Analytical Solution of an Eddy Current Problem for a Two-Layer Medium with Varying Electric Conductivity and Magnetic Permeability

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Abstract: - Method of integral transforms is used in the present paper in order to construct an analytical solution of an eddy current problem describing the interaction of a coil with alternating current located above a two-layer planar conducting medium. The electric conductivity and magnetic permeability of the upper layer are modeled by exponential functions of the vertical coordinate while the properties of the lower half-space are assumed to be constant. The change in impedance of the coil is calculated in closed form. Results of numerical computations are presented.

Key-Words: - vector potential, integral transform, Bessel function

1 Introduction

Theory of eddy current methods is well-developed in the literature for the case where a coil with alternating current is located above a conducting multilayer medium [1]-[4]. The properties of any conducting layer in [1]-[4] (for example, electric conductivity and magnetic permeability) are assumed to be constant.

In many engineering applications such as surface hardening, surface alloying, determination of thickness of metal coatings, de-carbonization and spot welding (see, for example, [5], [6]) the electric and magnetic properties of a conducting material can be modified by an external magnetic field. Mathematical models describing the interaction of an alternating current in a coil with eddy currents in a conducting medium with constant properties should be modified in order to take into account variability of the characteristics of the material. The following two methods are usually used in practice. The first approach is based on the assumption that a conducting medium with variable properties can be approximated by a multilayer medium with large number of layers of relatively small thickness where the electric conductivity and magnetic permeability of each layer are assumed to be constant. In other words, the electric conductivity and magnetic permeability are modeled as piecewise constant functions of the vertical coordinate. Such an approach is described, for example, in [7] where up to 50 layers in the vertical direction are used. The second method assumes that the electric conductivity and/or magnetic permeability can be modeled by a relatively simple functions such as power or exponential functions depending on some parameters. For some combinations of the parameters one can construct analytical solutions of the corresponding boundary value problems in terms of known special functions such as Bessel functions or hypergeometric functions (examples can be found in [4], [8]-[10]).

In the present paper we follow the second approach. Closed-form solution for the change in impedance of a coil with alternating current located above a two-layer planar conducting medium is found for the case where both the electric conductivity and magnetic permeability of the upper layer are exponential functions of a vertical coordinate. The properties of the lower half-space are assumed to be constant.

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2 Mathematical Formulation

Suppose that a single-turn coil of radius r_c is located at a distance h above a two-layer conducting medium (see Fig. 1).

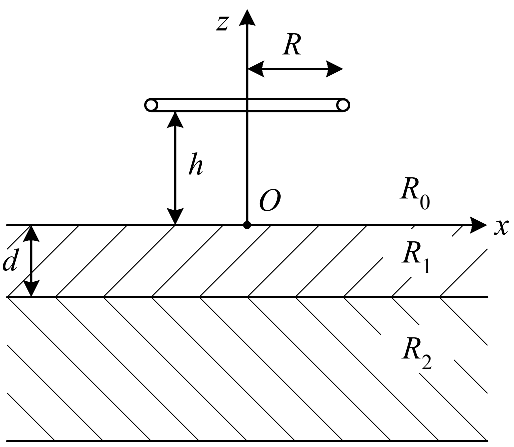

Fig.1. A single-turn coil above a conducting two-layer medium.

We consider a system of cylindrical polar coordinates (r,φ,z) centered at O. The electric conductivity σ and magnetic permeability μ of the upper layer of thickness d are modeled by the following exponential functions of the vertical coordinate z:

$$\sigma = \sigma_m \exp(\alpha z), \qquad \mu = \mu_0 \mu_m \exp(\beta z), \qquad (1)$$

where μ_0 is the magnetic constant, σ_m, μ_m, α and β are given constants. The properties of the medium in the lower half-space are assumed to be constant ($\sigma_2 = const,\ \mu_2 = const$).

We assume that the vector potential has only one non-zero component of the form

$$\vec{A} = A(r,z)\vec{e}_\varphi, \qquad (2)$$

where $\vec{e}_\varphi$ is a unit vector in the φ-direction. The current in the coil is given by

$$i(t)\vec{e}_\varphi = I\exp(j\omega t)\vec{e}_\varphi, \qquad (3)$$

where $j=\sqrt{-1}$, ω is the frequency of the current and I is the amplitude of the current.

