A geometric approach to a non stationary process

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Abstract— In this work, we consider a non stationary random process as a geometric space with a variable curvature. The latter can be, obviously, locally constant leading to a constant local metric tensor or equivalently, a constant auto-correlation for the same interval and hence a local stationary behaviour. As an application to clarify this point view, we have, therefore, used this local curvature approximation for modeling or compressing the image of V₂O₅/TeO₂ amorphous thin film structure with a Gaussian stationary ergodic random process.

Keywords— non stationary random process; metric tensor; ergodic; local curvature; image compression

1 INTRODUCTION

The auto-correlation can be used to compare two signals or two samples of the same process. Its value indicates how much two signals are correlated. For the random process, the auto-correlation is calculated by the inner product in Hilbert space using the mathematical Esperance as follows [1,…,4]

\[
\phi_X(i, j) = E[X_i, X_j] \tag{1}
\]

Where \(X_i\) and \(X_j\) are two vectors of the process at two different instants \(i\) and \(j\). If the terms of the same parallel diagonals to the principle of the auto-correlation matrix are different then the process is non stationary. The aim is to show that a non stationary process may be considered geometrically, as a surface evolving in the time space or, eventually, another different space with a variable curvature. This allows us to use simple notions of Euclidean tangent space on any local region of a curved space by considering any very small region as flat [5,6,7]. Similarly, as a consequence, any small interval of non stationary process can be treated as stationary and even ergodic. So, in the following we use, therefore, geometric terms such as the metric tensor in the same way as the auto-correlation, to study a non stationary random process.

2 Discussion

2.1. General discussion

A random process, which is defined as an infinite number of curves of the same repeated event, can be seen as a geometric surface in our approach. A stationary random process may correspond, geometrically, to a surface with constant curvature. If the latter vanishes, then the corresponding space is flat. As a consequence the metric tensor for the latter can be given by the following inner product of the orthonormal base [6,8]

\[
g_{kl} = \epsilon_k \cdot \epsilon_l = \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \tag{2}
\]

This relation is equivalent to the following auto-correlation in Hilbert space;

\[
\phi_X(i, j) = \sigma^2 \delta_{ij} = \sigma^2 g_{ij} \tag{3}
\]
In which $\sigma^2$ is constant representing the process variance.

An example of a non-stationary process may be represented by the curves (o , ., *, and +) of fig.(1). We can see that the process in the intervals [0 20] and [20 40] do not look the same; hence the autocorrelation is not constant for the same intervals. Similarly a metric tensor of a surface with variable curvature is not constant. If we focus on a small space or on a small interval of the process, then the curvature may not vary too much and, thus, it could be considered as approximately constant. This procedure of approximating a local curved space by a tangent flat space is known as Gauss approximation. It is possible, therefore, to divide the process into many small stationary intervals or, geometrically, into small spaces with constant curvature and hence stationary. This is, particularly, illustrated in fig.(2) by the two small intervals [20 25] and [45 50]. Furthermore, the smaller these intervals the more stationary and ergodic they will be since their mean value calculated vertically can be obtained horizontally on any curve in these intervals. This mean value which can be described by the mathematical Esperance is, therefore, almost constant $E[X,v] = cte = m_v$. If, in addition, the ergodic process samples are independent then each small interval can be described by the metric tensor (5). It should be, however, well noted, that it is not possible to obtain an ergodic process by reconstructing it from these small ergodic intervals.

In this example we have taken the mixed oxide film of 60%$V_2O_5$ and 40%$TeO_2$ which we prepared [9] by thermal co-evaporation in a vacuum better than $10^{-6}$ torr. The electron diffraction image of the $V_2O_5/TeO_2$ amorphous thin film structure is given below. Due to its circular symmetry, we focused on a small part from its top to obtain a smaller size (the image below the original). This leads to a reduction of its dimension from 1690x1556 to 333x366.

Fig. 2. Non stationary random process; Intervals [20 25] and [45 50] are smaller enough to be considered as stationary.

