each coalition considered there is a unique solution, we conclude that there is a unique solution for each player who is included in specific coalitions. This is calculated when all the solutions of the coalitions including him are multiplied. For instance, the solution for player i, who is included in N_B, N_{BA}, N_{BAA}, N_{BAAB} coalitions, is given: $R_i = (R_B)(R_{BA})(R_{BAA})(R_{BAAB})$, and $C_i = (C_B)(C_{BA})(C_{BAA})(C_{BAAB})$.

Obviously, each time we consider the division of a set N in a pair of coalitions with h and n-h players respectively, where $1 \leq h < n$, there are alternative possible combinations of the players in the coalitions and each combination results in a unique system's solution: $\left(R^*_1, R^*_2, \ldots, R^*_n\right)\left(C^*_1, C^*_2, \ldots, C^*_n\right)$. Thus, the number of the system's possible solutions, with which the grand-coalition's profits and risks are allocated equally among players, is equal to the number of possible combinations of the players in the coalitions, until all players are divided in a singleton coalition. Let $s(n)$ denote the number of solutions for a grand-coalition with $n \geq 2$ players. We mention that $s(n)$ is equal to the sum of possible combinations of the n players in coalitions, which is multiplied with the number of possible solutions for the specific coalitions. That is, $s(n)$ increases with the number of players, as for a grand-coalition with 3 players, i.e. $N = \{1,2,3\}$, there are 3 combinations in a 2- player coalition:

1) $\{\{1,2\}=\{2,1\},\{3\}\}$ that gives a unique solution: $\left(R_1^{*1}, R_2^{*1}, R_3^{*1}, C_1^{*1}, C_2^{*1}, C_3^{*1}\right)$
2) $\{\{1,3\}=\{3,1\},\{2\}\}$ that gives another solution: $\left(R_1^{*2}, R_2^{*2}, R_3^{*2}, C_1^{*2}, C_2^{*2}, C_3^{*2}\right)$
3) $\{\{2,3\}=\{3,2\},\{1\}\}$ that gives another solution: $\left(R_1^{*3}, R_2^{*3}, R_3^{*3}, C_1^{*3}, C_2^{*3}, C_3^{*3}\right)$

Each of these solutions allocates the grand-coalition's profits and risks equally among all players, i.e. the $s(3)=3$. Particularly, the mean values are equal for all players in all solutions: $\mu_i^j = \mu_N / n$, $\forall i = 1,2,3$, $j = 1,2,3$. However, the standard deviations are equal for all players in each solution: $\sigma_1^j = \sigma_2^j = \sigma_3^j, \forall j = 1,2,3$ and may be different among the three solutions: $\sigma_i^1 \neq \sigma_i^2 \neq \sigma_i^3, \forall i = 1,2,3$. This happens, because at least one of the revenue-sharing or cost-sharing ratios differs among the solutions. Moreover, for a grand-coalition with 4 players, i.e. $N = \{1,2,3,4\}$, there are $\frac{4!}{3!(4-3)!} = 4$ combinations of the 4 players in a 3- player coalition: $\{\{1,2,3\},\{4\}\}$, {{1,2,4},{3}}, {{1,3,4},{2}} and {{2,3,4},{1}}, where each one has 3 possible solutions, while there are also three combinations of the 4 players in a 2-player coalition respectively: {{1,2},{3,4}}, {{1,3},{2,4}}, {{1,4},{2,3}} and each one has 1 solution, thus: $s(4) = 4s(3) + 3s(2) \Leftrightarrow s(4) = 4(3) + 3(1) = 15$. Furthermore, the number of possible solutions, with which the system's profits and risks are allocated equally among all players, is calculated with (15) and (16), whether n is odd or even number respectively.