The components $A_i(r,z),\ i=0,1,2$ of the vector potential in regions R_0, R_1 and R_2 (see Fig. 1) satisfy the following system of partial differential equations:

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r-r_c)\delta(z-h), \qquad (4)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} + \frac{\partial^2 A_1}{\partial z^2} - \beta\frac{\partial A_1}{\partial z} - j\omega\sigma_m\mu_0\mu_m \exp[(\alpha+\beta)z]A_1 = 0, \qquad (5)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} - j\omega\sigma_2\mu_0\mu_2 A_2 = 0, \qquad (6)$$

where $\delta(x)$ is the Dirac delta-function.
The boundary conditions are

$$A_0\,|_{z=0} = A_1\,|_{z=0}, \quad \frac{\partial A_0}{\partial z}\Big|_{z=0} = \frac{1}{\mu_m}\frac{\partial A_1}{\partial z}\Big|_{z=0}, \qquad (7)$$

$$A_0\,|_{z=-d} = A_1\,|_{z=-d}, \quad \frac{1}{\mu_1}\frac{\partial A_1}{\partial z}\Big|_{z=-d} = \frac{1}{\mu_2}\frac{\partial A_2}{\partial z}\Big|_{z=-d}. \qquad (8)$$

The following conditions hold at infinity:

$$A_i, \ \frac{\partial A_i}{\partial r} \to 0 \text{ as } r \to \infty, \ i=0,1,2, \qquad (9)$$

$$A_0 \to 0 \text{ as } z \to +\infty, \ A_2 \to 0 \text{ as } z \to -\infty. \qquad (10)$$

Analytical solution to problem (4)-(10) is considered in the next section.

3 Solution of the Problem

Applying the Hankel transform of the form

$$\widetilde{A}_i(\lambda,z) = \int_0^\infty A_i(r,z) r J_1(\lambda r)dr, \ i=0,1,2, \qquad (11)$$

to (4)-(10) we obtain

$$\frac{d^2\tilde{A}_0}{dz^2}-\lambda^2\tilde{A}_0=-\mu_0 I r_c J_1(\lambda r_c)\delta(z-h), \qquad (12)$$

$$\frac{d^2\tilde{A}_1}{dz^2}-\lambda^2\tilde{A}_1-j\omega\mu_0\mu_m\sigma_m e^{(\alpha+\beta)z}\tilde{A}_1-\beta\frac{d\tilde{A}_1}{dz}=0, \qquad (13)$$

$$\frac{d^2\tilde{A}_2}{dz^2}-\lambda^2\tilde{A}_2-j\omega\mu_0\mu_2\sigma_2\tilde{A}_2=0, \qquad (14)$$

$$\tilde{A}_0|_{z=0}=\tilde{A}_1|_{z=0},\ \frac{d\tilde{A}_0}{dz}|_{z=0}=\frac{1}{\mu_m}\frac{d\tilde{A}_1}{dz}|_{z=0}, \qquad (15)$$

$$\tilde{A}_0|_{z=-d}=\tilde{A}_1|_{z=-d},\ \frac{1}{\mu_1}\frac{d\tilde{A}_1}{dz}|_{z=-d}=\frac{1}{\mu_2}\frac{d\tilde{A}_2}{dz}|_{z=-d}, \qquad (16)$$

$\tilde{A}_0\to 0$ as $z\to+\infty$, $\tilde{A}_2\to 0$ as $z\to-\infty$. (17)

In order to construct the solution to (12) we consider the following two regions: $0<z<h$ and $z>h$ (the solutions in these regions are denoted by $\tilde{A}_{00}$ and $\tilde{A}_{01}$, respectively). Thus,

$$\frac{d^2\tilde{A}_{00}}{dz^2}-\lambda^2\tilde{A}_{00}=0, \qquad 0<z<h, \qquad (18)$$

$$\frac{d^2\tilde{A}_{01}}{dz^2}-\lambda^2\tilde{A}_{01}=0, \qquad z>h. \qquad (19)$$

The general solution to (18) can be written in the form

$$\tilde{A}_{00}=C_1e^{\lambda z}+C_2e^{-\lambda z}. \qquad (20)$$

The solution to (19) satisfying (17) is

$$\tilde{A}_{01}=C_3e^{-\lambda z}. \qquad (21)$$

The solution to (13) can be expressed in terms of modified Bessel functions (see [12], formula 2.1.3.10, page 247):

$$\tilde{A}_1(\lambda,z)=C_4e^{\beta z/2}I_\nu\left(ce^{(\alpha+\beta)z/2}\right)+C_5e^{\beta z/2}K_\nu\left(ce^{(\alpha+\beta)z/2}\right), \qquad (22)$$

where $c=\dfrac{2\sqrt{j\omega\mu_0\mu_m\sigma_m}}{\alpha+\beta}$ and $\nu=\dfrac{\sqrt{\beta^2+4\lambda^2}}{\alpha+\beta}$.