2.2. Example of Gauss approximation application to the electron diffraction image of the $V_2O_5/TeO_2$ amorphous thin film structure.

Fig.1. Non stationary random process, interval [20 40] is very large, hence non stationary.

Fig. 3. The entire original image (above) and a portion of its top (below)
Our aim, in treating such example, is simply to see the quality of the Gauss approximation. The latter suggests that small intervals of any non-stationary process can be assimilated, geometrically, to a local tangent flat space corresponding, conventionally in signal processing, to a stationary process. So by assuming that every small interval, such as those [20, 25] and [45, 50] of fig. (2), is stationary and ergodic, only one curve is sufficient to study the whole process corresponding to the interval. This is, in fact, what we applied in this example to each matrix line of the above image fig. (3).

Since its structure was prepared by vacuum evaporation with a constant deposition rate, we have, naturally, proposed a Gaussian stationary model [10, ..., 13] to represent each interval of a matrix line. In the following figures (4-8), we have plotted, in each figure, four original lines and their corresponding reconstructed ones as well as the original image and its reconstructed by the proposed Gaussian model, using our approach. We have calculated the mean square error (MSE) between each original and its corresponding reconstructed line for five different numbers of intervals. The results are presented in table (1).

Table-1: MSE for different numbers \( N_i \) of intervals in each of the four matrix lines

<table>
<thead>
<tr>
<th>Line</th>
<th>MSE ( (N_i=2) )</th>
<th>MSE ( (N_i=5) )</th>
<th>MSE ( (N_i=10) )</th>
<th>MSE ( (N_i=15) )</th>
<th>MSE ( (N_i=25) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.0014</td>
<td>0.0008</td>
<td>0.0006</td>
<td>0.0003</td>
<td>0.0002</td>
</tr>
<tr>
<td>100</td>
<td>0.0054</td>
<td>0.0015</td>
<td>0.0006</td>
<td>0.0003</td>
<td>0.0003</td>
</tr>
<tr>
<td>150</td>
<td>0.0097</td>
<td>0.0034</td>
<td>0.002</td>
<td>0.0011</td>
<td>0.0007</td>
</tr>
<tr>
<td>300</td>
<td>0.0011</td>
<td>0.0006</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Fig (4) shows the reconstructed (compressed) image obtained by dividing each matrix line only into two intervals \( N_i=2 \). We can see that the reconstructed lines are very noisy and the reconstructed image is completely different from the original. Fig (5) shows that, when the interval length is slightly decreased by increasing the intervals number to \( N_i=5 \), the reconstructed lines are becoming less noisy and the compressed image looks more like the original but not smooth. As the interval length is more reduced by dividing each matrix line into ten intervals fig (6), we observe that the compressed image gets more homogeneous and the reconstructed lines are close to the originals as indicated by the MSE in table (1).
In figures (7) and (8) corresponding to \( Ni=15 \) and \( Ni=25 \) respectively, we can hardly distinguish between the original and the compressed images. But in looking at the MSE in table (1), we can confirm, in general, that the smaller the interval the more closely the reconstructed image to the original will be. This is in a good agreement with the Gauss approximation; a local region in a curved space [14] could be approximated by a tangent flat space. This corresponds, in signal processing, to a small interval in a non stationary process, could be considered as stationary and ergodic.

Notice that the size of the matrix representing the Gaussian model in each case is \( 333 \times (2xNi) \), while the original matrix size is \( 333 \times 366 \). \( (2xNi) \) is the number of the model parameters; the variances and mean values.

**4 Conclusion**

In this work, we have shown that it is possible to assimilate a non stationary process to a curved space with variable curvature. More importantly, we have verified, using the image processing example, that a local small interval of the non stationary process, can be considered as stationary and ergodic. This is in a good agreement with our geometric approach which is, a local curved space can be approximated by a tangent flat space, according to Gauss approximation.

**References**