C. *Computation of the precise number of possible solutions*

Taking into consideration that the grand-coalition is a finite set of players and the number of possible solutions is calculated with (14) and (15), whether n is odd or even number respectively, we conclude that there are finite possible solutions that define the equal profit and risk allocation among players. However, the number of possible solutions is rapidly increased with the number of players. Particularly, from the Proof of Theorem 1 we get $s(2)=1$ and from the Proof of Theorem 3, we get $s(3)=3$ and $s(4)=15$ and thus, for $n=5,6,\ldots,10$ we compute:

$$s(5) = \frac{5!}{4!}s(4) + \frac{5!}{3!2!}s(3)s(2) = 5(15) + 10(3)(1) = 105$$

$$s(6) = \frac{6!}{5!}s(5) + \frac{6!}{4!2!}s(4)s(2) + \frac{6!}{(3!)^2}(s(3))^2(\frac{1}{2}) = 945$$

$$s(7) = \frac{7!}{6!}s(6) + \frac{7!}{5!2!}s(5)s(2) + \frac{7!}{4!3!}s(4)s(3) = 10{,}395$$

$$s(8) = \frac{8!}{7!}s(7) + \frac{8!}{6!2!}s(6)s(2) + \frac{8!}{5!3!}s(5)s(3) + \frac{8!}{(4!)^2}(s(4))^2(\frac{1}{2}) = 135{,}135$$

$$s(9) = \frac{9!}{8!}s(8) + \frac{9!}{7!2!}s(7)s(2) + \frac{9!}{6!3!}s(6)s(3) + \frac{9!}{5!4!}s(5)s(4) = 2{,}027{,}025$$

$$s(10) = \frac{10!}{9!}s(9) + \frac{10!}{8!2!}s(8)s(2) + \frac{10!}{7!3!}s(7)s(3) + \frac{10!}{6!4!}s(6)s(4) + \frac{10!}{(5!)^2}(s(5))^2(\frac{1}{2}) = 34{,}429{,}425$$

```
In[1]:= s[n_] := Piecewise[
          {{n!/(n - 1)! s[n - 1] + Sum[(n!/((n - k)! k!) s[n - k] s[k]), {k, 2, (n - 1)/2}],
            (n - 1)/2 ∈ Integers && n ≥ 3},
           {n!/(n - 1)! s[n - 1] + Sum[(n!/((n - k)! k!) s[n - k] s[k]), {k, 2, n/2 - 1}] +
            n!/((n/2)!)^2 s[n/2]^2 1/2, n/2 ∈ Integers && n ≥ 4}}
          , 1]
        s[2]
Out[2]= 1
```

Fig. 1: Calculations of possible solutions with the Wolfram Mathematica, (2008).

TABLE II
POSSIBLE SOLUTIONS $s(n)$ FOR THE GRAND-COALITION $N = \{1,2,3,\ldots n\}$

n	s(n)
$n = 2$	1
$n = 3$	3
$n = 4$	15
$n = 5$	105
$n = 6$	945
$n = 7$	10,395
$n = 8$	135,135
$n = 9$	2,027,025
$n = 10$	34,429,425
$n = 11$	654,729,075
$n = 12$	13,749,310,575
$n = 13$	31,623,414,225
$n = 14$	7,905,853,580,625
$n = 15$	213,458,046,676,875
$n = 16$	6,190,283,353,629,375
$n = 17$	191,898,783,962,510,625
$n = 18$	6,332,659,870,762,850,625
$n = 19$	221,643,095,476,699,771,875
$n = 20$	8,200,794,532,637,891,559,375
$n = 21$	319,830,986,772,877,770,815,625
$n = 22$	13,113,070,457,687,988,603,440,625
$n = 23$	563,862,029,680,583,509,947,946,875
$n = 24$	25,373,791,335,626,257,947,657,609,375
$n = 25$	1,192,568,192,774,434,123,539,907,640,625

That is, in Fig. 1 we introduce a code that can be used in the Wolfram Research, Inc., Mathematica, Version 7.0, Champaign, IL (2008), for the calculation of the precise number of possible solutions.

Moreover, in Table II we illustrate the results arising for n = 2,3,4,…., and 25.

V. CONCLUSIONS

In situations where individual players examine to cooperate by forming a grand-coalition, the system's coordination can be achieved with revenue-cost-sharing mechanisms. Herein, we focus on cases where parts of the system's costs are undertaken individually by players and the remaining costs and revenues are shared between them, in order to allocate equally the grand-coalition's profits and risks. A cooperative game theory approach is used, in order to estimate the possible coalitions of players and to compute the finite set of system's solutions. Each of these solutions is a pair of vectors r, c ∈ $\mathbb{R}^N$, with which the profits of all players are normally distributed, having equal mean values and variances. The development of an algorithm for the computation of the system's solutions can be a subject for future research. Future papers can also be focused on the comparison between the system's solutions for the equal profit and risk allocation among all players, with the Nash-bargaining solution.

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