Finally, the solution to (14) which is bounded as $z\to-\infty$ is

$$\tilde{A}_2(\lambda,z)=C_6e^{qz}, \qquad (23)$$

where $q=\sqrt{\lambda^2+j\omega\sigma_2\mu_0\mu_2}$.

Continuity of the vector potential at $z=h$ gives

$$C_1e^{\lambda h}+C_2e^{-\lambda h}=C_3e^{-\lambda h}. \qquad (24)$$

Integrating equation (12) with respect to z from $h-\varepsilon$ to $h+\varepsilon$, considering the limit as $\varepsilon\to+0$ and using continuity of the function $\tilde{A}_0$ at $z=h$ we obtain

$$-C_3\lambda e^{-\lambda h}-C_1\lambda e^{\lambda h}+C_2\lambda e^{-\lambda h}=-\mu_0 I r_c J_1(\lambda r_c). \qquad (25)$$

Arbitrary constants C_1-C_6 in (20)-(23) are determined using conditions (15), (16), (24) and (25). In particular, the constant C_2 has the form

$$C_2=\frac{\mu_0 I r_c J_1(\lambda r_c)e^{-\lambda h}}{2\lambda}\cdot\frac{B}{D}, \qquad (26)$$

where

$$B=-p[(\lambda\mu_m-\beta/2)I_\nu(c)-c(\alpha+\beta)I_\nu'(c)/2]+(\lambda\mu_m-\beta/2)K_\nu(c)-c(\alpha+\beta)K_\nu'(c)/2, \qquad (27)$$

$$D=-p[(\lambda\mu_m+\beta/2)I_\nu(c)+c(\alpha+\beta)I_\nu'(c)/2]+(\lambda\mu_m+\beta/2)K_\nu(c)+c(\alpha+\beta)K_\nu'(c)/2, \qquad (28)$$

and

$$p=\frac{E}{F}, \qquad (29)$$

where

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$$E=(\mu_2\beta-2\mu_1 q)K_\nu\left(ce^{-(\alpha+\beta)d/2}\right)+\mu_2 c(\alpha+\beta)e^{-(\alpha+\beta)d/2}K_\nu^{'}\left(ce^{-(\alpha+\beta)d/2}\right), \quad (30)$$

$$F=(\mu_2\beta-2\mu_1 q)I_\nu\left(ce^{-(\alpha+\beta)d/2}\right)+\mu_2 c(\alpha+\beta)e^{-(\alpha+\beta)d/2}I_\nu^{'}\left(ce^{-(\alpha+\beta)d/2}\right). \quad (31)$$

The induced vector potential (in the transformed space) due to the presence of a two-layer medium can be written in the form

$$\widetilde{A}_0^{ind}(\lambda,z)=C_2 e^{-\lambda z}, \quad (32)$$

where C_2 is given by (26).

Applying the inverse Hankel transform

$$A_i(r,z)=\int_0^\infty \widetilde{A}_i(\lambda,z)\lambda J_1(\lambda r)d\lambda,\ i=0,1,2$$

to (32) we obtain the induced vector potential of the form

$$A_0^{ind}(r,z)=\frac{\mu_0 I r_c}{2}\int_0^\infty \frac{B}{D}J_1(\lambda r)J_1(\lambda r_c)e^{-\lambda(z+h)}d\lambda, \quad (33)$$

where B and D are given by (27) and (28), respectively. The induced change in impedance of the coil is computed by means of the formula (see [4]) as follows

$$Z_{ind}=\frac{j\omega}{I}A_0^{ind}(r_c,h)\cdot 2\pi r_c. \quad (34)$$

Using (33) and (34) we obtain the following formula for the computation of the change in impedance

$$Z_{ind}=j\omega\mu_0\pi r_c^2\int_0^\infty \frac{B}{D}J_1^2(\lambda r_c)e^{-2\lambda h}d\lambda. \quad (35)$$

4 Numerical results

Numerical computations of the integral in (35) are done with Mathematica. It is convenient to rewrite (35) in dimensionless form. Introducing new variable $s=\lambda r_c$ we transform (35) to the following form:

$$Z_{ind}=j\omega\mu_0\pi r_c Z,$$

where

$$Z=\int_0^\infty \frac{\widetilde{B}}{\widetilde{D}}J_1^2(s)e^{-\delta s}ds. \quad (36)$$

Here

$$\widetilde{B}=-\widetilde{p}[(s\mu_m-\eta/2)I_\nu(\widetilde{c})-\beta_1\sqrt{j}I_\nu^{'}(\widetilde{c})]+(s\mu_m-\eta/2)K_\nu(\widetilde{c})-\beta_1\sqrt{j}K_\nu^{'}(\widetilde{c}),$$

$$\widetilde{D}=-\widetilde{p}[(s\mu_m+\eta/2)I_\nu(\widetilde{c})+\beta_1\sqrt{j}I_\nu^{'}(\widetilde{c})]+(s\mu_m+\eta/2)K_\nu(\widetilde{c})+\beta_1\sqrt{j}K_\nu^{'}(\widetilde{c}),$$

$$\widetilde{p}=\frac{\widetilde{E}}{\widetilde{F}},$$

$$\widetilde{E}=(\mu_2\eta-2\mu_1\sqrt{s^2+j\beta_2^2})K_\nu\left(\widetilde{c}e^{-(\xi+\eta)\gamma/2}\right)+2\mu_2\beta_1\sqrt{j}e^{-(\xi+\eta)\gamma/2}K_\nu^{'}\left(\widetilde{c}e^{-(\xi+\eta)\gamma/2}\right),$$

$$\widetilde{D}=(\mu_2\eta-2\mu_1\sqrt{s^2+j\beta_2^2})I_\nu\left(\widetilde{c}e^{-(\xi+\eta)\gamma/2}\right)+2\mu_2\beta_1\sqrt{j}e^{-(\xi+\eta)\gamma/2}I_\nu^{'}\left(\widetilde{c}e^{-(\xi+\eta)\gamma/2}\right),$$

$$\beta_1=r_c\sqrt{\sigma_m\mu_0\mu_m},\ \delta=\frac{2h}{r_c},\ \ \xi=\alpha r_c,\ \ \eta=\beta r_c,$$

$$\widetilde{c}=\frac{2\beta_1\sqrt{j}}{\xi+\eta},\ \beta_2=\beta_1\theta,$$

$$\nu=\frac{\sqrt{\eta^2+4s^2}}{\xi+\eta},\ \theta=\sqrt{\frac{\sigma_2\mu_2}{\sigma_m\mu_m}}.$$

The change in impedance, Z, computed by means of (36) is shown in Fig. 2 for the following values of the parameters:
$\gamma=0.2,\ \theta=1.5,\ \mu_m=1,\ \mu_2=100,$
$\delta=0.05,\ \eta=-4.$

Calculated points on each curve correspond to different values of β_1 (in the range from 3 to 10), larger β values correspond to larger values of the real part of Z. It is seen from the graph that for larger negative ξ the modulus of the change in impedance is getting smaller.

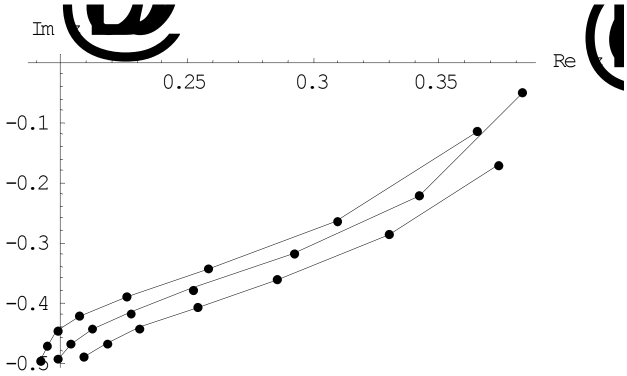

Fig.2. The real and imaginary parts of Z for three values of $\xi : -4, -2$ and 0 (from top to bottom). Calculated points on each curve correspond to different values of β_1.

5 Conclusion

Analytical solution for the change in impedance of a single-turn coil located above a two-layer conducting medium with varying electric conductivity and magnetic permeability in the upper layer is obtained in the present paper. The properties of the lower layer are assumed to be constant. Numerical results demonstrate that computations can easily be done for different values of the parameters of the problem. Such a model can be quite useful in applications. It is shown in [6] that under certain conditions (for example, as a result of surface hardening of ferromagnetic materials) a thin layer of reduced magnetic permeability appears near the surface. Experimental data show that the relative magnetic permeability can be approximated by an exponential function of the vertical coordinate. Mathematical model developed in this paper can be used in such cases for the calculation of the change in impedance of a coil placed above a ferromagnetic material with continuously varying electric conductivity and magnetic permeability.

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